Representations of $\mathfrak{sl}_n$ & SL_n

1) $\dim L(\lambda)<\infty$ for $\lambda\in\Lambda_+$

1.0) Recap & goal.

Recall that $\mathbb{F}$ is an algebraically closed field of char 0, $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{F})$, $\mathfrak{h}=\{\operatorname{diag}(x_1,\dots,x_n)\mid x_1+\dots+x_n=0\}$, the Cartan subalgebra w. basis $h_i=E_{ii}-E_{i+1,i+1}$ for $i=1,\dots,n-1$. Consider subset of dominant weights

$$\Lambda_+=\{\lambda\in\mathfrak{h}^*\mid \langle\lambda,h_i\rangle\in\mathbb{Z}_{\geq 0}\ \forall i\Leftrightarrow \lambda=\sum_{i=1}^n\lambda_i\varepsilon_i \text{ w. } \lambda_i\in\mathbb{Z}\ \&\ \lambda_1\geq\dots\geq\lambda_n\}$$

We have a partial order on $\mathfrak{h}^*$: $\lambda\leq\mu\Leftrightarrow \mu-\lambda\in\operatorname{Span}_{\mathbb{Z}_{\geq 0}}(\varepsilon_i-\varepsilon_j\mid i<j)$

For a positive root $\beta=\varepsilon_i-\varepsilon_j$ $(i<j)$, we write e_β for E_{ij} & f_β for E_{ji}.

For $\lambda\in\mathfrak{h}^*$ we have defined the Verma module

$$\Delta(\lambda):=\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\operatorname{Span}_{\mathbb{F}}\{x-\langle\lambda,x\rangle, e_\beta\mid \text{all } x\in\mathfrak{h}\ \&\ \text{positive } \beta\}\ni v_\lambda:=\text{coset of } 1.$$

In Lec 14, Sec 2, we've seen that:

1) $\forall\ \mathfrak{g}$-rep V, $\operatorname{Hom}_{\mathfrak{g}}(\Delta(\lambda),V)\xrightarrow{\sim}\{v\in V\mid xv=\langle\lambda,x\rangle v, e_\beta v=0\}$, $\varphi\mapsto\varphi(v_\lambda)$
2) The vectors $f^{\vec{k}}v_\lambda$ form a weight basis of $\Delta(\lambda)$ $(\vec{k}\in\mathbb{Z}_{\geq 0}^N,\ N:=n(n-1)/2)\Rightarrow$
2') $\Delta(\lambda)=\bigoplus_{\mu\leq\lambda}\Delta(\lambda)_\mu$ - wt. spaces; $\{f^{\vec{k}}v_\lambda\mid \lambda-\mu=\sum_{i=1}^N k_i\beta_i\}$ form basis in $\Delta(\lambda)_\mu$.
3) $\exists!$ unique irreducible quotient $L(\lambda)$ of $\Delta(\lambda)$ $\forall\ \lambda\in\mathfrak{h}^*$.
4) $\forall$ fin. dim. $\mathfrak{g}$-irrep V $\exists!$ dominant weight λ w. $V\simeq L(\lambda)$.

The following will finish the classification of fin. dim $\mathfrak{g}$-irreps.

Thm: $\forall\ \lambda\in\Lambda_+\Rightarrow\dim L(\lambda)<\infty$.

To prove this we'll construct an a priori larger quotient $\tilde{L}(\lambda)$ of $\Delta(\lambda)$ & show $\dim\tilde{L}(\lambda)<\infty$. Later on, we'll show $\tilde{L}(\lambda)=L(\lambda)$.

1.1) Construction of $\tilde{L}(\lambda)$

Recall that in the case of $\mathfrak{sl}_2$, we had $L(\lambda)=\Delta(\lambda)/f^{\lambda+1}\Delta(\lambda)$ (Sec. 1.4 of Lec 6), the construction of $\tilde{L}(\lambda)$ for $\mathfrak{g}=\mathfrak{sl}_n$ generalizes the r.h.s. Set $\alpha_i:=\varepsilon_i-\varepsilon_{i+1}$. The roots $\alpha_1,\dots\alpha_{n-1}$ are called simple.

