MATH 6030, PROBLEM SET 3, DUE APRIL 24

There 3 problems worth 16 points total. Your formal score is the minimum of the actual score and 10. You can use previous parts of the problems to prove the subsequent parts and get full credit for the subsequent parts even if you haven't solved the previous parts.

Problem 1, 4pts total. Let λ be a dominant weight. *Turns out one can describe the set of weights of $L(\lambda)$ quite explicitly.* The purposes of this problem is to prove that

(*) any <u>dominant</u> weight μ that is $\leqslant \lambda$ is a weight of $L(\lambda)$.

a, 1pt) For a positive root $\beta = \epsilon_i - \epsilon_j$ (with $i < j$) we write h_β for $E_{ii} - E_{jj}$. Let ν be a weight and V be a finite dimensional $\mathfrak{g}$-representation. Prove that if $v \in V_\nu$ is a nonzero vector, and $\langle \nu, h_\beta \rangle > 0$, then $f_\beta v \neq 0$.

b, 2pts) Let μ be as in (*). Write $\lambda - \mu$ as $\gamma_1 + \ldots + \gamma_k$, where γ_i's are positive roots, and k is minimal among all such expressions. Prove that $\langle \lambda - \sum_{j=1}^{i-1} \gamma_j, h_{\gamma_i} \rangle > 0$ for all $i = 1, \ldots, k$.

c, 1pt) Deduce (*).

As a comment, recall that if μ is a weight if and only if $w\mu$ is a weight for any $w \in W$ (the Weyl group). So the previous paragraph allows to completely describe the set of weights.

Problem 2, 8pts total. *The point of this outrageously long problem is to study a part of the center of $U(\mathfrak{sl}_n)$ in the case when the field has positive characteristic. As we have seen in the case of $\mathfrak{sl}_2$, the center is much bigger than in characteristic* 0. *Still there's an important subalgebra in the center whose description mirrors that in characteristic* 0. Below $\mathbb{F}$ denotes an algebraically closed field, $G = \mathrm{SL}_n, \mathfrak{g} = \mathfrak{sl}_n$. We assume that $\operatorname{char} \mathbb{F} = p$ with $p > n$. We will be studying the subalgebra $U(\mathfrak{g})^G$ and its description via a direct analog of the Harish-Chandra homomorphism.

a, 1pt) Justify that there is an analog of the homomorphism $z \mapsto \mathrm{HZ}_z : U(\mathfrak{g})^G \to U(\mathfrak{h})$, and that it is injective.

b, 1pt) Use the assumption on the characteristic of $\mathbb{F}$ to show that the map $\mathrm{gr}(U(\mathfrak{g})^G) \to S(\mathfrak{g})^G$ is an isomorphism. *Hint: one cannot symmetrize an arbitrary element of $S(\mathfrak{g})^G$ but one can symmetrize generators.*

c, 1pt) Reduce the proof of the claim that the image of $z \mapsto \mathrm{HC}_z : U(\mathfrak{g})^G \to \mathbb{F}[\mathfrak{h}^*]$ is $\mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$ to the claim that the image is invariant under $s_i\cdot$ for every $i = 1, \ldots, n-1$. *This invariance will be established in the subsequent parts.*

d, 1pt) Show that the subalgebra $U(\mathfrak{sl}_2)^{\mathrm{SL}_2} \subset U(\mathfrak{sl}_2)$ is generated by the Casimir element C.

e, 1pt) For $i = 1, \ldots, n-1$, consider the subgroup $L_i \subset G = \mathrm{SL}_n$ consisting of all block diagonal matrices of the form $\mathrm{diag}(a_1, \ldots, a_{i-1}, g_i, a_{i+2}, \ldots, a_n)$, where $a_j \in \mathbb{F} \setminus \{0\}$ and $g_i \in \mathrm{GL}_2$ with $\det(g_i) \prod a_j = 1$. Let $\mathfrak{l}_i \subset \mathfrak{g}$ be the Lie algebra of L_i. Show that $U(\mathfrak{l}_i)^{L_i}$ is the algebra of polynomials in the following $n-1$ variables: $h_j, j \neq i, i+1, h_i + h_{i+1}, C_i$, where C_i is the Casimir element of $\mathfrak{g}_i$.

f, 1pt) We write $U(\mathfrak{g})_0, U(\mathfrak{l}_i)_0$ for the subalgebras of weight 0 elements. Prove that any element $a \in U(\mathfrak{g}_0)$ is uniquely written in the following form: $a = a(i) + \sum_\beta a_\beta e_\beta$, where the summation is over all positive roots β different from $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $a(i) \in U(\mathfrak{l}_i)_0$, and $a_\beta \in U(\mathfrak{g})$. Further prove that $a \mapsto a(i) : U(\mathfrak{g})_0 \to U(\mathfrak{l}_i)_0$ is an algebra homomorphism.

g, 1pt) Prove that $z(i) \in U(\mathfrak{l}_i)^{L_i}$ for all $z \in U(\mathfrak{g})^G$ and the map $z \mapsto \mathrm{HC}_z$ factors as the composition of $z \mapsto z(i)$ and the analog of Harish-Chandra homomorphism $U(\mathfrak{l}_i)^{L_i} \to U(\mathfrak{h})$.

h, 1pt) Use c) and g) to show that $z \mapsto \mathrm{HC}_z$ gives an isomorphism $U(\mathfrak{g})^G \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$.

This way of thinking about the Harish-Chandra homomorphism was discovered when the instructor and G. Dhillon worked on describing the "Harish-Chandra center" for the affine Lie algebras.

Problem 3, 4pts total. Let $\mathbb{F}_q^\times$ denote the multiplicative group of $\mathbb{F}_q$ and let $\chi_1, \ldots, \chi_n$ be pairwise distinct group homomorphisms $\mathbb{F}_q^\times \to \mathbb{C}^\times$. Consider the subgroup of upper triangular matrices $B_q \subset G_q := \mathrm{GL}_n(\mathbb{F}_q)$ and let $\chi : B_q \to \mathbb{C}^\times$ be the homomorphism sending the matrix (a_{ij}) to $\prod_{i=1}^n \chi_i(a_{ii})$ that we view as a 1-dimensional B_q-representation.

a, 2pts) Prove $\mathrm{Ind}_{B_q}^{G_q} \chi$ is irreducible.

b, 2pts) Prove that, up to an isomorphism, $\mathrm{Ind}_{B_q}^{G_q} \chi$ only depends on $\chi_1, \ldots, \chi_n$ up to permutation.

Hint: look at functions on G_q that transform in a suitable way under the actions of B_q from the left and from the right.