Lec 13: Hopf algebras, filtrations & gradings, IV

1) $\mathbb{F}[G]$-comodules vs $D(G)$-modules

2) Faithful representation of G.

1.0) Introduction

Let G be an algebraic group over an algebraically closed field $\mathbb{F}$. Recall that in Lec 3 we have constructed a unital associative algebra of distributions, $D(G)$.

Let V be a rational G-representation. Let $c_{\alpha,v}\in \mathbb{F}[G]$ be its matrix coefficients ($\alpha\in V^*, v\in V$). We claim that:

(i) $\exists!$ $D(G)$-module structure on V s.t. $\langle\alpha,\delta v\rangle = \langle\delta, c_{\alpha,v}\rangle$ $\forall$ $\alpha\in V^*, v\in V, \delta\in D(G)$. $\forall$ G-linear map is $D(G)$-linear & $\forall$ G-stable subspace is $D(G)$-stable

(ii) If G is irreducible as variety, then any $D(G)$-linear map is G-linear & any $D(G)$-stable subspace is G-stable.

Recall, Exercise 1 in Sec 1.2 of Lec 4 that the representation of $\mathfrak{g}$ in V also satisfies $\langle\alpha, xv\rangle = \langle x, c_{\alpha,v}\rangle$. Remark in Sec 1 of Lec 5 says that $\mathfrak{g}\subset D(G)$ as a Lie subalgebra & so the claimed $D(G)$-module structure on V restricted to $\mathfrak{g}$ gives the usual $\mathfrak{g}$-representation in V. Now consider the homomorphism $\mathcal{U}(\mathfrak{g})\to D(G)$ Turns out that if char $\mathbb{F}=0$, then it is an isomorphism. This implies that in (ii) we can replace $D(G)$ w. $\mathfrak{g}$. This implies Fact C in Sec 1.0 of Lec 10 (originally in Sec 2 of Lec 5).

1.1) Pairings

By a pairing between vector spaces $V, V^\vee$ we mean a bilinear map $\langle\cdot,\cdot\rangle: V\times V^\vee\to\mathbb{F}$. We say it's nondegenerate in V if $\langle v,\alpha\rangle=0$ $\forall\alpha\in V^\vee\Rightarrow v=0$. We similarly define the non-degeneracy in $V^\vee$.

Also note that from $\langle\cdot,\cdot\rangle$ we get a pairing $V\otimes V\times V^\vee\otimes V^\vee\to\mathbb{F}$ determined by $\langle u\otimes v,\alpha\otimes\beta\rangle=\langle u,\alpha\rangle\langle v,\beta\rangle$.

Definition: Let (A,Δ,η) be a coalgebra & $A^\vee$ be an algebra. We say that a pairing $\langle\cdot,\cdot\rangle: A\times A^\vee\to\mathbb{F}$ is invariant: if

(1) $\langle\Delta(f),a\otimes b\rangle=\langle f,ab\rangle$ $\quad\forall f\in A,\ a,b\in A^\vee$

(2) $\langle f,1\rangle=\eta(f)$

Example: Let $A=\mathbb{F}[G]$ & $A^\vee=\mathcal{D}(G)$. Set $\langle f,\delta\rangle:=\delta(f)$. (1) reads $[\delta_1\otimes\delta_2](m^*(f))=[\delta_1*\delta_2](f)$, which is the definition of the product in $\mathcal{D}(G)$. Similarly, (2) is the definition of $1\in\mathcal{D}(G)$. So $\langle\cdot,\cdot\rangle$ is invariant. It is non-degenerate in $\mathcal{D}(G)$ (b/c $\mathcal{D}(G)$ is constructed as a subspace in $\mathbb{F}[G]^*$). In fact, if G is irreducible, then $\langle\cdot,\cdot\rangle$ is non-degenerate in $\mathbb{F}[G]$ as well. Namely, let $f\in\mathbb{F}[G]$ be s.t. $\delta(f)=0$ $\forall\delta\in\mathcal{D}(G)$. Let $A:=\mathbb{F}[G]$ & $\mathfrak{m}$ be the max. ideal of e. Recall that $\mathcal{D}(G)=\bigcup_{k\geqslant0}(A/\mathfrak{m}^{k+1})^*$. So $f\in\bigcap_{k\geqslant0}\mathfrak{m}^{k+1}$. For $G=GL_n$, this an exercise (hint: $\mathfrak{m}^k\subset\{f\in\mathbb{F}[GL_n]\mid$ Taylor series of f starts w. terms of deg $\geqslant k$). For general G, this a consequence of the Krull intersection theorem: Let A be a Noetherian domain & $I\subsetneq A$ an ideal. Then $\bigcap_{k\geqslant0}I^k=\{0\}$. See Corollary 5.4 in Eisenbud's "Commutative algebra with a view..."

1.2) $A^\vee$- module from A-comodule.

Let $A\times A^\vee \longrightarrow \mathbb{F}$ be an invariant pairing. So, $\forall\, a\in A^\vee$, we have a linear map $\langle\cdot, a\rangle : A\to\mathbb{F}$.