Lemma: Suppose $\langle\lambda,h_i\rangle=:n\in\mathbb{Z}_{\geqslant 0}$. $\exists!$ nonzero homomorphism $\varphi_i:\Delta(\lambda-(n+1)\alpha_i)\longrightarrow\Delta(\lambda)$

Proof: Consider $v:=f_i^{n+1}v_\lambda\in\Delta(\lambda)_{\lambda-(n+1)\alpha_i}$. Thx to 1) in Sec 1.0, it's enough to show $e_\beta f_i^{n+1}v_\lambda=0$ $\forall$ positive roots β. Consider two cases:

1) $\beta=\alpha_i$, so $e_\beta, f_i\in\mathfrak{g}_i\simeq\mathfrak{sl}_2$. By Lemma in Sec. 3.2 in Lec 5,
$e_if_i^{n+1}=(n+1)f_i^n(h_i-n)+f_i^{n+1}e_i \Rightarrow e_if_i^{n+1}v_\lambda=0$

2) $\beta\neq\alpha_i$. We'll need

Exercise: $[f_i,e_\beta]$ is proportional to e_γ for a positive root $\gamma\neq\alpha_i$

We claim $e_\beta f_i^m v_\lambda=0$ $\forall m\geqslant 0$. The proof is induction on m w. step $e_\beta f_i^m v_\lambda=f_ie_\beta f_i^{m-1}v_\lambda+a e_\gamma f_i^{m-1}v_\lambda$ $(a\in\mathbb{F})$. $\square$

Of course, if $\lambda\in\Lambda_+$, then we get $\varphi_i:\Delta(\lambda-(\langle\lambda,h_i\rangle+1)\alpha_i)\rightarrow\Delta(\lambda)$ $\forall i\in\{1,\dots,n-1\}$. We set

$$\tilde{L}(\lambda)=\Delta(\lambda)/\sum_{i=1}^{n-1}\operatorname{im}\varphi_i$$

The next property of $\tilde{L}(\lambda)$ follows from 2') in Sec 1.0).

(*) $\tilde{L}(\lambda)=\bigoplus_{\mu\leqslant\lambda}\tilde{L}(\lambda)_\mu$ $\quad(\tilde{L}(\lambda)_\mu=\{v\in\tilde{L}(\lambda)\mid xv=\langle\mu,x\rangle v\ \forall x\in\mathfrak{h}\})$

1.2) Finite dimensional weight spaces

Lemma: We have $\dim \Delta(\lambda)_\mu < \infty$ $\forall \lambda \in \mathfrak{h}^*, \mu \leq \lambda$. Hence $\dim \tilde{L}(\lambda)_\mu < \infty$.

Proof:

$\dim \Delta(\lambda)_\mu = [2')$ in Sec 1.0$] = \#\{(k_1,\ldots k_N) \mid \sum_{i=1}^{N} k_i \beta_i = \mu - \lambda\}$.

Set $\rho^\vee = \mathrm{diag}\left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}\right) \in \mathfrak{h}$ so that for $\beta = \varepsilon_i - \varepsilon_j \Rightarrow \langle \rho^\vee, \beta \rangle = i-j$ [if $i<j$] $\in \mathbb{Z}_{>0}$. So $\sum k_i \beta_i = \mu - \lambda \Rightarrow \sum k_i \langle \beta_i, \rho^\vee \rangle = \langle \mu - \lambda, \rho^\vee \rangle$ & $\dim \Delta(\lambda)_\mu \leq (\langle \mu - \lambda, \rho^\vee \rangle + 1)^N$. $\square$

To prove $\dim \tilde{L}(\lambda) < \infty$ it remains to establish

(**) $Wt(\tilde{L}(\lambda)) := \{\mu \mid \tilde{L}(\lambda)_\mu \neq \{0\}\}$ is finite

1.2) Weyl group

To show (**) we'll need a certain symmetry of $Wt(\tilde{L}(\lambda))$. The symmetry is w.r.t. a subgroup of $GL(\mathfrak{h}^*)$.

