Lec 12: Hopf algebras, filtrations & gradings, III

1) Proof of PBW Thm & related claims

2) Comodules.

1) Proof of PBW Thm & related claims

1.1) Proof of PBW in char 0

Let $\mathbb{F}$ be an algebraically closed field, G an algebraic group/$\mathbb{F}$ & $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\pi: S(\mathfrak{g}) \twoheadrightarrow \mathrm{gr}\, \mathcal{U}(\mathfrak{g})$ be the natural graded algebra homomorphism (Sec 1.2 of Lec 11). In Sec 1.4 of Lec 11 we've seen that π is a coalgebra homomorphism. Our main result in this section is:

Prop.: Let $\mathfrak{g}$ be a (finite dimensional) Lie algebra & $\pi: S(\mathfrak{g}) \to A$ be a homomorphism of graded coalgebras s.t. $(\ker \pi)_{\leqslant 1} = \{0\}$.

1) If $\mathrm{char}\, \mathbb{F} = 0$, then π is injective.

2) If $\mathrm{char}\, \mathbb{F} = p > 0$, then $(\ker \pi)_{\leqslant p-1} = \{0\}$.

Proof:

Assume the contrary & pick $f \in (\ker \pi)_m \setminus \{0\}$ w. minimal m. By the minimality assumption: $\pi: S(\mathfrak{g})_{\leqslant m-1} \hookrightarrow A_{\leqslant m-1}$ & so $\pi \otimes \pi$: $(S(\mathfrak{g})_{\leqslant m-1})^{\otimes 2} \hookrightarrow (A_{\leqslant m-1})^{\otimes 2}$. Note that $(\pi \otimes \pi) \circ \Delta(f) = \Delta(\pi(f)) = 0$. Also $\Delta(f) \in (S(\mathfrak{g})^{\otimes 2})_m = \bigoplus_{i=0}^{m} S(\mathfrak{g})_i \otimes S(\mathfrak{g})_{m-i}$. Write $\Delta(f) = \sum_{i=0}^{m} \Delta(f)_i$ w. $\Delta(f)_i \in S(\mathfrak{g})_i \otimes S(\mathfrak{g})_{m-i}$. Since $\pi \otimes \pi: S(\mathfrak{g})_i \otimes S(\mathfrak{g})_{m-i} \hookrightarrow A_i \otimes A_{m-i}$ $\forall$ $0 < i < m \Rightarrow \Delta(f)_i = 0$. Now choose a basis $x_1, \ldots x_n \in \mathfrak{g}$.

Claim: $\forall\, m>1$, $F\in S(\mathfrak{g})_m$, $\Delta(F)_1=\sum_{i=1}^{n} x_i\otimes \partial_i F$

Proof of Claim: enough to consider $F=x_1^{\alpha_1}\cdots x_n^{\alpha_n}\Rightarrow \Delta(F)=\prod_{i=1}^{n}(x_i\otimes 1+1\otimes x_i)^{\alpha_i}$

By the binomial formula, $(x_i\otimes 1+1\otimes x_i)^{\alpha_i}=\sum_{j=0}^{\alpha_i}\binom{\alpha_i}{j}x_i^{j}\otimes x_i^{\alpha_i-j}$. Then the term of the form $x_i\otimes ?$ is $x_i\otimes \alpha_i x_1^{\alpha_1}\cdots x_i^{\alpha_i-1}\cdots x_n^{\alpha_n}=x_i\otimes\partial_i F$ $\square$

We conclude that $\partial_i f=0\ \forall i$. Recall $m>0$. If $\operatorname{char}\mathbb{F}=0$, then $f=0$. And if $\operatorname{char}\mathbb{F}=p$, $\partial_i f=0\ \forall i\Rightarrow f\in\mathbb{F}[x_1^p,\dots x_n^p]\Rightarrow \deg f\geq p$. $\square$

We now provide sufficient conditions for $(\ker\pi)_{\leq 1}=\{0\}$.

Lemma: Let $G\subset GL_n$ be an algebraic subgroup

1) If $x\in\mathfrak{g}$, $a\in\mathbb{F}$ are s.t. $\varphi(x)=a\cdot id_V$ $\forall$ rational representation $\Phi: G\to GL(V)$ ($\&\ \varphi=T_e\Phi$), then $x=0\ \&\ a=0$.

