Lec 14: Representations of $\mathfrak{sl}_n$ & SL_n, I

0) Introduction.

1) Weight decomposition & highest weights

2) Verma modules and their irreducible quotients.

0) We now proceed to understanding the representation theory of simple algebraic groups & their Lie algebras. It turns out that the case of SL_n & $\mathfrak{sl}_n$ is already representative enough (outside the setting of characteristic p Lie algebra representations, where the case of $\mathfrak{sl}_n$ is significantly easier than the general case). We will concentrate on the characteristic 0 case and briefly discuss the char p case (area of active recent & current interest)

The three problems we are going to address for $\mathfrak{sl}_n$:

(I) The classification of finite dimensional irreducible representations.

(II) Complete reducibility of finite dimensional representations.

(III) Computation of characters of finite dimensional irreps.

We start with (I) - based on highest weight theory.

1) Weight decomposition & highest weights

1.0) Reminder on reps of $\mathfrak{sl}_2$

Many questions about representations of $\mathfrak{g}=\mathfrak{sl}_n$ reduce to those about $\mathfrak{sl}_2$ thanks to the following subalgebras $\mathfrak{g}_i=\mathrm{Span}_{\mathbb{F}}(e_i,h_i,f_i)$ with $e_i=E_{i,i+1}$, $h_i=E_{ii}-E_{i+1,i+1}$, $f_i=E_{i+1,i}$, $i=1,..n-1$ (the 1st index is the row # & the 2nd is the column #). Each of $\mathfrak{g}_i\simeq\mathfrak{sl}_2$.

Another importent property is:

(*) $h_1,\dots h_{n-1}$ pairwise commute.

We'll need the following claim about fin. dim. $\mathfrak{sl}_2$-reps

Lemma: Let V be a finite dimensional $\mathfrak{sl}_2$-representation. Then

1) $V=\bigoplus_{n\in\mathbb{Z}} V_n(h)$, where $V_n(h)=\{v\in V \mid hv=nv\}$

2) $e: V_n\to V_{n+2}$ & is injective for $n<0$.

Proof: The main classification result of Lec 6 is isomorphism $V=\bigoplus_i L(\lambda_i)$, where $\lambda_i\in\mathbb{Z}_{\geqslant 0}$

Both 1) & 2) reduce to the case when $V=L(\lambda)$. There V has basis v_i, $i=0,\dots\lambda$, with $hv_i=(\lambda-2i)v_i$, $ev_i=i(\lambda-i+1)v_{i-1}$ $\Rightarrow$ 1) & 2) $\square$

1.1) Weight decomposition

Definition: 1) The subalgebra of all diagonal matrices in $\mathfrak{sl}_n$: $\{diag(x_1,\dots x_n) \mid x_1+x_2+\dots+x_n=0\}$ is called a **Cartan subalgebra**. We denote this subalgebra by $\mathfrak{h}$. Note that $h_1,\dots h_{n-1}$ form a basis in $\mathfrak{h}$.

2) A **weight** is an element $\lambda\in\mathfrak{h}^*$ s.t. $\langle\lambda,h_i\rangle\in\mathbb{Z}$ $\forall i=1,\dots n-1$. The set of weights, a lattice in $\mathfrak{h}^*$, is denoted by Λ

3) For $\lambda\in\Lambda$, and a $\mathfrak{g}$-rep. V we define the **weight space** $V_\lambda:=\{v\in V \mid xv=\langle\lambda,x\rangle v\ \forall x\in\mathfrak{h}\}=\bigcap_{i=1}^{n-1}V_{\langle\lambda,h_i\rangle}(h_i)$.

Examples: 1) Let $V=\mathbb{F}^n$ be the tautological representation of $\mathfrak{sl}_n$ w. tautological basis $e_1,\dots e_n$, weight vectors. Their weights are denoted by $\varepsilon_1,\dots,\varepsilon_n$ so that $\varepsilon_i : \mathrm{diag}(x_1,\dots x_n) \mapsto x_i$.

2) Consider the adjoint representation, $\mathfrak{g}$. For $X=\mathrm{diag}(x_1,\dots,x_n)$ & $Y=(y_{ij})\in\mathfrak{sl}_n$, we have $[X,Y]=((x_i-x_j)y_{ij})$. So $\mathfrak{h}=\mathfrak{g}_0$, and for $i\neq j$, we have that $\alpha:=\varepsilon_i-\varepsilon_j$ is a weight of $\mathfrak{g}$ w. $\mathfrak{g}_\alpha=\mathbb{F}E_{ij}$. The nonzero weights are called roots, they are $\varepsilon_i-\varepsilon_j$ $(i\neq j)$.

Exercise: The weight spaces in $\Lambda^k\mathbb{F}^n$ are 1-dimensional, the weights are of the form $\varepsilon_{i_1}+\dots+\varepsilon_{i_k}$, $i_1<i_2<\dots<i_k$, and the corresponding weight vectors are $e_{i_1}\wedge e_{i_2}\wedge\dots\wedge e_{i_k}$.