Definition: Define $s_i \in GL(\mathfrak{h}^*)$ by $s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i$ & let the **Weyl group** W be the subgroup of $GL(\mathfrak{h}^*)$ generated by $s_1, \ldots s_{n-1}$.

Note that for $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i$ we have $\langle \lambda, h_i \rangle = \lambda_i - \lambda_{i+1}$ &

$$s_i \lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i - (\lambda_i - \lambda_{i+1})(\varepsilon_i - \varepsilon_{i+1}) = \lambda_1 \varepsilon_1 + \ldots + \lambda_{i+1}\varepsilon_i + \lambda_i \varepsilon_{i+1} + \ldots + \lambda_n \varepsilon_n.$$

It follows that $W = S_n$ acting on $\mathfrak{h}^*$ by permuting the coordinates & $s_i = (i, i+1)$.

We will prove the following

Proposition: For any finite dimensional $\mathfrak{g}$-rep V & also for $V=\tilde{L}(\lambda)$, we have $\dim V_\mu = \dim V_{w\mu}$ $\forall w\in W$.

Proposition implies (**). Namely note that $W\mu \cap \Lambda^+$ is a single element $\forall \mu\in\Lambda$ (the weight lattice): we order the entries $\mu_1,\dots,\mu_n$ in decreasing order. So we need to show $\{\mu\in\Lambda^+ \mid \mu\leq\lambda\}$ is finite. Note that $\langle\rho^\vee,\mu\rangle\in\frac{1}{n}\mathbb{Z}_{\geq 0}$ $\forall\mu\in\Lambda_+$ (exercise) so $\langle\rho^\vee,\lambda-\mu\rangle\in\mathbb{Z}_{\geq0}$ & $\langle\rho^\vee,\lambda-\mu\rangle\leq\langle\rho^\vee,\lambda\rangle$. As in the proof of Lemma in Sec 1.2, this leaves fin. many options for $\lambda-\mu$. So we have $\dim\tilde{L}(\lambda)<\infty$ modulo Proposition.

1.3) Proof of Proposition

Lemma: $\forall v\in\tilde{L}(\lambda)$, $i=1,\dots,n-1$, $\exists$ fin. dim. $\mathfrak{g}_i$-stable subspace $V^\circ\subset V$ w. $v\in V^\circ$

Proof:

Let $\underline{v}_\lambda$ be the image of $v_\lambda\in\Delta(\lambda)$ in $\tilde{L}(\lambda)$. Set

$$V_i^0 := \operatorname{Span}_{\mathbb{F}}(f^i\underline{v}_\lambda \mid i=0,\dots,\langle\lambda,h_i\rangle)$$

fin. dim. & $\mathfrak{g}_i$-stable. Note that $U(\mathfrak{g})v_\lambda=\Delta(\lambda)\Rightarrow U(\mathfrak{g})\underline{v}_\lambda=\tilde{L}(\lambda)$ $\Rightarrow U(\mathfrak{g})V_i^0=\tilde{L}(\lambda)$. Note that $U(\mathfrak{g})V_0^i=\sum_{i=0}^\infty \operatorname{im}\varphi_i$, where

$$\varphi_i:\mathfrak{g}^{\otimes i}\otimes V_i^0\longrightarrow\tilde{L}(\lambda),\ x_1\otimes\ldots\otimes x_i\otimes v\longmapsto x_1\ldots x_i v.$$

We've seen (Lemma in Sec 1.1 of Lec 14) that the action map $\mathfrak{g}\otimes U\to U$ is $\mathfrak{g}$-linear ($\forall$ $\mathfrak{g}$-rep U), so the iterated map $\mathfrak{g}^{\otimes i}\otimes V\to V$ is $\mathfrak{g}$-linear (exercise: induct on i). Since φ_i is the restriction of this map to a $\mathfrak{g}_i$-stable subspace $\mathfrak{g}^{\otimes i}\otimes V_i^0\subset\mathfrak{g}^{\otimes i}\otimes V$, it's $\mathfrak{g}_i$-linear. So $\operatorname{im}\varphi_i$ is $\mathfrak{g}_i$-stable & fin. dim. We take $V_0^i:=\sum_{i=0}^j \operatorname{im}\varphi_i$ for suitable j. $\square$