2) For $\pi: S(\mathfrak{g})\twoheadrightarrow \operatorname{gr}\mathcal{U}(\mathfrak{g})$, $(\ker\pi)_{\leq 1}=\{0\}$.

Proof: 1) Since $G\subset GL_n$ we have $\mathfrak{g}\subset\mathfrak{gl}_n$ & $\mathbb{F}^n$ is a rational G-rep. Take $V=\mathbb{F}^n\oplus\mathbb{F}_{triv}$ ("triv" for trivial). Since x acts by 0 on $\mathbb{F}_{triv}\Rightarrow$ $a=0$. And since $\mathfrak{g}\subset\mathfrak{gl}_n$, $x=0$.

2): $(\ker\pi)_0=\{0\}$ b/c $1\neq 0$ in $\mathcal{U}(\mathfrak{g})$.

Let $x\in\mathfrak{g}=S(\mathfrak{g})_1$ be s.t. $x\in\ker\pi\Leftrightarrow\exists\ a\in\mathcal{U}(\mathfrak{g})_{\leq 0}=\mathbb{F}1$ s.t. $x-a$ is 0 in $\mathcal{U}(\mathfrak{g})$. Then $x-a$ acts by 0 on any $\mathcal{U}(\mathfrak{g})$-module $=\mathfrak{g}$-rep contradicting 1). $\square$

Corollary: Assume $\mathfrak{g}$ is as in Lemma

1) If $\operatorname{char}\mathbb{F}=0$, then $\pi: S(\mathfrak{g}) \xrightarrow{\sim} \operatorname{gr} U(\mathfrak{g})$, so PBW Thm hods.

2) If $\operatorname{char}\mathbb{F}=p-1$, then $\pi: S(\mathfrak{g})_{\leq p-1} \xrightarrow{\sim} \operatorname{gr} U(\mathfrak{g})_{\leq p-1}$.

Proof: combine 2) of Lemma w. Proposition.

1.2) Additivity of $x \mapsto x^p - x^{[p]}$

Corollary: Assume $\operatorname{char}\mathbb{F}=p$ & let G be as in Lemma. Then

$$(x+y)^p - (x+y)^{[p]} = x^p - x^{[p]} + y^p - y^{[p]} \quad \forall\, x,y \in \mathfrak{g}.$$

Proof:

Let $z \in U(\mathfrak{g})$ be the difference of l.h.s & r.h.s. WTS $z=0$. Notation: for $u \in U(\mathfrak{g})_{\leq i}$ we write $\bar{u}$ for its class in $U(\mathfrak{g})_{\leq i}/U(\mathfrak{g})_{\leq i-1}$. E.g., $U(\mathfrak{g})_{\leq 1}/U(\mathfrak{g})_{\leq 0} = $ [2) of Lemma] $= \mathfrak{g}$ & $\bar{\xi}=\xi \ \forall\, \xi \in \mathfrak{g}$.

Step 1: We claim $z \in U(\mathfrak{g})_{\leq p-1}$. Note that $\xi^p \in U(\mathfrak{g})_{\leq p}$, $\xi^{[p]} \in U(\mathfrak{g})_{\leq 1}$ $\Rightarrow z \in U(\mathfrak{g})_{\leq p}$. But $z + U(\mathfrak{g})_{\leq p-1} = (\bar{x}+\bar{y})^p - \bar{x}^p - \bar{y}^p = $ [$\operatorname{gr} U(\mathfrak{g})$ is commutative as quotient of $S(\mathfrak{g})$] $=0 \Rightarrow z \in U(\mathfrak{g})_{\leq p-1}$. Choose min'l k w. $z \in U(\mathfrak{g})_{\leq k}$. $\Rightarrow k \leq p-1$.

Step 2: We claim that $\Delta(z) = z\otimes 1 + 1 \otimes z$. Indeed, $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$, $\Delta(\xi^p) = (\xi\otimes 1 + 1\otimes \xi)^p = $ [$\xi \otimes 1$ & $1 \otimes \xi$ commute] $= \xi^p \otimes 1 + 1 \otimes \xi^p \Rightarrow \Delta(z) = z\otimes 1 + 1 \otimes z$. Now we'll use claims before the lemma to finish the proof.

