Lec 10: Hopf algebras, filtrations & gradings, I.

0) Completion of classification of SL_2-irreps
1) Hopf algebras.

0) Let $\mathbb{F}$ be an algebraically closed field w. char $\mathbb{F} \neq 2$ & $G = SL_2$. Consider $B = \{\begin{pmatrix} t & 0 \\ u & t^{-1}\end{pmatrix}\} \subset G$ Let $\mathbb{F}_n$ be the representation of B in $\mathbb{F}$, where $\begin{pmatrix} t & u \\ 0 & t^{-1}\end{pmatrix}$ acts by t^n In Sec 2.3 of Lec 9 we proved that $M(n) = \mathrm{Ind}_B^G \mathbb{F}_n$. In particular, Frobenius reciprocity implies

(1) $\mathrm{Hom}_G(V, M(n)) = \mathrm{Hom}_B(V, \mathbb{F}_n)$ $\forall$ rational G-rep V.

Last time we reduced the classification of G-irreps to

Claim: $\mathrm{Hom}_G(V, M(n)) \neq 0$ $\forall$ G-irrep. V w. max. weight n.

To prove this we need:

Lemma: $\forall$ rational G-rep V & any $m \in \mathbb{Z}$:

- $V_{\leqslant m} = \bigoplus_{k \leqslant m} V_k$ is a B-subrepresentation
- $V_{\leqslant m}/V_{\leqslant m-1} \simeq \mathbb{F}_m^{\oplus \dim V_m}$ as a B-representation

Proof:

$V_{\leqslant m}$ is stable under $\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$. So in 1) we need to show that $V_{\leqslant m}$ is stable under $\{\begin{pmatrix} 1 & 0 \\ u & 1\end{pmatrix}\}$ &, for 2), $\begin{pmatrix} 1 & 0 \\ u & 1\end{pmatrix}$ acts by id on $V_{\leqslant m}/V_{\leqslant m-1}$.

Pick a basis $v_1, \dots v_n \in V$ of weight vectors w. weights $m_1, \dots m_n$. Let $\alpha_1, \dots \alpha_n$ be the dual basis of V^* (of weights $-m_1, \dots, -m_n$). We claim that

(*) $f(u) := c_{\alpha_i, v_j}\begin{pmatrix} 1 & 0 \\ u & 1\end{pmatrix} \in \mathbb{F}[u]$ is homogeneous of deg. $(m_j - m_i)/2$

To prove (*) observe that $\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t^2u & 1 \end{pmatrix}$ $\forall t \in \mathbb{F}\setminus\{0\}$, $u \in \mathbb{F}$. So

$f(t^2u) = \langle \alpha_i, \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} v_j \rangle = \langle \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\alpha_i, \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} t^{m_j} v_j \rangle$

$= t^{m_j - m_i} \langle \alpha_i, \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} v_j \rangle = t^{m_j - m_i} f(u) \rightsquigarrow (*)$.

In particular, $[m_j < m_i \Rightarrow c_{\alpha_i, v_j}(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}) = 0] \Leftrightarrow (1)$. And if $m_j = m_i = m$ then $c_{\alpha_i, v_j}(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix})$ is constant, meaning that all elements $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ are represented by the same operator in $V_{\leq m}/V_{\leq m-1}$. It has to be the identity (b/c it is for $u=0$). $\square$

Proof of Claim: $\mathrm{Hom}_G(V, M(n)) = [(1)] = \mathrm{Hom}_B(V, \mathbb{F}_n)$ so it's enough to show the r.h.s. is $\neq 0$. Note that by 1) of Lemma, $V_{\leq n-1} \subset V$ is a B-subrep, so $\mathrm{Hom}_B(V, \mathbb{F}_n) \hookleftarrow \mathrm{Hom}_B(V/V_{\leq n-1}, \mathbb{F}_n) = [2)$ of Lemma$] = \mathrm{Hom}_B(\mathbb{F}_n^{\oplus \dim V_n}, \mathbb{F}_n) \neq 0$. $\square$

1) Hopf algebras

1.0) Introduction.

So far, we've left a number of claims w/o proof:

A (Sec 2 of Lec 2) Every affine algebraic group is isomorphic to an algebraic subgroup of GL_n for some n.

B (PBW Theorem, Sec 1.1 of Lec 5): If $x_1, \dots x_n$ is a basis in a Lie algebra $\mathfrak{g}$, then the ordered monomials $x_1^{d_1} \dots x_n^{d_n}$ ($d_i \geq 0$) form a basis in $U(\mathfrak{g})$.

C (Answer 3 in Sec 2 of Lec 5) If char $\mathbb{F}=0$, G is an algebraic group irreducible as a variety, $\mathfrak{g}=Lie(G)$, then $\forall$ $\mathfrak{g}$-linear map between rational G-reps is also G-linear, & $\forall$ $\mathfrak{g}$-subrepresentation is also a G-subrepresentation.

