MATH 6030, PROBLEM SET 2, DUE MARCH 30

There 4 problems worth 14 points total. Your formal score is the minimum of the actual score and 10.

Problem 1, 4pts. *We have seen in Lecture 8 that over characteristic* 0 *fields the rational representations of* SL_2 *and the finite dimensional representations of* $\mathfrak{sl}_2$ *are basically the same. This miserably fails in positive characteristic. A purpose of this problem is to illustrate this.* Let $\mathbb{F}$ denote an algebraically closed field of characteristic $p > 2$. Recall baby Verma modules $\underline{\Delta}^z(\lambda)$ over $\mathfrak{sl}_2$ introduced in Section 2.1 of Lecture 7. Show that $\underline{\Delta}^z(\lambda)$ is obtained from a rational representation of SL_2 if and only if $z = 0$ and $\lambda = p-1$. *Hint: an easier task in the "if" part is to show that* $z = 0$ *and* $\lambda \in \mathbb{F}_p$. *Then to show that* $\lambda = p-1$ *one should look at the weights and the action of the center of* SL_2.

Problem 2, 5pts total. *In this problem we will examine two classes of representations of* $G = \mathrm{SL}_2$*:* $M(n)$ *(a.k.a. the dual Weyl modules) and their duals, Weyl modules,* $W(n) := M(n)^*$.

1, 1pt) Let V be a rational representation of G whose maximal weight is at most n. Construct an embedding $V_n \hookrightarrow \mathrm{Hom}_G(W(n), V)$.

2, 1pt) Show that $\mathrm{Hom}_G(M(k), M(\ell))$ is zero for $k < \ell$ and is 1-dimensional for $k = \ell$. State and prove an analog of this result for the Weyl modules $W(n)$.

3, 1pt) Show that $\dim \mathrm{Hom}_G(W(k), M(\ell)) = \delta_{k,\ell}$ (the Kronecker symbol).

4, 2pts) Show that every short exact sequence

$$0 \to M(k) \to V \to W(\ell) \to 0$$

of rational representations of G splits (i.e., $V \twoheadrightarrow W(\ell)$ has a right inverse).

One can give an abstract definition of a category, where "highest weight theory works". We get so called "highest weight categories". In such categories we have a bunch of families of objects, including "standard" and "costandard" objects. It turns out that in the category of rational representations of SL_2 *these objects are the Weyl and dual Weyl modules. The properties in the problem are general properties of standard and costandard objects.*

Problem 3, 2pts. Let U, V be ($\mathbb{Z}_{\geqslant 0}$-)filtered vector spaces and $\varphi : U \to V$ be a filtered linear map.

1, 1pt) Show that if $\operatorname{gr}\varphi$ is injective, then φ is injective.

2, 1pt) Show that if $\operatorname{gr}\varphi$ is surjective, then φ is surjective.

Problem 4, 3pts. Let A be a Hopf algebra over a field $\mathbb{F}$ that is cocommutative meaning that for the swap isomorphism $\sigma : A^{\otimes 2} \to A^{\otimes 2}, a \otimes b \mapsto b \otimes a$, we have $\sigma \circ \Delta = \Delta$. An element $a \in A$ is called primitive if $\Delta(a) = a \otimes 1 + 1 \otimes a$.

1, 1pt) Prove that the primitive elements form a Lie subalgebra in A.

2, 2pts) Now suppose that $\mathrm{char}\,\mathbb{F} = 0$. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Prove that all primitive elements in $U(\mathfrak{g})$ lie in $\mathfrak{g}$.