Lec 11: Hopf algebras, filtrations & gradings, II.

1) Graded & filtered vector spaces

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1.1) Main definitions

Let $\mathbb{F}$ be a field & V be a vector space.

Definition 1: • By a $(\mathbb{Z}_{\geq 0}\text{-})$grading on V we mean a decomposition $V = \bigoplus_{i \geq 0} V_i$.

• By a $(\mathbb{Z}_{\geq 0})$-filtration on V we mean a collection of subspaces $V_{\leq i} \subset V$ $(i \in \mathbb{Z}_{\geq 0})$ s.t. $V_i \subseteq V_{i+1}$ $\forall i$ & $V = \bigcup_{i \geq 0} V_{\leq i}$.

Example: • $\mathbb{F}$ is considered as a graded vector space w. $\mathbb{F}_0 = \mathbb{F}$ & as a filtered vector space w. $\mathbb{F}_{\leq i} = \mathbb{F}$ $\forall i \geq 0$.

• Let W be a vector space, then $T(W) = \overset{\infty}{\underset{i=0}{\bigoplus}} W^{\otimes i}$ is graded: $T(W)_i = W^{\otimes i}$.

Definition 2: Let U, V be graded (resp. filtered) vector spaces & $\varphi: U \longrightarrow V$ be a linear map. We say φ is graded (resp. filtered) if $\varphi(U_i) \subset V_i$ (resp. $\varphi(U_{\leq i}) \subset V_{\leq i}$) $\forall i$.

Now let's explain a connection between graded & filtered. Every graded vector space $V = \bigoplus_{i \geq 0} V_i$ is naturally filtered: $V_{\leq i} = \overset{i}{\underset{j=0}{\bigoplus}} V_j$ & every graded linear map $\varphi: U \to V$ is filtered.

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Conversely, let $V := \bigcup_{i \geq 0} V_i$ be a filtered vector space. It ==associated graded space== is
$$\mathrm{gr}\, V := \bigoplus_{i \geq 0} V_{\leq i} / V_{\leq i-1} \quad (V_{\leq -1} := \{0\})$$
& to a filtered linear map $\varphi: U \to V$ we assign a graded linear map
$$\mathrm{gr}\, \varphi : \mathrm{gr}\, U \to \mathrm{gr}\, V, \quad [\mathrm{gr}\, \varphi](u + U_{\leq i-1}) = \varphi(u) + V_{\leq i-1} \quad (u \in U_{\leq i})$$
(this defines $\mathrm{gr}\, \varphi$ on $(\mathrm{gr}\, U)_i$ & then we extend to $\mathrm{gr}\, U = \bigoplus_{i \geq 0} (\mathrm{gr}\, U)_i$)

Exercise 0: $\mathrm{gr}(\mathrm{id}_U) = \mathrm{id}_{\mathrm{gr}\, U}$ & $\mathrm{gr}(\psi \circ \varphi) = [\mathrm{gr}\, \psi] \circ [\mathrm{gr}\, \varphi]$.

1.2) Subspaces & quotients

We discuss the compatibility of these constructions w. sub- & quotient spaces.

Let V be a graded space. A subspace $U \subset V$ is a ==graded subspace== if $U = \bigoplus_{i=0} (U \cap V_i)$. It's graded w. $U_i = U \cap V_i$. And the quotient V/U in this case is graded w. $(V/U)_i = V_i / U_i$

Now let V be a filtered space, $U \subset V$ be a subspace. We equip U & V/U w. filtrations: $U_{\leq i} := U \cap V_{\leq i}$ & $(V/U)_{\leq i} := (V_{\leq i} + U)/U$

Exercise 1: $\mathrm{gr}\, U$ is identified w. the graded subspace $\bigoplus_{i \geq 0} \{u \in V_i \mid u + v' \in U$ for some $v' \in V_{\leq i-1}\} / V_{\leq i-1} \subset \mathrm{gr}\, V$ & $\mathrm{gr}(V/U) \xrightarrow{\sim} \mathrm{gr}\, V / \mathrm{gr}\, U$

Example: Consider the symmetric algebra
$$S(W) = T(W)/J; \quad J := (u \otimes v - v \otimes u \mid u, v \in W) \text{ is graded (exercise)} \& S(W)$$
is graded w. $S(W)_i = S^i(W)$.