The following summarizes basic properties of weight decompositions.

Lemma: 1) $V=\bigoplus_{\lambda\in\Lambda} V_\lambda$

2) If $\varphi: U\to V$ is a $\mathfrak{g}$-linear map, then $\varphi(U_\lambda)\subset V_\lambda$ $\forall \lambda\in\Lambda$.

3) If $u\in U_\lambda, v\in V_\mu \Rightarrow u\otimes v\in (U\otimes V)_{\lambda+\mu}$, so $(U\otimes V)_\nu=\bigoplus_{\lambda\in\Lambda} U_\lambda\otimes V_{\nu-\lambda}$.

4) The map $\mathfrak{g}\otimes V\to V$, $x\otimes v\mapsto xv$, is $\mathfrak{g}$-linear

5) If $x\in\mathfrak{g}_\alpha$ & $v\in V_\lambda$, then $xv\in V_{\lambda+\alpha}$

Proof:

1): pairwise commuting diagonalizable operators, h_i $(i=1,\dots n-1)$, are diagonalizable simultaneously.

2) & 3): exercise.

4): Let $\varphi:\mathfrak{g}\otimes V\to V$ be the map: $\varphi(y(x\otimes v))=\varphi([y,x]\otimes v+x\otimes yv)$ $=[y,x]v+xyv=yxv=y\varphi(x\otimes v)$.

5) Follows from 2), 3), 4) $\square$

1.2) Highest & dominant weights.

Definition: • positive roots are those of the form $\varepsilon_i-\varepsilon_j$ w. $i<j$.

- For $\lambda,\mu\in\mathfrak{h}^*$ set $\lambda\leq\mu$ if $\mu-\lambda=\sum_{i=1}^{k}\beta_i$ for some k & positive roots β_i.
- A highest weight of V is a maximal element of $\{\lambda \mid V_\lambda\neq\{0\}\}$.
- We say $\lambda\in\Lambda$ is dominant if $\langle\lambda,h_i\rangle\geq 0\ \forall\ i=1,\ldots n-1$. The set of dominant weights is denoted by Λ_+.

We note that every $\lambda\in\Lambda$ can be (non-uniquely) written as $\sum_{i=1}^{n}\lambda_i\varepsilon_i$ w $\lambda_i\in\mathbb{Z}$, the condition that $\lambda\in\Lambda_+$ means then $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n$.

Example: $\mathbb{F}^n$ has a unique highest weight, it is ε_1.

Lemma: 1) Every nonzero representation has a highest weight
2) If λ is highest & $v\in V_\lambda$, then $yv=0\ \forall\ y\in\mathfrak{g}_\beta$ if β is positive.
3) Every highest weight, λ, is dominant.

Proof:

1) follows b/c the set of weights is finite, 2) is an exercise.

3) $e_i\in\mathfrak{g}_{\varepsilon_i-\varepsilon_{i+1}}$ & $\varepsilon_i-\varepsilon_{i+1}$ is positive $\Rightarrow e_iv=0\ \forall\ v\in V_\lambda\subset V_{\langle\lambda,h_i\rangle}(h_i)$
Applying Lemma in Sec 1.0) to $\mathfrak{g}_i=\mathrm{Span}_{\mathbb{F}}(e_i,h_i,f_i)\simeq\mathfrak{sl}_2$, we see that $\langle\lambda,h_i\rangle\geq 0$. This holds $\forall\ i\Rightarrow\lambda$ is dominant. □

Exercise: 1) $\varepsilon_1+\ldots+\varepsilon_i$ is the unique highest weight of $\Lambda^i\mathbb{F}^n$ $(i=1,\ldots,n-1)$.
2) If λ & μ are highest weights of U,V, then $\lambda+\mu$ is a highest weight of $U\otimes V$.

Our final classification result for finite dimensional irreducible $\mathfrak{g}$-representations is that such has a unique highest weight & that taking the highest weight gives a bijection between the irreps & Λ_+. This generalizes the $\mathfrak{sl}_2$-case.

2) Verma modules and their irreducible quotients.

Our general construction follows what we did for $\mathfrak{sl}_2$: we construct an irrep. w. highest weight λ as a quotient of the corresponding Verma module.

Notation: Let $\beta_1, \dots, \beta_N$ $(N = \frac{n(n-1)}{2})$ be the positive roots in some order. For $\beta_k = \varepsilon_i - \varepsilon_j$ $(i<j)$, we write $e_{\beta_k} = E_{ij}$ & $f_{\beta_k} = E_{ji}$. The elements $f_{\beta_j}, h_i, e_{\beta_j}$ form a basis in $\mathfrak{g}$. For $\vec{k} \in \mathbb{Z}^N_{\geq 0}$ we set $e^{\vec{k}} := e^{k_1}_{\beta_1} \dots e^{k_N}_{\beta_N}$ & define $f^{\vec{k}}$ similarly. And for $\vec{\ell} \in \mathbb{Z}^{n-1}_{\geq 0}$ we write $h^{\vec{\ell}}$ for $\prod_{i=1}^{n-1} h_i^{\ell_i}$. So the PBW Thm tells us that the monomials $e^{\vec{k}} h^{\vec{\ell}} f^{\vec{m}}$ $(\vec{k}, \vec{m} \in \mathbb{Z}^N_{\geq 0}, \vec{\ell} \in \mathbb{Z}^{n-1}_{\geq 0})$ form a basis in $\mathcal{U}(\mathfrak{g})$.

