MATH 6030, PROBLEM SET 1, DUE FEBRUARY 18

There 2 problems worth 14 points total. Your formal score is the minimum of the actual score and 10.

Problem 1, 7pts total. *This problem investigates the distribution spaces $D_\alpha(X)$. Its ultimate goal, as we will see later, is to equip $D(G)$ with a Hopf algebra structure.*

a, 2pts) Let X_i be affine varieties over $\mathbb{F}$, $\alpha_i \in X_i$, and $\delta_i \in D_{\alpha_i}(X_i), i = 1, 2$. Arguing as in Section 1.2 of Lecture 3, one can view $\delta_1 \otimes \delta_2$ as an element of $D_{(\alpha_1,\alpha_2)}(X_1 \times X_2)$. Show that the map sending $\delta_1 \otimes \delta_2 \in D_{\alpha_1}(X_1) \otimes D_{\alpha_2}(X_2)$ to the eponymous element of $D_{(\alpha_1,\alpha_2)}(X_1 \times X_2)$ is an isomorphism $D_{\alpha_1}(X_1) \otimes D_{\alpha_2}(X_2) \xrightarrow{\sim} D_{(\alpha_1,\alpha_2)}(X_1 \times X_2)$. In what follows we identify these two spaces using this isomorphism.

b, 2pt) We write d_X for the diagonal embedding $X \hookrightarrow X \times X, \alpha \mapsto (\alpha, \alpha)$. Let

$$\Delta := d_{X,*} : D_\alpha(X) \to D_{(\alpha,\alpha)}(X \times X) = D_\alpha(X) \otimes D_\alpha(X),$$

and $\eta : D_\alpha(X) \to \mathbb{F}$ be given by $\delta \mapsto \delta(1)$, where 1 is the unit in $\mathbb{F}[X]$. Prove the following identities

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \quad (\eta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \eta) \circ \Delta = \mathrm{id},$$

where in the above equalities id stands for the identity endomorphism of $D_\alpha(X)$.

c, 3pts) Now let X be an algebraic group G and $\alpha = e$. As we know from Lecture 3, Sec. 1.2, $D(G)$ becomes a unital associative algebra with respect to convolution. Let $\epsilon : \mathbb{F} \to D(G)$ be the map sending $z \in \mathbb{F}$ to $z1 \in D(G)$. Let $i : G \to G$ denote the inversion map, and $S := i_* : D(G) \to D(G)$. Finally, let $m : D(G) \otimes D(G) \to D(G)$ be given by $\delta_1 \otimes \delta_2 \mapsto \delta_1 * \delta_2$. Prove the following claims:

(1) Δ is a homomorphism of unital algebras.
(2) $S(\delta_1) * S(\delta_2) = S(\delta_2 * \delta_1)$ for all $\delta_1, \delta_2 \in D(G)$.
(3) $m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \epsilon \circ \eta$, an equality of maps $D(G) \to D(G)$.

Hint: modulo a), parts b) and c) are about equalities between certain morphisms.

Problem 2, 7pts total. *This problem concerns the classical harmonic polynomials and shows the ubiquity of $\mathfrak{sl}_2$ in Math!* Consider the Laplace operator $\Delta := \sum_{i=1}^n \partial_i^2$, the Euler operator $\mathsf{eu} = \sum_{i=1}^n x_i\partial_i$, and the operator F of multiplication by $-\frac{1}{4}\sum_{i=1}^n x_i^2$ acting on the space $\mathbb{C}[x_1, \ldots, x_n]$.

a, 1pt) Show that the assignment $e \mapsto \Delta, f \mapsto F, h \mapsto -\mathsf{eu} - \frac{n}{2}$ defines a representation of $\mathfrak{sl}_2$ in $\mathbb{C}[x_1, \ldots, x_n]$.

b, 3pts) Set $\Lambda_n := -\frac{n}{2} - \mathbb{Z}_{\geqslant 0}$. Consider an $\mathfrak{sl}_2$-module M, where h acts diagonalizably with finite dimensional eigenspaces and eigenvalues in Λ_n. Show that M is isomorphic to the direct sum of (irreducible) Verma modules $\Delta(\lambda)$ with $\lambda \in \Lambda_n$.

c, 2pt) Deduce that every polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ can be uniquely written as $\sum_{k=0}^\infty F^k h_k$, where h_k is a *harmonic* polynomial, i.e., a polynomial killed by Δ.

d, 1pt) Prove that the projection $\mathbb{R}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{R}[x_1, \ldots, x_n]/(x_1^2 + \ldots + x_n^2 - 1)$ restricts to an isomorphism between the subspace of all harmonic polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ and $\mathbb{R}[x_1, \ldots, x_n]/(x_1^2 + \ldots + x_n^2 - 1)$. *This is an algebraic counterpart of the theorem in PDE* *that a solution to the Laplace equation in a domain with fixed boundary condition exists and is unique; the domain of interest is a ball.*