D (2) of Thm in Sec 1.1 of Lec 8) Let char $\mathbb{F}=p>0$, G be an algebraic group, $\mathfrak{g}=Lie(G)$ & $\iota:\mathfrak{g}\rightarrow Z(U(\mathfrak{g}))$ be given by $\iota(x)=x^p-x^{[p]}$. Then $\iota(x+y)=\iota(x)+\iota(y)$ $\forall$ $x,y\in\mathfrak{g}$.

This facts do not look related. However one can get (at least) partial proofs for all of them using two techniques

1) Hopf algebras & related objects
2) Gradings & filtrations on vector spaces.

Today we start our discussion of Hopf algebras. These are associative unital algebras w. additional structure. Here are two motivating questions to consider them.

I) Algebraic groups are affine varieties w. additional (group) structure. How does this reflect in their algebras of functions?

II) The representations of (finite) groups, G, & of Lie algebras, $\mathfrak{g}$, can be tensored & dualized. On the other hand, those are the representations of $\mathbb{F}G$ & $U(\mathfrak{g})$, associative algebras. For associative algebras, there are no operations of tensoring & dualizing representations. So $\mathbb{F}G$ & $U(\mathfrak{g})$ should come w. additional structures that enable those operations.

1.1) Definitions

Let A be an (associative unital) $\mathbb{F}$-algebra. From the product $A\times A\to A$, a bilinear map, we produce a linear map $\mu: A\otimes A\to A$, $\mu(a\otimes b)=ab$. And from $1\in A$ we produce a linear map $\varepsilon: \mathbb{F}\to A$, $z\mapsto z1$. Clearly one can recover the product from μ & 1 from ε. The axioms of an associative unital algebra in this language are the claims that the following diagrams are commutative:

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\mu\otimes id"'] \arrow[d, "id\otimes\mu"'] & A\otimes A \arrow[d, "\mu"] \\ A\otimes A \arrow[r, "\mu"] & A \end{tikzcd} \qquad \begin{tikzcd} & A\otimes A \arrow[d, "\mu"] & \\ \mathbb{F}\otimes A \arrow[ur, "\varepsilon\otimes id"] \arrow[r, "="] & A & A\otimes\mathbb{F} \arrow[l, "="'] \arrow[ul, "id\otimes\varepsilon"'] \end{tikzcd}$$

If we reverse all the arrow we get to the following

Definition 1: A (counital coassociative) coalgebra A over $\mathbb{F}$ is a vector space$/\mathbb{F}$ w. linear maps $\Delta: A\to A\otimes A$ (coproduct) & $\eta: A\to\mathbb{F}$ (counit) s.t. that the diagrams obtained from the above by reversing the arrows commute:

$$\begin{tikzcd} A\otimes A\otimes A & A\otimes A \arrow[l, "\Delta\otimes id"] \\ A\otimes A \arrow[u, "id\otimes\Delta"] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} & A\otimes A \arrow[dl, "\eta\otimes id"'] \arrow[dr, "id\otimes\eta"] & \\ \mathbb{F}\otimes A & A \arrow[l, "="'] \arrow[r, "="] \arrow[u, "\Delta"'] & A\otimes\mathbb{F} \end{tikzcd}$$

Now note that if A is a (unital associative) algebra, then $A \otimes A$ is so as well (w. maps $\mu \otimes \mu$ & $\varepsilon \otimes \varepsilon$).

Definition 2: A bialgebra over $\mathbb{F}$ is an algebra A over $\mathbb{F}$ equipped w. a coalgebra structure (Δ, η) s.t. $\Delta : A \to A \otimes A$ & $\eta : A \to \mathbb{F}$ are (unital) algebra homomorphisms.

Definition 3: A Hopf algebra is a bialgebra together with an antipode map, an algebra homomorphism $S : A \to A^{op}$ (= A w. opposite multiplication) making the following commutative:

$$\begin{tikzcd}
 & A \otimes A \arrow[rr, "S \otimes id"] & & A \otimes A \arrow[dr, "\mu"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\eta"] \arrow[dr, "\Delta"'] & & \mathbb{F} \arrow[rr, "\varepsilon"] & & A \\
 & A \otimes A \arrow[rr, "id \otimes S"'] & & A \otimes A \arrow[ur, "\mu"'] &
\end{tikzcd}$$

Remark: Let A be a Hopf algebra & U, V be A-modules. We can define their tensor product: $U \otimes V$ is an $A \otimes A$-module & we turn it into an A-module via Δ: $a.(u \otimes v) = \Delta(a)(u \otimes v) = [\Delta(a) = \sum a^i_{(1)} \otimes a^i_{(2)}]$ $= \sum a^i_{(1)} u \otimes a^i_{(2)} v$. Similarly, V^* is an A-module via $\langle a.\alpha, v \rangle =$ $= \langle \alpha, S(a) v \rangle$. We also have a distinguished trivial representation: A acts on $\mathbb{F}$ via η. The resulting tensor product of modules has many (but not all) familiar properties, e.g. the natural map $(U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$ is A-linear (from coassociativity). But $U \otimes V \xrightarrow{\sim} V \otimes U$ ($u \otimes v \mapsto v \otimes u$) may fail to be A-linear (this would require "cocommutativity" of Δ).