Proof of Proposition: Suppose V is a $\mathfrak{g}$-rep'n & $i\in\{1,..n\}$

(i) $V=\bigoplus_{\mu\in\Lambda} V_\mu$ $V_\mu=\{v\in V \mid xv=\langle\mu,x\rangle v\ \forall x\in\mathfrak{h}\}$ w. $\dim V_\mu<\infty\ \forall\mu$

(ii) $\forall v\in V$, $\exists$ fin. dim. $\mathfrak{g}_i$-stable $V^\circ\subset V$ s.t. $v\in V^\circ$.

We claim that $\dim V_\mu=\dim V_{s_i\mu}$ $\forall i\in\Lambda$. Conditions (i) & (ii) hold for both kinds of V in the proposition & all i. Hence W is generated by the s_i's, the claim of proposition follows.

Let $n:=\langle\mu,h_i\rangle$ so that $s_i\mu=\mu-n\alpha_i$. We can assume $n>0$, otherwise replace μ w. $s_i\mu$. Then $f_i^n:V\to V$ sends V_μ to $V_{s_i\mu}$. We claim that $f_i^n: V_\mu\to V_{s_i\mu}$ is injective. Assume the contrary: let $v\in V_\mu$ be s.t. $f_i^n v=0$. Take $V^\circ\ni v$ as in (ii) so that $v\in(V^\circ)_n$ (the weight space for $\mathfrak{g}_i$). Decompose V° as the direct sum of $\mathfrak{g}_i$-irreps $V^\circ=\bigoplus_{i=1}^k L(m_i)$ $(m_i\in\mathbb{Z}_{\geqslant 0})$. The vector v can have nontrivial projection to $L(m_i)$ only if $m_i-n\in 2\mathbb{Z}_{\geqslant 0}$. And in this case both $L(m_i)_n$ & $L(m_i)_{-n}$ have $\dim=1$ & $f_i^n: L(m_i)_n\xrightarrow{\sim} L(m_i)_{-n}$ (exercise). We arrive at a contradiction w. $f_i^n v=0\Rightarrow f_i^n: V_\mu\hookrightarrow V_{s_i\mu}$. Similarly, $e_i^n: V_{s_i\mu}\hookrightarrow V_\mu$ finishing the proof. $\square$

1.4) $\mathfrak{sl}_n$-specific proof

The above argument works for all simple Lie algebras. Here's an easier proof of $\dim L(\lambda)<\infty$ for $\mathfrak{g}=\mathfrak{sl}_n$ that also classifies the rational SL_n-irreps.

Define the fundamental weights $\omega_i\in\mathfrak{h}^*$, $i=1,\dots,n-1$, $\omega_i:=\sum_{j=1}^i\varepsilon_j$ so that $\langle\omega_i,h_j\rangle=\delta_{ij}$. Every $\lambda\in\Lambda_+$ is uniquely written as $\sum_{i=1}^{n-1}n_i\omega_i$ w. $n_i=\langle\lambda,h_i\rangle\in\mathbb{Z}_{\geqslant 0}$.

Proposition: The SL_n-representation $\bigotimes_{i=1}^{n-1}(\Lambda^i \mathbb{F}^n)^{\otimes n_i}$ has an irreducible constituent that when viewed as an $\mathfrak{sl}_n$-rep is irreducible of highest weight λ.

Proof: exercise – observe that the highest weight of $\bigotimes_{i=1}^{n-1}(\Lambda^i \mathbb{F}^n)^{\otimes n_i}$ is λ.

Another exercise: prove that the irreducible rational SL_n-reps are in bijection w. Λ_+ (by taking the highest weight of the corresponding Lie algebra representation.