Lec 1: algebraic groups & Lie algebras I

1) Algebraic geometry basics.
2) Algebraic groups & their representations.

1) Algebraic geometry basics

Our first topic is the introduction & study of basics for fundamental objects for this course: algebraic groups, their Lie algebras & their representations.

In the first lecture, we will introduce algebraic groups that, for us, are affine algebraic varieties w. a compatible group structure. For this, we will need basics about affine varieties.

Let $\mathbb{F}$ be an algebraically closed field (of any characteristic).

Definition: • By an <u>embedded affine variety</u> we mean a subset of $\mathbb{F}^n$ (for some n) defined by polynomial equations.

• Let $X \subset \mathbb{F}^n$, $Y \subset \mathbb{F}^m$ be embedded affine varieties. A map $\varphi: X \to Y$ is called polynomial (a.k.a. a morphism) if $\exists f_1, \ldots f_m \in \mathbb{F}[x_1, \ldots x_n]$ s.t. $\varphi = (f_1, \ldots f_m)|_X$.

• The algebra of polynomial functions $\mathbb{F}[X]$ consists of polynomial maps $X \to \mathbb{F}$ w. usual addition and multiplication of functions.

As usual in Geometry, we should try to view geometric objects in a coordinate free way. For affine varieties, this means "irrespective of an embedding into $\mathbb{F}^n$ & can be done as follows.

i) Set $I(X)=\{f\in\mathbb{F}[x_1,..x_n] \mid f|_X=0\}$, an ideal s.t. $I(X)=\sqrt{I(X)}$. We have $\mathbb{F}[X]\xrightarrow{\sim}\mathbb{F}[x_1,..x_n]/I(X)$ so $\mathbb{F}[X]$ is an $\mathbb{F}$-algebra w/o nonzero nilpotent elements & fixed finite collection of generators $\bar{x}_i=x_i+I(X)$ $(i=1,..n)$. Conversely to a finitely generated commutative $\mathbb{F}$-algebra A w/o nonzero nilpotent elements we assign embedded affine variety $X\subset\mathbb{F}^n$ w. $\mathbb{F}[X]=A$ once we choose n generators in A: the choice of generators gives $\varphi:\mathbb{F}[x_1,..x_n]\twoheadrightarrow A$ & $X:=\{\alpha\in\mathbb{F}^n \mid f(\alpha)=0, \forall f\in\ker\varphi\}$.

ii) We get an algebra homomorphism $\varphi^*:\mathbb{F}[Y]\longrightarrow\mathbb{F}[X]$, $g\mapsto g\circ\varphi \Rightarrow \varphi^*(y_i+I(Y))=f_i+I(X)$, $i=1,..m$. The assignment $\varphi\mapsto\varphi^*$ defines a bijection

(*) $\{\text{morphisms } X\to Y\}\xrightarrow{\sim}\mathrm{Hom}_{\mathbb{F}\text{-}\mathcal{A}lg}(\mathbb{F}[Y],\mathbb{F}[X])$

(*) is functorial: $(id_X)^*=id_{\mathbb{F}[X]}$ & for $\varphi:X\to Y$, $\psi:Y\to Z\Rightarrow$ $(\psi\circ\varphi)^*=\varphi^*\circ\psi^*$.

ii) allows us to talk about "abstract" affine varieties, X. They correspond to fin. generated $\mathbb{F}$-algebras w/o nilpotent elements. The choice of generators corresponds to an embedding of X into some $\mathbb{F}^n$ but we view X irrespective of an embedding. The notion of a morphism still makes sense in this setting.

Here are two important constructions:

(1) Let X be an affine variety & $f \in \mathbb{F}[X]$. Then $X_f := \{x \in X \mid f(x) \neq 0\}$ is an affine variety w. $\mathbb{F}[X_f] = \mathbb{F}[X][f^{-1}]$.

(2) Let X, Y be affine varieties. Then $X \times Y$ is also an affine variety w. $\mathbb{F}[X] \otimes_{\mathbb{F}} \mathbb{F}[Y] \xrightarrow{\sim} \mathbb{F}[X \times Y]$, $[f \otimes g](x,y) := f(x)g(y)$.

The following example of the notions appearing above is of crucial importance for us.

