Lec 11: Hopf algebras, filtrations & gradings, II.

1) Graded & filtered vector spaces

1.1) Main definitions

$\mathbb{F}$ is field, V vector space/$\mathbb{F}$.

Definition: 1) $\mathbb{Z}_{\geqslant 0}$-grading on V: $V=\bigoplus_{i=0}^{\infty} V_i$

2) $\mathbb{Z}_{\geqslant 0}$-filtration on V: subspaces $V_{\leqslant i}\subset V$ $(i\geqslant 0)$:

$V_{\leqslant i}\subset V_{\leqslant i+1}$ $\forall i$ & $V=\bigcup_{i\geqslant 0} V_{\leqslant i}$

Examples: • $\mathbb{F}$ is graded w. $\mathbb{F}_0=\mathbb{F}$

filtered w. $\mathbb{F}_{\leqslant i}=\mathbb{F}$ $\forall i$.

• W vector space $\rightsquigarrow$ tensor algebra $T(W)=\bigoplus_{i=0}^{\infty} \underbrace{W^{\otimes i}}_{= T(W)_i}$

grading

Connection: $\forall$ graded v.s. is filtered

$V=\bigoplus_{i\geqslant 0} V_i \rightsquigarrow V_{\leqslant i}=\bigoplus_{j=0}^{i} V_j$

• Filtered $V=\bigcup_{i\geqslant 0} V_{\leqslant i} \rightsquigarrow$ associated graded

$\operatorname{gr} V = \bigoplus_{i=0}^{\infty} V_{\leq i}/V_{\leq i-1}$ $(V_{-1} := \{0\})$ – graded v.s.

Def: U, V graded (resp. filtered) v.s. $\varphi \in \operatorname{Hom}_{\mathbb{F}}(U,V)$:

- Say φ is graded if $\varphi(U_i) \subset V_i$ $\forall i$
  filtered if $\varphi(U_{\leq i}) \subset V_{\leq i}$ $\forall i$

- Graded $\varphi: U \to V$, then φ is also filtered.
- U, V are filtered & $\varphi: U \to V$ is filtered $\rightsquigarrow$
  graded $\operatorname{gr}\varphi: \operatorname{gr} U \longrightarrow \operatorname{gr} V$

$$\operatorname{gr} U = \bigoplus_i \underbrace{U_{\leq i}/U_{\leq i-1}}, \qquad \operatorname{gr} V = \bigoplus_i V_{\leq i}/V_{\leq i-1}$$

here $[\operatorname{gr}\varphi](u + U_{\leq i-1}) := \varphi(u) + V_{\leq i-1}$ $\forall i$.

Exercise 0: $\operatorname{gr} \mathrm{Id}_U = \mathrm{Id}_{\operatorname{gr} U}$ & $\operatorname{gr}(\varphi \circ \psi) = \operatorname{gr}\varphi \circ \operatorname{gr}\psi$.

1.2) Subspaces & quotients

$V = \bigoplus_i V_i$ graded v.s. Say subspace $U \subset V$ is graded if $U = \bigoplus_{i \geq 0} (U \cap V_i) \longleftarrow$ graded as a space w. $U_i = U \cap V_i$.

$V/U = \bigoplus_{i\geq 0} V_i/U_i$ – grading

Filtered $V = \bigcup_i V_{\leq i}$, $U \subset V$ is filtered by $U_{\leq i} = U \cap V_{\leq i}$
& V/U is filtered: $(V/U)_{\leq i} := (V_{\leq i} + U)/U$.

Exercise 1: V is filtered, $U \subset V$

$\mathrm{gr}\, U = \bigoplus_{i\geq 0} \{u \in V_{\leq i} \mid \exists\ v' \in V_{\leq i-1} \text{ s.t. } u + v' \in U\} / V_{\leq i-1} \subset \mathrm{gr}\, V$
graded subspace

$\mathrm{gr}\, V / \mathrm{gr}\, U \xrightarrow{\sim} \mathrm{gr}(V/U)$

Example: W vect. space $\rightsquigarrow$ graded $T(W)$
$T(W) \supset J = (x\otimes y - y \otimes x \mid x, y \in W)$ ⟵ graded (exercise)
symmetric algebra $S(W) = T(W)/J = \bigoplus_{i=0}^{\infty} S^i(W)$ – graded.

Now $W = \mathfrak{g}$ (Lie algebra)
$T(\mathfrak{g}) \supset I = (x\otimes y - y\otimes x - [x,y] \mid x, y \in \mathfrak{g}) \rightsquigarrow \mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/I$
is filtered w. $\mathcal{U}(\mathfrak{g})_{\leq i} = \mathrm{Span}_{\mathbb{F}}(x_1 \ldots x_j \mid j \leq i \ \&\ x_1, \ldots x_j \in \mathfrak{g})$.