Step 3: Consider $\bar{z} \in (\operatorname{gr} U(\mathfrak{g}))_k = S(\mathfrak{g})_k$. Let $\Delta_S: S(\mathfrak{g}) \to S(\mathfrak{g}) \otimes S(\mathfrak{g})$ be the coproduct. Then $\Delta(z) + (U(\mathfrak{g})^{\otimes 2})_{\leq k} = \Delta_S(\bar{z}) \Rightarrow \Delta_S(\bar{z}) = \bar{z}\otimes 1 + 1 \otimes \bar{z}$

By Claim in the proof of Proposition, if $k>1$, then all partials of $\bar{z}$

vanish & since $k<p \Rightarrow \bar{z}=0$. Hence $z\in U(\mathfrak{g})_{\leq k-1}$, we arrive at a contradiction w. choice of k. So $k\leq 1 \Rightarrow z\in U(\mathfrak{g})_{\leq 1}$.

Step 4: Let $\Phi: G\to GL(V)$ be a rational G-representation & $\varphi = T_e\Phi: \mathfrak{g}\to\mathfrak{gl}(V)$. By Thm in Sec 2 of Lec 2, $\varphi(\xi)^p = \varphi(\xi^{[p]})\ \forall \xi\in\mathfrak{g}$. So z acts by 0 on V. By 1) of Lemma in Sec 1.1, $z=0$. $\square$

1.3) PBW theorem in char p

Here we sketch a proof of the following (based on base change). Details are left as an extended exercise.

Proposition: Suppose $\mathfrak{g}\subset\mathfrak{gl}_n$. Then PBW Thm holds for $\mathfrak{g}$.

Sketch of proof:

1) If $S(\mathfrak{gl}_n)\hookrightarrow \operatorname{gr} U(\mathfrak{gl}_n)$, then $S(\mathfrak{g})\hookrightarrow \operatorname{gr} U(\mathfrak{g})$. This reduces the proof to the case of $\mathfrak{gl}_n$.

2) Set $\mathfrak{gl}_n(\mathbb{Z}) = Mat_n(\mathbb{Z})$ so that $\mathfrak{gl}_n(\mathbb{F}) = \mathbb{F}\otimes_{\mathbb{Z}}\mathfrak{gl}_n(\mathbb{Z})\ \forall$ fields $\mathbb{F}$. Consider the ring $U_{\mathbb{Z}}(\mathfrak{gl}_n(\mathbb{Z}))$ defined as in the case of fields. Use universal properties of $U_?(?)$ to establish isomorphism

$$(1)\qquad \mathbb{F}\otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{gl}_n(\mathbb{Z})) \xrightarrow{\sim} U(\mathfrak{gl}_n(\mathbb{F})).$$

3) Let $x_1,\dots x_m$ be a basis of the free abelian group $\mathfrak{gl}_n(\mathbb{Z})$. Then $U_{\mathbb{Z}}(\mathfrak{gl}_n(\mathbb{Z})) = Span_{\mathbb{Z}}(x_1^{d_1}\dots x_m^{d_m} \mid d_i\geq 0)$.

4) Use (1): with $\mathbb{F}=\mathbb{C}$ to show $x_1^{d_1}\dots x_m^{d_m}\in U_{\mathbb{Z}}(\mathfrak{gl}_n(\mathbb{Z}))$ are linearly independent & for arbitrary $\mathbb{F}$ to finish the proof. $\square$

2) Comodules

2.1) Definition

In the tensor product language a module over an algebra (A,μ,ε) can be defined as a vector space M together w. linear map $\mu_M: A\otimes M \to M$ making the following diagrams commutative:

$$\begin{tikzcd} A\otimes A\otimes M \arrow[r, "\mu\otimes id_M"'] \arrow[d, "id_A\otimes \mu_M"] & A\otimes M \arrow[d, "\mu_M"] \\ A\otimes M \arrow[r, "\mu_M"'] & M \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F}\otimes M \arrow[rr, "="] \arrow[dr, "\varepsilon\otimes id_M"'] & & M \\ & A\otimes M \arrow[ur, "\mu_M"'] & \end{tikzcd}$$

Reversing the arrows we get the following:

Definition: Let (A,Δ,η) be a coalgebra. By a (right) comodule over A is a vector space M w. a linear map $M \xrightarrow[\Delta_M]{} M\otimes A$ making the following diagrams commutative:

$$\begin{tikzcd} M\otimes A\otimes A & M\otimes A \arrow[l, "id_M\otimes\Delta"'] \\ M\otimes A \arrow[u, "\Delta_M\otimes id_A"] & M \arrow[l, "\Delta_M"] \arrow[u, "\Delta_M"'] \end{tikzcd} \qquad \begin{tikzcd} M\otimes\mathbb{F} & & M \arrow[ll, "="'] \arrow[dl, "\Delta_M"] \\ & M\otimes A \arrow[ul, "id_M\otimes\eta"] & \end{tikzcd}$$

Example: A w. $\Delta: A\to A\otimes A$ is a comodule over itself (regular comodule).