Example: Consider the group $GL_n = \{A \in \mathrm{Mat}_n(\mathbb{F}) \mid \det A \neq 0\}$.
By (1), it's an affine variety w. $\mathbb{F}[GL_n] = \mathbb{F}[x_{ij} \mid i,j = 1,..n][\det^{-1}]$
By (2), $GL_n \times GL_n$ is also an affine variety. Let
$m: GL_n \times GL_n \longrightarrow GL_n$, $(A,B) \mapsto AB$, be the product map.
It's a morphism w. m^* given by:

det (AB) = det (A) det (B)

$$m^*(x_{ij}) = \sum_{k=1}^{n} x^1_{ik} \otimes x^2_{jk}, \quad m^*(\det^{-1}) = (\det^1)^{-1} \otimes (\det^2)^{-1}$$

(the superscript in the r.h.s. is the # of copy of $\mathbb{F}[GL_n]$)

Similarly, the inversion map $i: GL_n \to GL_n$, $i(A) = A^{-1}$ is a morphism.

Rem: A subset, X, in an affine variety, Y, is called Zariski closed if it can be defined by polynomial equations, such subsets are indeed the closed subsets in a topology, the Zariski topology. Note that X is again an affine variety. The homomorphism $i^*: \mathbb{F}[Y] \to \mathbb{F}[X]$ corresponding to $i: X \hookrightarrow Y$ is surjective (exercise)

2) Algebraic groups & their representations.

2.1) Affine algebraic groups.

Definition:

By an (affine) algebraic group G we mean an affine variety equipped w. morphisms $m: G\times G\to G$ (multiplication) & $i: G\to G$ (inversion) satisfying axioms of a group.

Examples (of algebraic groups)

0) GL_n, the general linear group, see Example in Sec 1.

Here's a coordinate-free way to think about this group. If V is an n-dimensional vector space, we can talk about the group $GL(V)$ of invertible linear maps $V\to V$. A choice of a basis in V identifies $GL(V)$ w. GL_n.

To get more examples, notice that every Zariski closed (= given by polynomial equations) subgroup G in an algebraic group $\tilde{G}$ (an algebraic subgroup) is an algebraic group on its own: $m: G\times G\to G$ is a morphism: the composition $G\times G\hookrightarrow \tilde{G}\times\tilde{G}\to \tilde{G}$ is a morphism hence (exercise) $G\times G\to G$ is a morphism.

1) $SL_n=\{A\in GL_n \mid \det(A)=1\}$ is an algebraic subgroup of GL_n, as it's given by a single polynomial equation. This is the special linear group. And since det is independent of the choice of a basis, can talk about $SL(V)\subset GL(V)$.

2) Assume char $\mathbb{F} \neq 2$. Set $O_n = \{A \in GL_n \mid AA^T = I\}$. The matrix entries of AA^T are (quadratic) polynomials in those of A, so O_n is an algebraic subgroup of GL_n.

More conceptually, let B be a non-degenerate symmetric form on a vector space V of $\dim = n$ (all these forms have an orthonormal basis so there's no difference between them). Then we can consider $O(V,B) := \{g \in GL(V) \mid B(gu,gv) = B(u,v)\ \forall u,v \in V\}$ A choice of an orthonormal basis for B identifies $O(V,B)$ w. O_n. The group O_n (or $O(V,B)$) is called the orthogonal group.

Note that $\det(A) = \pm 1$ for $A \in O_n$. Set $SO_n := \{A \in O_n \mid \det A = 1\}$. This is also an algebraic subgroup called the special orthogonal group.

3) Similarly, for a non-degenerate skew-symmetric form ω on a finite dimensional vector space V (then, automatically, $\dim V$ is even) we can similarly consider the symplectic group $Sp(V,\omega) = \{g \in GL(V) \mid \omega(gu,gv) = \omega(u,v)\ \forall u,v \in V\}$. One can find a basis $v_1, \dots, v_{2n} \in V$ s.t $\omega(v_i, v_j) = \pm \delta_{i+j, 2n+1}$, where we have a "+" $\Leftrightarrow i \leq n$. Let J be the matrix of ω in this basis:
$J = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & 0 \end{pmatrix}$ so that $Sp(V,\omega) \xrightarrow{\sim} \{A \in GL_{2n} \mid A^T J A = J\} =: Sp_{2n}$.

The groups in Examples 0-3 are called classical. They are extremely important.

4) The subgroups of upper-triangular, $\left\{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\}$, upper-uni-triangular, $\left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\}$, and diagonal, $\{\mathrm{diag}(z_1, \dots, z_n)\}$ matrices in GL_n are algebraic.

5) The multiplicative group $\mathbb{F}^\times = GL_1(\mathbb{F})$ often denoted by $\mathbb{G}_m$, and the additive group $(\mathbb{F}, +)$, $\mathbb{G}_a$, are algebraic groups.