- $x\otimes y - y \otimes x \in (\mathrm{gr}\, I)_2 \subset T(\mathfrak{g})_2$

- $\mathrm{gr}\, I$ is a 2-sided ideal: $ab, ba \in \mathrm{gr}\, I$ $\forall a \in T(\mathfrak{g})_i$; $b \in (\mathrm{gr}\, I)_j$ $\forall i,j \geq 0$: $b \in (\mathrm{gr}\, I)_j \Leftrightarrow \exists\, c \in T(\mathfrak{g})_{\leq j-1} \mid b+c \in I$ $\Rightarrow a(b+c), (b+c)a \in I \Rightarrow [ac, ca \in T(\mathfrak{g})_{\leq i+j-1}] \Rightarrow ab, ba \in (\mathrm{gr}\, I)_{i+j}$.

- $\mathrm{gr}\, I \ni x\otimes y - y\otimes x \Rightarrow \mathrm{gr}\, I \supset J \rightsquigarrow$
  graded algebra epimorphism
  $$\pi: S(\mathfrak{g}) = T(\mathfrak{g})/J \twoheadrightarrow T(\mathfrak{g})/\mathrm{gr}\, I = \mathrm{gr}\, U(\mathfrak{g})$$

PBW Thm ($x_1, \ldots x_n \in \mathfrak{g}$, basis, $\Rightarrow x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ basis in $U(\mathfrak{g})$)

$\Uparrow$

π is iso ($\Leftrightarrow$ injective):

Prove: $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ w. $\sum \alpha_k \leq j$ form basis in $U(\mathfrak{g})_{\leq j}$

induction on j

Base: $j = -1$

Step: from $j = i$ to $j = i+1$ uses

- $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ w. $\sum \alpha_k = i+1$ form basis in $S^{i+1}(\mathfrak{g})$
- $S^{i+1}(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g})_{\leq i+1} / U(\mathfrak{g})_{\leq i}$

(exercise).

1.3) Tensor products

To prove PBW need $\ker \pi = \{0\}$:

• see $S(\mathfrak{g})$, $\operatorname{gr} \mathcal{U}(\mathfrak{g})$ are "graded Hopf algebras" & π is homomorphism of such.

• next lecture: (under some restrictions on $\mathfrak{g}$) & $\operatorname{char} \mathbb{F} = 0$ this force $\ker \pi = \{0\}$.

· Hopf algebras: via maps between tensor products & equalities of such $\rightsquigarrow$ tensor products of filtered/graded vector space

Definition: • graded $U = \bigoplus_{i \geq 0} U_i$, $V = \bigoplus_{i \geq 0} V_i$ $\rightsquigarrow$
$U \otimes V = \bigoplus_{i,j \geq 0} U_i \otimes V_j$ is graded via

$$(U \otimes V)_k = \bigoplus_{i=0}^{k} U_i \otimes V_{k-i}$$

• filtered $U = \bigcup_{i \geq 0} U_{\leq i}$, $V = \bigcup_{j \geq 0} V_{\leq j}$ $\rightsquigarrow$

$$(U \otimes V)_{\leq k} = \sum_{i=0}^{k} U_{\leq i} \otimes V_{\leq k-i}$$

Lemma: U, V filtered $\rightsquigarrow \operatorname{gr} U \otimes \operatorname{gr} V \xrightarrow[\text{natural}]{\sim} \operatorname{gr}(U \otimes V)$

Proof:

$$\underbrace{\operatorname{gr} U \otimes \operatorname{gr} V = \bigoplus_{i,j \geq 0} (U_{\leq i}/U_{\leq i-1}) \otimes (V_{\leq j}/V_{\leq j-1})}\ni (u + U_{\leq i-1}) \otimes (v + V_{\leq j-1})$$

$$\downarrow \tau \qquad\qquad\qquad\qquad \swarrow$$

$$(U \otimes V)_{\leq i+j} / (U \otimes V)_{\leq i+j-1} \ni u \otimes v + (U \otimes V)_{\leq i+j-1}$$

$\rightsquigarrow \tau : \operatorname{gr} U \otimes \operatorname{gr} V \longrightarrow \operatorname{gr}(U \otimes V)$

To show τ is iso:

pick basis in U: u_j^i s.t. u_j^i w $i \leq k$ basis in $U_{\leq k}$

V: v_m^ℓ (similar)

Bases in:

$$\operatorname{gr} U \otimes \operatorname{gr} V \ni (u_j^i + U_{\leq i-1}) \otimes (v_m^\ell + V_{\leq \ell-1})$$

$$\downarrow \tau \qquad\qquad \Longrightarrow \tau \text{ is iso.}$$

$$\operatorname{gr}(U \otimes V) \ni u_j^i \otimes v_m^\ell + (U \otimes V)_{\leq i+\ell-1} \qquad \square$$

Exercise 2: filtered linear $\varphi_1 : U_1 \to V_1$, $\varphi_2 : U_2 \to V_2$

Then $\varphi_1 \otimes \varphi_2 : U_1 \otimes U_2 \longrightarrow V_1 \otimes V_2$ is filtered &

$\operatorname{gr}(\varphi_1 \otimes \varphi_2) = \operatorname{gr} \varphi_1 \otimes \operatorname{gr} \varphi_2$.