Proposition: 1) Any A-comodule (V,Δ_V) becomes an $A^\vee$-module via $av=[id_V\otimes\langle\cdot, a\rangle]\Delta_V(v)$.

2) Any A-comodule homomorphism is $A^\vee$-linear & any $A^\vee$-sub-comodule is $A^\vee$-stable. Moreover, if $\langle\cdot,\cdot\rangle$ is non-degenerate in $A^\vee$, then the converses are true as well.

Proof:

1) We need to show $a(bv)=(ab)v$; $1v=v$ is exercise. Let $v_i, i\in I$, be a basis in V & $\Delta_V(v_j)=\sum_{i\in I} v_i\otimes c_{ij}$ $(c_{ij}\in A$ w. $|\{i \mid c_{ij}\neq 0\}|<\infty\ \forall j)$ so $av_j=\sum_i\langle c_{ij},a\rangle v_i$. The comodule coassociativity, $[id_V\otimes\Delta]\circ\Delta_V$ $=[\Delta_V\otimes id_A]\circ\Delta_V$ reads $\Delta(c_{ij})=\sum_{k\in I}c_{ik}\otimes c_{kj}$. Then (1) reads $\sum_{k\in I}\langle c_{ik},a\rangle\langle c_{kj},b\rangle=\langle c_{ij},ab\rangle$ & we have $a(bv_j)=a(\sum_k\langle c_{kj},b\rangle v_k)$ $=\sum_{i,k}\langle c_{ik},a\rangle\langle c_{kj},b\rangle v_i=\sum_i\langle c_{ij},ab\rangle v_i=(ab)v_j$.

2) Let $\tau: V^1\to V^2$, in a basis, $\tau(v_j^1)=\sum_{i\in I}t_{ij}v_i^2$; τ is a comodule homomorphism

$$\Leftrightarrow [\tau\otimes id_A]\circ\Delta_{V^1}(v_j^1)=\Delta_{V^2}\circ\tau(v_j)\ \forall j$$

$$\Leftrightarrow [\tau\otimes id_A](\sum_k v_k\otimes c^1_{kj})=\Delta_{V^2}(\sum t_{kj}v_k^2)$$

$$\Leftrightarrow \sum_{i,k}v_i\otimes t_{ik}c^1_{kj}=\sum_{i,k}v_i\otimes c^2_{ik}t_{kj}$$

$$\Leftrightarrow \sum_k t_{ik}c^1_{kj}=\sum_k c^2_{ik}t_{kj}\ \ \forall\, i,j$$

Similarly, τ is $A^\vee$-linear $\Leftrightarrow\langle\sum_k t_{ik}c^1_{kj},a\rangle=\langle\sum_k c^2_{ik}t_{kj},a\rangle\ \forall\, i,j\in I$ $\forall a\in A^\vee$. Part 2) about homomorphisms follows, a part about submodules is left as an exercise. □

Exercise: Let V be a rational representation of G, hence an $\mathbb{F}[G]$-comodule. The resulting $D(G)$-module satisfies $\langle \alpha, \delta v\rangle = \delta(c_{\alpha,v})$, hence the characterization from Sec 1.0.

Example: $V = A = \mathbb{F}[G]$, $A^\vee = D(G)$. Then we can write $\Delta(f) = m^*(f)$ as $\sum_{i=1}^{k} f_i \otimes f_i'$ with $f(gh) = \sum_i f_i(g) f_i'(h)$. So $[\delta f](g) = [\sum_i \delta(f_i') f_i](g)$ e.g. for $g=e$ we get $f(h) = \sum_i f_i(e) f_i'(h) \Rightarrow$

(3) $$[\delta f](e) = \delta(f).$$

In general, consider the left action $G \curvearrowright G \rightsquigarrow G \curvearrowright \mathbb{F}[G]$, so $[\delta f](g) = \delta(g^{-1}.f)$ (exercise).

1.3) $\mathcal{U}(\mathfrak{g})$ vs $D(G)$ in char 0

Proposition: Assume char $\mathbb{F} = 0$. Then the homomorphism $\mathcal{U}(\mathfrak{g}) \xrightarrow{\psi} D(G)$ induced by $\mathfrak{g} = T_e G \hookrightarrow D(G)$ is an isomorphism.

Ideas of proof (details are extended exercise)

Recall that $D(G) = \bigcup_{k \geq 0} (A/\mathfrak{m}^{k+1})^*$ ($\mathfrak{m} \subset A := \mathbb{F}[G]$ is the max. ideal of e). One can show that this is a Hopf algebra filtration & ψ is filtered & Hopf homom'm. Arguments like those of Sec 1 of Lec 12 show ψ is injective (this requires char 0). It remains to show $\dim \mathcal{U}(\mathfrak{g})_{\leq n} \geq \dim D(G)_{\leq n}$ $\forall n$. Note that

$$\dim D(G)_{\leq n} = \dim A/\mathfrak{m}^{n+1} = \sum_{i=0}^{n} \dim \mathfrak{m}^i/\mathfrak{m}^{i+1} = [S^i(\mathfrak{m}/\mathfrak{m}^2) \overset{\text{multiplication in } A}{\twoheadrightarrow} \mathfrak{m}^i/\mathfrak{m}^{i+1}]$$
$$\leq \sum_{i=0}^{n} \dim S^i(\mathfrak{m}/\mathfrak{m}^2) = \dim S(\mathfrak{g})_{\leq n} = \dim \mathcal{U}(\mathfrak{g})_{\leq n} \qquad \square$$

Rem: This fails in char p, already for $G = \mathbb{G}_a$, cf. Example in Secs

1.1 and 1.2 of Lec 3.