Exercise: The product of two algebraic groups is algebraic.

2.2) Homomorphisms & representations

Let G & H be algebraic groups.

Definition 1: A group homomorphism $\rho: G \to H$ is called an **algebraic group homomorphism** if it's a morphism of varieties.

Example: Let $G \subset \tilde{G}$ be an algebraic subgroup. Then inclusion map $G \hookrightarrow \tilde{G}$ is an algebraic group homomorphism.

Rem: The composition of algebraic group homomorphisms is also an algebraic group homomorphism.

Definition 2: Let V be a finite dimensional space. By a **rational representation** of G in V we mean an algebraic group homomorphism $G \to GL(V)$.

Studying rational representations is one of important goals of this course. Here's an equivalent condition of being rational.

Important exercise: Let $\rho: G \to GL(V)$ be some representation. For $\alpha \in V^*$, $v \in V$ define the matrix coefficient $c_{\alpha,v}: G \to \mathbb{F}$ by

$$c_{\alpha,v}(g) := \alpha(gv)$$

Then ρ is rational $\Leftrightarrow c_{\alpha,v} \in \mathbb{F}[G]$ $\forall \alpha \in V^*, v \in V$.

Hint: if $v_1, .. v_n \in V$ is a basis & $\alpha_1, .. \alpha_n \in V^*$ is the dual basis, then $\rho: G \to GL(V) \xrightarrow{\sim} GL_n$ is given by $\rho(g) = (c_{\alpha_i, v_j}(g))_{i,j=1}^n$; $\rho: G \to GL_n$ is a morphism $\Leftrightarrow$ the composition $G \xrightarrow{\rho} GL_n \hookrightarrow Mat_n$ is a morphism.

Examples (of rational representations)

0) Any algebraic subgroup $G \subset GL_n$ comes w. a tautological rational representation in $\mathbb{F}^n$. Can apply this to $G = GL_n, SL_n, SO_n, Sp_n$...

1) Usual operations w. representations preserve the rationality:

i) if $U \subset V$ is G-stable, then the representations of G in $U, V/U$ are rational if V is (their matrix coeffs are subsets of those for V).

ii) For similar reasons if V^1, V^2 are rational G-representations, then so is $V^1 \oplus V^2$.

iii) $V := V^1 \otimes V^2$ is rational: indeed if $\rho^i: G \to GL(V^i)$ are the corresponding representations, then the representation in $V^1 \otimes V^2$ is given by $\rho(g) := \rho^1(g) \otimes \rho^2(g)$ hence for $\alpha^i \in (V^i)^*$, $v^i \in V^i$ we have

$$c_{\alpha^1 \otimes \alpha^2, v^1 \otimes v^2} = c_{\alpha^1, v^1} c_{\alpha^2, v^2} \in \mathbb{F}[G].$$

Combining this w. i), we see that the representations in $Sym^n(V^1)$,

$\Lambda^n(V^*)$ are rational.

iv) If V is a rational representation of G, then so is V^* – indeed $\rho^*: G \longrightarrow GL(V^*)$ is the composition of $\rho: G \to GL(V)$ & isomorphism $GL(V) \xrightarrow{\sim} GL(V^*)$ given (after choosing a basis) by $g \mapsto (g^{-1})^t$

2) Suppose char $\mathbb{F} = p > 0$. In this case the map $x \mapsto x^p$ is an automorphism of the field $\mathbb{F}$ (the Frobenius automorphism). The map $Fr: GL_n(\mathbb{F}) \longrightarrow GL_n(\mathbb{F}), (a_{ij}) \mapsto (a_{ij}^p)$ is therefore an algebraic group homomorphism. It's an automorphism of an abstract group but not of an algebraic group (exercise)

Now suppose $\rho: G \longrightarrow GL_n$ be a rational representation. Then it's Frobenius twist $\rho^{(1)} := Fr \circ \rho : G \to GL_n$ is also rational.

Premium exercise: realize $\rho^{(1)}$ using i) & iii) of Example 1)

Rem (on terminology): For $G = GL_n(\mathbb{F})$ one talks about the polynomial representations: $c_{\alpha,v}(g)$ is a polynomial in the matrix entries of g. E.g. operations i)-iii) in Example 1 applied to the tautological representation in $\mathbb{F}^n$ give polynomial representations, while iv) does not. The name "rational" is used b/c we allow det in the denominator.