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Now let $\mathfrak{g}$ be a Lie algebra. We view $T(\mathfrak{g})$ as a filtered space getting a filtration on its quotient $U(\mathfrak{g})$, explicitly

$$U(\mathfrak{g})_{\leq i} = \mathrm{Span}_{\mathbb{F}}(x_1 \dots x_j \mid j \leq i \;\&\; x_1, \dots x_j \in \mathfrak{g})$$

We would like to understand $\mathrm{gr}\, U(\mathfrak{g})$. Recall that

$$U(\mathfrak{g}) = T(\mathfrak{g})/I, \quad I := (x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g}).$$

So $x \otimes y - y \otimes x \in (\mathrm{gr}\, I)_2 \subset T(\mathfrak{g})_{\leq 2}/T(\mathfrak{g})_{\leq 1} = \mathfrak{g}^{\otimes 2}$

Now note that $\mathrm{gr}\, I \subset T(\mathfrak{g})$ is an ideal. Indeed, it's enough to show $ba, ab \in (\mathrm{gr}\, I)_{i+j} \;\forall\; a \in T(\mathfrak{g})_i \;\&\; b \in (\mathrm{gr}\, I)_j$. Take $c \in T(\mathfrak{g})_{\leq j-1}$, w. $b + c \in I_{\leq j}$, then $(b+c)a, a(b+c) \in I_{\leq i+j}$. Since $ca, ac \in T(\mathfrak{g})_{\leq i+j-1}$, by Exercise 1, $ba, ab \in (\mathrm{gr}\, I)_{i+j}$.

So $\mathrm{gr}\, I \supset x \otimes y - y \otimes x \Rightarrow \mathrm{gr}\, I \supset J \rightsquigarrow$ graded algebra epimorphism $\pi: S(\mathfrak{g}) \twoheadrightarrow T(\mathfrak{g})/\mathrm{gr}\, J = \mathrm{gr}\, U(\mathfrak{g})$.

We claim that PBW theorem follows from $S(\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}\, U(\mathfrak{g})$ (in fact, equivalent to it). Indeed, let $x_1, \dots x_n \in \mathfrak{g}$ be a basis. We induct on j to show $x_1^{d_1} \dots x_n^{d_n}$ w. $\sum d_i \leq j$ form a basis of $U(\mathfrak{g})_{\leq j}$. The base is $j = -1$ & the step is based on the observations that $x_1^{d_1} \dots x_n^{d_n}$ (w. $\sum d_i = j+1$) form a basis in $S(\mathfrak{g})_{j+1}$, while $S(\mathfrak{g})_{j+1} \xrightarrow{\sim} U(\mathfrak{g})_{\leq j+1}/U(\mathfrak{g})_{\leq j}$. The details are an exercise.

1.3) Tensor products

Our task in proving the PBW theorem is therefore to show that $\pi: S(\mathfrak{g}) \twoheadrightarrow \mathrm{gr}\, U(\mathfrak{g})$ is injective. To do this we show that both $S(\mathfrak{g})$ & $\mathrm{gr}\, U(\mathfrak{g})$ are "graded Hopf algebras" & π is a homomorphism

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of such. Using this (& some other observations) we'll show in the next lecture that if char $\mathbb{F} = 0$, then $\ker \pi = \{0\}$.

Since the data of co-, bi-, Hopf algebras involves maps between tensor products & equalities between them, we'll need tensor products of graded & filtered vector spaces & linear maps.

$\underline{\text{Definition}}$: • Let $U = \bigoplus_{i \geq 0} U_i$, $V = \bigoplus_{j \geq 0} V_j$ be graded vector spaces so that $U \otimes V = \bigoplus_{i,j \geq 0} U_i \otimes V_j$. We define a grading on $U \otimes V$ by $(U \otimes V)_k = \bigoplus_{i=0}^{k} U_i \otimes V_{k-i}$.