The following generalizes Definition in Sec 1.3 of Lec 6.

Definition: Let $\lambda \in \mathfrak{h}^*$. The Verma module $\Delta(\lambda)$ is $\mathcal{U}(\mathfrak{g})/I_\lambda$, where

$$I_\lambda = \mathcal{U}(\mathfrak{g})\{x - \langle \lambda, x \rangle, e_\alpha \mid x \in \mathfrak{h}, \alpha \text{ is positive root}\}$$

Set $v_\lambda := 1 + I_\lambda$. We have the following, similarly to Proposition in Sec 1.3 in Lec 6. The proofs are similar & are omitted.

Lemma: (a) $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\Delta(\lambda), V) \xrightarrow{\sim} \{v \in V \mid xv = \langle\lambda, x\rangle, e_\alpha v = 0\}, \varphi \mapsto \varphi(v_\lambda)$, $\forall$ $\mathfrak{g}$-representation V.

(b) The elements $f^{\vec{k}} v_\lambda$ form a basis in $\Delta(\lambda)$. Moreover, we have

$$x f^{\vec{k}} v_\lambda = \langle \lambda - \sum_{j=1}^{N} k_j \beta_j, x\rangle f^{\vec{k}} v_\lambda \quad , \forall x \in \mathfrak{h}.$$

(c) In particular, $\Delta(\lambda) = \bigoplus_{\mu \in \mathfrak{h}^*, \mu \leq \lambda} \Delta(\lambda)_\mu$ w. $\Delta(\lambda)_\lambda = \mathbb{F} v_\lambda$.

(d) For any $\mathcal{U}(\mathfrak{g})$-submodule $M \subset \Delta(\lambda)$, we have $M = \bigoplus_\mu M_\mu$, $M_\mu := M \cap \Delta(\lambda)_\mu$.

When $n=2$, one can completely describe all submodules of $\Delta(\lambda)$. In general, this is impossible. However, we have the following.

Proposition: $\forall \lambda \in \mathfrak{h}^*$ $\Delta(\lambda)$ has a unique <u>maximal</u> (w.r.t. $\subseteq$) submodule (hence unique irreducible quotient, to be denoted by $L(\lambda)$).

Proof: Consider a $\mathcal{U}(\mathfrak{g})$-submodule M. Consider the subspace $\Delta(\lambda)^+ = \bigoplus_{\mu < \lambda} \Delta(\lambda)_\mu$. Note that $\dim \Delta(\lambda)_\lambda = \mathbb{F} v_\lambda$ & $\Delta(\lambda) = \mathcal{U}(\mathfrak{g}) v_\lambda$ ((c) of Lemma)) & thx to d) of Lemma, $M \subsetneq \Delta(\lambda) \Leftrightarrow M \subset \Delta(\lambda)^+$. $\forall$ subspace $V \subset \Delta(\lambda)$ contains the unique maximal $\mathcal{U}(\mathfrak{g})$-submodule: $\bigcap_{a \in \mathcal{U}(\mathfrak{g})} \{v \in V \mid av \in V\}$ (exercise). Apply this to $V = \Delta(\lambda)^+$ & finish the proof. $\square$

Corollary: Let V be an irreducible finite dimensional representation of $\mathfrak{g}$. Then $V \simeq L(\lambda)$ for a unique $\lambda \in \Lambda_+$. Moreover, $\dim V_\lambda = 1$.

Proof:

By Lemma in Sec 1.2, V has a highest weight, λ, & $\forall$ $v \in V_\lambda$ satisfies $xv = \langle\lambda, x\rangle v$ & $e_\alpha v = 0$ $\forall$ positive α. By a) of Lemma above, we have nonzero $\Delta(\lambda) \to V$ hence V is an irred. quotient of $\Delta(\lambda)$, so $V \simeq L(\lambda)$. Note that by the construction of $L(\lambda)$, we have $\dim L(\lambda)_\lambda = \dim \Delta(\lambda)_\lambda = 1$. Also by (c) above, we have $L(\lambda)_\mu \neq \{0\} \Rightarrow \mu \leq \lambda$. This implies the uniqueness of the highest weight. $\square$

Conclusion: We have embedded the set $\mathrm{Irr}_{fd}(\mathfrak{g})$ of finite dimensional irreducible $\mathfrak{g}$-reps into the set Λ_+ of dominant weights. What remains is to prove that the image is Λ_+ - for each dominant weight there is a finite dimensional irrep. w. that highest weight $\Leftrightarrow$ for $\lambda \in \Lambda_+$, $\dim L(\lambda) < \infty$ - to be done in Lec 15.