1.4) Filtered & graded (co-, bi-, Hopf) algebras

A graded (co-, bi-, Hopf) algebra = graded v.s. w. corresponding structures that are graded linear maps

(bialgebra: $\mu, \varepsilon, \Delta, \eta$)

Similar: for filtered v.s.

Proposition: 1) $U(\mathfrak{g})$ is filtered Hopf algebra

2) $S(\mathfrak{g})$ is graded Hopf algebra

3) $\operatorname{gr} U(\mathfrak{g})$ is graded Hopf algebra

4) $\sigma: S(\mathfrak{g}) \twoheadrightarrow \operatorname{gr} U(\mathfrak{g})$ is graded Hopf algebra homomorphism.

Proof:

1) "$A = \bigcup_{i \geq 0} A_{\leq i}$ is filtered algebra" means

i) $\mu: A \otimes A \to A$ is filtered $\Leftrightarrow ab = \mu(a \otimes b) \in A_{\leq i+j}$ $\forall a \in A_{\leq i}$, $b \in A_{\leq j}$

ii) $\varepsilon: \mathbb{F} \to A$ is filtered $\Leftrightarrow 1 \in A_{\leq 0}$

$A = U(\mathfrak{g})$: $U(\mathfrak{g})_{\leq i} = \operatorname{Span}_{\mathbb{F}}(x_1 \dots x_j \mid j \leq i,\ x_1, \dots x_j \in \mathfrak{g})$
$\Downarrow$
i) & ii)

• need Δ, η, S

prove Δ is filtered

$\mathcal{U}(\mathfrak{g})^{\otimes 2}$ is filtered algebra, (e.g. similar to above)

$\Delta(x_1 \dots x_j) = \Delta(x_1)\Delta(x_2)\dots\Delta(x_j) = [\Delta(x) = x\otimes 1 + 1\otimes x \in \mathcal{U}(\mathfrak{g})^{\otimes 2}_{\leq 1}] \subset (\mathcal{U}(\mathfrak{g})^{\otimes 2})_{\leq j} \Rightarrow \Delta$ is filtered

2) $S(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}$ w. zero $[\cdot,\cdot]) \rightsquigarrow$ Hopf alg. structure on $S(\mathfrak{g})$.
It's graded is similar to 1).

3) If $(A, \mu, \varepsilon, \Delta, \eta, S)$ is filtered Hopf algebra $\Rightarrow$
$(\mathrm{gr}\, A, \mathrm{gr}\,\mu, \mathrm{gr}\,\varepsilon, \dots)$ is graded Hopf algebra

Use gr is compositions & $\otimes$

e.g. coassociativity for $\mathrm{gr}\,\Delta$

For Δ: $(\Delta\otimes id)\circ\Delta = (id\otimes\Delta)\circ\Delta$

$$([\mathrm{gr}\,\Delta]\otimes id)\circ \mathrm{gr}\,\Delta = \mathrm{gr}((\Delta\otimes id)\circ\Delta) = \mathrm{gr}((id\otimes\Delta)\circ\Delta) =$$
$$= (id\otimes[\mathrm{gr}\,\Delta])\circ\mathrm{gr}\,\Delta$$

4) π is graded algebra homom by Sec 1.2.

Compatibility w. coproducts:

$\Delta: S(\mathfrak{g}) \to S(\mathfrak{g})^{\otimes 2}$, $\Delta': \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$:

$$[gr\,\Delta'] \circ \pi = (\pi \otimes \pi) \circ \Delta \qquad (2)$$

algebra homom's $S(\mathfrak{g}) \to gr\,\mathcal{U}(\mathfrak{g})^{\otimes 2}$

enough to check (2) on generators of $S(\mathfrak{g})$: $x \in \mathfrak{g}$

$\bar{x}$ = image of x in $\mathcal{U}(\mathfrak{g})_{\leq 1} / \mathcal{U}(\mathfrak{g})_{\leq 0}$, $\pi(x) = \bar{x}$.

$[gr\,\Delta'] \circ \pi(x) = [gr\,\Delta'](\bar{x}) = [\Delta'(x) = x \otimes 1 + 1 \otimes x] = \bar{x} \otimes 1 + 1 \otimes \bar{x}$

$(\pi \otimes \pi) \circ \Delta(x) = [\pi \otimes \pi](x \otimes 1 + 1 \otimes x) = \bar{x} \otimes 1 + 1 \otimes \bar{x}$. $\square$

Rem: advantage of $A \rightsquigarrow gr\,A$:

easier to understand

e.g. $gr\,\mathcal{U}(\mathfrak{g}) \xleftarrow{\sim} S(\mathfrak{g})$

info on $S(\mathfrak{g}) \rightsquigarrow Z(\mathcal{U}(\mathfrak{g}))$ for $\mathfrak{g} = \mathfrak{sl}_n$ (in char = 0)

later in class.