• Let $U = \bigcup_{i \geq 0} U_{\leq i}$, $V = \bigcup_{j \geq 0} V_{\leq j}$ be filtered vector spaces. We define a filtration on $U \otimes V$ by $(U \otimes V)_{\leq k} = \sum_{i=0}^{k} U_{\leq i} \otimes V_{\leq k-i}$ (note that $U_{\leq i} \otimes V_{\leq k-i}$ is naturally a subspace of $U \otimes V$).

$\underline{\text{Lemma}}$: Let U, V be filtered vector spaces. Then $\exists$ natural isom'm
$$(\mathrm{gr}\, U) \otimes (\mathrm{gr}\, V) \xrightarrow{\sim} \mathrm{gr}(U \otimes V) \text{ of graded vector spaces}$$

Proof:
$$(\mathrm{gr}\, U) \otimes (\mathrm{gr}\, V) = \bigoplus_{i,j \geq 0} (U_{\leq i}/U_{\leq i-1}) \otimes (V_{\leq i}/V_{\leq i-1}) \qquad (1)$$
We define $\tau: (U_{\leq i}/U_{\leq i-1}) \otimes (V_{\leq j}/V_{\leq j-1}) \longrightarrow \mathrm{gr}(U \otimes V)_{i+j}$ by
$(u + U_{\leq i-1}) \otimes (v + V_{\leq j-1}) \longmapsto u \otimes v + (U \otimes V)_{\leq i+j-1}$ (& then extend to the direct sum (1), denote the extension also by τ). To show it's an isom'm choose a basis $u_j^i \in U$ s.t the vectors u_j^i w. $i \leq k$ are a basis in $U_{\leq k}$, & an analogous basis v_m^ℓ in V. Then the vectors $(u_j^i + U_{\leq i-1}) \otimes (v_m^\ell + V_{\leq \ell-1})$ form a basis in $\mathrm{gr}\, U \otimes \mathrm{gr}\, V$, while the vectors $u_j^i \otimes v_m^\ell + (U \otimes V)_{\leq i+\ell-1}$ form a basis in $\mathrm{gr}(U \otimes V)$ (exercise). Since τ

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sends one basis to another, it's an isomorphism. □

Exercise 2: Let $\varphi_1 : U_1 \to V_1$, $\varphi_2 : U_2 \to V_2$ be filtered linear maps between filtered vector spaces. Then $\varphi_1 \otimes \varphi_2 : U_1 \otimes V_1 \to U_2 \otimes V_2$ is filtered & under the isomorphism of Lemma, $\mathrm{gr}(\varphi_1 \otimes \varphi_2) = (\mathrm{gr}\,\varphi_1) \otimes (\mathrm{gr}\,\varphi_2)$.

1.4) Filtered (co-, bi-, Hopf) algebras

By a graded algebra (coalgebra, bialgebra or Hopf algebra) structure on a graded vector space V we mean such a structure, where the structure maps (e.g. $\mu, \varepsilon, \Delta, \eta$ for a bialgebra) are graded. The analogous definition works in the filtered setting.

Prop'n : 1) $U(\mathfrak{g})$ (w. filtration from Sec 1.2) is a filtered Hopf algebra.
  2) $S(\mathfrak{g})$ is graded Hopf algebra.
  3) So is $\mathrm{gr}\,U(\mathfrak{g})$.
  4) $\mathrm{gr}: S(\mathfrak{g}) \to \mathrm{gr}\,U(\mathfrak{g})$ is a graded Hopf algebra homomorphism.

Sketch of proof:

1): "$A = \bigcup_{i \geq 0} A_{\leq i}$ is a filtered algebra" means
 i) $\mu : A \otimes A \to A$ is filtered $\Leftrightarrow ab = \mu(a \otimes b) \in A_{\leq i+j}$ $\forall\, a \in A_{\leq i}, b \in A_{\leq j}$
 ii) $\varepsilon : \mathbb{F} \to A$ is filtered $\Leftrightarrow 1 \in A_{\leq 0}$.