2.2) Comodules vs rational representations

Let G be an algebraic group w. product $m: G\times G\to G$ & unit e that we view as a map $pt\to G$. As we've seen in Sec 1.2 of Lec 10, $\mathbb{F}[G]$ is a Hopf algebra w. $\Delta = m^*$ & $\eta = e^*$.

Proposition: Let V be a finite dimensional vector space over $\mathbb{F}$. To equip V w. a $\mathbb{F}[G]$-comodule structure is the same as to equip it w. a structure of a rational G-representation. In particular,

(a) $\mathrm{Hom}_G(V^1, V^2) = \{\mathbb{F}[G]\text{-comodule homomorphisms } V^1 \to V^2\}$

(b) $V' \subset V$ is G-stable $\Leftrightarrow$ $V' \subset V$ is a sub-comodule.

Proof:

First we claim that the following data are equivalent:

(i) A morphism $\rho: G \to \mathrm{End}(V)$

(ii) a linear map $\Delta_V: V \to V \otimes \mathbb{F}[G]$

(iii) a bilinear map $(\alpha, v) \mapsto c_{\alpha,v}: V^* \times V \to \mathbb{F}[G]$

(i) $\xrightarrow{\sim}$ (iii): $\rho \mapsto c_{\alpha,v}(g) = \langle \alpha, \rho(g)v \rangle$ (matrix coefficient)

(ii) $\xrightarrow{\sim}$ (iii): $\Delta_V \mapsto c_{\alpha,v} = \langle \alpha \otimes id, \Delta_V(v) \rangle$ $(\alpha \otimes id : V \otimes \mathbb{F}[G] \to V \otimes \mathbb{F} = V)$.

Now we claim that ρ is a representation $\Leftrightarrow$ Δ_V is a comodule structure. Indeed, let $v_1, \dots v_n \in V$ be basis & $\alpha_1, \dots \alpha_n \in V^*$ be dual basis. Write $c_{ij} := c_{\alpha_i, v_j}$. Then ρ is a representation $\Leftrightarrow$

(*) $c_{ij}(gh) = \sum_{k=1}^n c_{ik}(g) c_{kj}(h)$ $\forall g, h \in G$ & $c_{ij}(e) = \delta_{ij}$.

Δ_V is a comodule structure $\Leftrightarrow$ coassociativity & counit axioms hold. Coassociativity says $(id_M \otimes \Delta) \circ \Delta_M(v_j) = (\Delta_M \otimes id_A) \circ \Delta_M(v_j) \in V \otimes \mathbb{F}[G]^{\otimes 2}$ $\forall j$. Note that $\Delta_M(v_j) = \sum_i v_i \otimes c_{ij}$. So we arrive at: l.h.s. $= \sum_i v_i \otimes m^*(c_{ij})$, r.h.s. $= \sum \Delta_M(v_k) \otimes c_{kj} = \sum_{k,i} v_i \otimes c_{ik} \otimes c_{kj}$. So l.h.s = r.h.s. $\Leftrightarrow$ $m^*(c_{ij}) = c_{ik} \otimes c_{kj}$ $\Leftrightarrow$ $c_{ij}(gh) = [m^*(c_{ij})](g,h) = \sum_{k=1}^n c_{ik}(g) c_{kj}(h)$, which the 1st part of (*). The 2nd part is similar but easier.

To prove (a) take $\tau \in \mathrm{Hom}(V^1, V^2)$ & write it as a matrix (t_{ij}).

Then τ is G-linear $\Leftrightarrow \tau\circ\rho^1(g)=\rho^2(g)\circ\tau\ \forall g \Leftrightarrow$

$$(**)\qquad \sum_j t_{ij}c^1_{jk}=\sum_j c^2_{ij}t_{jk}$$

And τ is a comodule homomorphism $\Leftrightarrow (\tau\otimes id)\circ\Delta_{V^1}=\Delta_{V^2}\circ\tau \Leftrightarrow$ [exercise: write in bases] $(**)$.

And (b) is left as an exercise (both conditions are expressed as vanishing of the matrix coefficients). $\square$