2) Faithful representation of G.

Here we explain why any algebraic group G is isomorphic to an algebraic subgroup of GL_n. We'll construct a rational representation $\Phi: G \to GL(V')$ s.t. Φ & $\varphi = T_e\Phi: \mathfrak{g} \to \mathfrak{gl}(V')$ are injective

We start w. a general result on comodules.

Lem: Let (A, Δ, η) be a coalgebra & (V, Δ_V) be an A-comodule. $\forall\, v \in V$ is contained in a fin. dim. A-subcomodule.

Proof:

We can write $\Delta_V(v) = \sum_{i=1}^{k} v_i \otimes a_i$ w. linearly independent $a_i \in A$. Set $V' := \operatorname{Span}_{\mathbb{F}}(v_i \mid i = 1, .. k)$. First, we claim that $v \in V'$. By the counit axiom of a comodule: $(id_V \otimes \eta) \circ \Delta_V(v) = v$ (under $V \otimes \mathbb{F} \xrightarrow{\sim} V$) so $v = [id_V \otimes \eta](\sum_{i=1}^{k} v_i \otimes a_i) = \sum_{i=1}^{k} \eta(a_i) v_i \in V'$

Second, we claim V' is a subcomodule. Note that any linear map $\operatorname{Span}_{\mathbb{F}}(a_i) \to \mathbb{F}$ can be extended to $A \to \mathbb{F}$. So it's enough to prove that $\forall\, \alpha \in A^*$ & $u = \sum_{i=1}^{k} \alpha(a_i) v_i = [id \otimes \alpha](\Delta_V(v))$ we have $\Delta_V(u) \in V' \otimes A$. We have $(\Delta_V \otimes id_A) \circ \Delta_V(v) = \sum \Delta_V(v_i) \otimes a_i$, so

$$\Delta_V(u) = \sum_{i=1}^{k} \alpha(a_i) \Delta_V(v_i) = [id_V \otimes id_A \otimes \alpha] \circ [\Delta_V \otimes id_A] \circ \Delta_V(v) =$$

$$= [(\Delta_V \otimes id_A) \circ \Delta_V = (id_V \otimes \Delta) \circ \Delta_V] = [id_V \otimes id_A \otimes \alpha]\Big(\sum_{i=1}^{k} v_i \otimes ([id_A \otimes \alpha](\Delta(a_i)))\Big)$$

$\in V' \otimes A.$ □

We apply this lemma to $V = A = \mathbb{F}[G]$. Let $f_1, .. f_k \in \mathbb{F}[G]$ be algebra generators. Let V_i' be fin. dim. subcomodules containing f_i

& $V' := \sum_{i=1}^{k} V_i'$. So we can view V' as a rational representation of G. Explicitly, we write $\Delta(f) = \sum_{i=1}^{k} f_{(1),i} \otimes f_{(2),i}$. Then $\forall \alpha \in V'^*$ we have $\langle \alpha, g.f \rangle = \sum_{i=1}^{k} \langle \alpha, f_{(1),i} \rangle f_{(2),i}(g)$. E.g. take $h \in G$ & $\alpha : V' \to \mathbb{F}$, $f \mapsto f(h)$. We get $[g.f](h) = \sum_{i=1}^{k} f_{(1),i}(h) f_{(2),i}(g) = m^*(f)(h,g) = f(hg)$.

Hence V' is stable under the action of G on $\mathbb{F}[G]$ by right translations. If $g \in G$ is such that g acts trivially on V', then $f_i(hg) = f_i(h)$ $\forall i$, $\forall h \in G$. Since f_i's generate $\mathbb{F}[G]$, $F(hg) = F(h)$ $\forall F \in \mathbb{F}[G] \Rightarrow hg = h \Rightarrow g = e$.

Now suppose $x \in \mathfrak{g}$ acts on V' by 0. We get $x.f_i = 0$ $\forall i \Rightarrow 0 = [x.f_i](e) = [\text{Example in Sec } 1.2] = x(f_i)$. Since f_i's generate $\mathbb{F}[G]$ & x is an e-derivation, we get thx to the Leibniz identity that $x(f) = 0$ $\forall f \Rightarrow x = 0$.

Rem: With a bit more Algebraic geometry one can show that the image of every algebraic group homomorphism is Zariski closed. Once $\varphi : G \to GL(V')$ & $\phi : \mathfrak{g} \to \mathfrak{gl}(V')$ are injective, $\varphi : G \to \varphi(G)$ is an algebraic group isomorphism. See [H1], Sec 8.6 for details.