For $A = U(\mathfrak{g})$, have $A_{\leq i} = \mathrm{Span}_{\mathbb{F}}(x_1 \dots x_j \mid j \leq i, x_k \in \mathfrak{g}) \Rightarrow$ i) & ii).

Let's show $\Delta: U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes 2}$ is coassociative, the axioms for S & η are similar but easier. Similarly to the above, $U(\mathfrak{g})^{\otimes 2}$ is a filtered algebra; $\Delta(x_1 \dots x_j) = \Delta(x_1) \dots \Delta(x_j) \in [\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i \in (U(\mathfrak{g})^{\otimes 2})_{\leq 1}]$

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$[U(\mathfrak{g})^{\otimes 2}]^j_{\leqslant 1} \subset (U(\mathfrak{g})^{\otimes 2})_{\leqslant j} \Rightarrow \Delta(U(\mathfrak{g})_{\leqslant j}) \subset (U(\mathfrak{g})^{\otimes 2})_{\leqslant j} \; \forall j$, i.e. Δ is filtered.

2): Equip $S(\mathfrak{g})$ w. Hopf algebra structure of $U(\mathfrak{g}$. w. zero $[\cdot,\cdot])$
The proof that it's graded is similar to 1) & left as an exercise.

3) If $(A,\mu,\Delta\dots)$ is a filtered Hopf algebra, then $\mathrm{gr}\,A$ is a graded Hopf algebra w. $\mathrm{gr}\,\mu, \mathrm{gr}\,\Delta$, etc. This is a formal consequence of the form of axioms (equality of compositions of tensor products of maps) & Exercises 0 in Sec 1.1 & 2 in Sec 1.3.

E.g., the coassociativity axiom for A is $(\Delta\otimes id)\circ\Delta = (id\otimes\Delta)\circ\Delta$.
Then $(\mathrm{gr}\,\Delta\otimes id)\circ\mathrm{gr}\,\Delta = \mathrm{gr}((\Delta\otimes id)\circ\Delta)) = \mathrm{gr}((id\otimes\Delta)\circ\Delta) = (id\otimes\mathrm{gr}\,\Delta)\circ\mathrm{gr}\,\Delta$,
which is the coassociativity axiom for $\mathrm{gr}\,A$.

4) By construction in Sec 1.1, π is a graded algebra homomorphism. We·ll show that it's a coalgebra homomorphism (which is the only property we are going to use). Let $\Delta: S(\mathfrak{g}) \to S(\mathfrak{g})^{\otimes 2}$ & $\Delta': U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes 2}$ denote the coproducts. We need to show that
$$\mathrm{gr}\,\Delta'\circ\pi = (\pi\otimes\pi)\circ\Delta \qquad\qquad (2)$$
Both sides of (2) are algebra homomorphisms $S(\mathfrak{g}) \to (\mathrm{gr}\,U(\mathfrak{g}))^{\otimes 2}$.
To show (2) amounts to checking equality on generators, i.e. $x\in\mathfrak{g}\subset S(\mathfrak{g})$.
Let $\bar{x}$ be the image of $x\in\mathfrak{g}$ in $(\mathrm{gr}\,U(\mathfrak{g}))_1$ so that $\pi(x)=\bar{x}$. Then
$\mathrm{gr}\,\Delta'\circ\pi(x) = \mathrm{gr}\,\Delta'(\bar{x}) = [\Delta'(x)= x\otimes 1 + 1\otimes x] = \bar{x}\otimes 1 + 1\otimes \bar{x}$
$(\pi\otimes\pi)\circ\Delta(x) = (\pi\otimes\pi)(x\otimes 1 + 1\otimes x) = \bar{x}\otimes 1 + 1\otimes \bar{x}$
This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

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Remark: One point of taking the associated graded is that it simplifies the structure, cf. isomorphism $S(\mathfrak{g}) \xrightarrow{\sim} \mathrm{gr}\, U(\mathfrak{g})$ that we'll prove later. One can use this to get an information about $\mathrm{gr}\, A$ to get some information about A, we'll use this strategy later to describe the center of $U(\mathfrak{sl}_n)$ (for char $\mathbb{F} = 0$).