1.2) Examples.

1) Let G be an affine algebraic group w. unit e, product map $m: G\times G\longrightarrow G$ & inversion map $i: G\to G$. Then $A=\mathbb{F}[G]$ is a Hopf algebra w. $\Delta=m^*$, $\eta=e^*$ (where we view e as a morphism $pt\to G$) & $S=i^*$. The coassociativity, counit & antipode axiom follow from associativity, unit & inverse axioms for G. For example, the inverse axiom reads $m\circ(id\times i)\circ d=m\circ(i\times id)\circ d=e\circ p$, where $d: G\longrightarrow G\times G$ sends g to (g,g) & $p: G\longrightarrow pt$ is the only map. Passing to homomorphisms between algebras of functions & identifying $\mathbb{F}[G\times G]$ w. $\mathbb{F}[G]\otimes\mathbb{F}[G]$ we get e.g.

$$d^*\circ(id\otimes S)\circ m^*=p^*\circ e^*.$$

Note that $\mu=d^*$: $[d^*(f_1\otimes f_2)](g)=[f_1\otimes f_2](g,g)=f_1(g)f_2(g)$, & $\varepsilon=p^*$ so we've established one half of the antipode axiom.

Remark: For $g\in G$, let $\mathbb{F}_g$ be the 1-dimensional $\mathbb{F}[G]$-module, where $f\in\mathbb{F}[G]$ acts by multiplication by $f(g)$. Then on $\mathbb{F}_g\otimes\mathbb{F}_h$, $f\in\mathbb{F}[G]$ acts by $\Delta(f)=f\circ m$. If we write $f\circ m$ as $\sum f^i_{(1)}\otimes f^i_{(2)}$ w. $f^i_{(1)}, f^i_{(2)}\in\mathbb{F}[G]$, then the action is by $\sum f^i_{(1)}(g)f^i_{(2)}(h)=f(m(g,h))=$
$=f(gh)$. So $\mathbb{F}_g\otimes\mathbb{F}_h\simeq\mathbb{F}_{gh}$. If $gh\neq hg$, we get $\mathbb{F}_g\otimes\mathbb{F}_h\not\simeq\mathbb{F}_h\otimes\mathbb{F}_g$.

2) In the same setting the distribution algebra $D(G)$ w. operations $\mu=m_*$, $\varepsilon=e_*$, $\Delta=d_*$, $\eta=p_*$, $S=i_*$ is a Hopf algebra. The check that this is a Hopf algebra is Problem 1 in HW1.

2') A parallel example for finite groups G is the group algebra $\mathbb{F}G$. Here $\Delta(g)=g\otimes g$, $\eta(g)=\delta_{g,e}$ & $S(g)=g^{-1}$. More generally, if $(A,\mu,\varepsilon,\Delta,\eta,S)$ is a Hopf algebra & $\dim A<\infty$, then $(A^*,\Delta^*,\eta^*,\mu^*,\varepsilon^*,S^*)$ is a Hopf algebra) (premium exercise) & $\mathbb{F}G=\mathbb{F}[G]^*$ as Hopf algebra.

3) Let $\mathfrak{g}$ be a Lie algebra. Then $\mathcal{U}(\mathfrak{g})$ acquires a Hopf algebra structure. Namely, define $\Delta:\mathfrak{g}\to\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$, $x\mapsto x\otimes 1+1\otimes x$. This is a Lie algebra homomorphism (exercise: hint the check is the same as for tensor product representations of $\mathfrak{g}$ in Sec 1.2 of Lec 4). So it extends to an associative algebra homomorphism $\Delta:\mathcal{U}(\mathfrak{g})\to\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$. It's coassociative: indeed both $(\Delta\otimes id)\circ\Delta$ & $(id\otimes\Delta)\circ\Delta$ $\mathcal{U}(\mathfrak{g})\to\mathcal{U}(\mathfrak{g})^{\otimes 3}$ are algebra homomorphisms so it's enough to show that they coincide on $\mathfrak{g}$, where both give

$$x\mapsto x\otimes 1\otimes 1+1\otimes x\otimes 1+1\otimes 1\otimes x$$

Similarly, define $\eta:\mathfrak{g}\to\mathbb{F}$, $x\mapsto 0$, $S:\mathfrak{g}\to\mathcal{U}(\mathfrak{g})^{op}$, $x\mapsto -x$ & extend to associative algebra homomorphisms. Then (Δ,η,S) equip $\mathcal{U}(\mathfrak{g})$ w. a Hopf algebra structure. We note that this definition is given that the operations of tensoring, dualizing & taking the trivial representation for $\mathfrak{g}$- & $\mathcal{U}(\mathfrak{g})$-modules agree.