## Lec 16: Representations of $SL_n$ & $\mathfrak{sl}_n$, III

### 1) Harish-Chandra (HC) isomorphism for the center of $\mathcal{U}(\mathfrak{g})$.

#### 1.0) Intro.

$\mathbb{F}$: alg. closed char 0 field, $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{F})$, $Z:=Z(\mathcal{U}(\mathfrak{g}))$.

Goal: Describe the algebra $Z$ and understand its action on $\Delta(\lambda)$, and its unique irred. quotient $L(\lambda)$ $(\lambda\in\mathfrak{h}^*)$. Apply this description to prove that every finite dimensional $\mathfrak{g}$-representation is completely reducible.

#### 1.1) Homomorphism $Z\longrightarrow \mathcal{U}(\mathfrak{h})$.

To describe $Z$ we construct an algebra homomorphism $Z\to\mathcal{U}(\mathfrak{h})$. Later we'll see it's injective and describe the image, hence describing $Z$. This homomorphism will also be used to describe how $Z$ acts on $\Delta(\lambda)$.

Recall: for $\beta=\varepsilon_i-\varepsilon_j$ $(i<j)$ we write $f_\beta:=E_{ji}$, $e_\beta=E_{ij}$. For $i=1,\dots,n-1$, $h_i:=E_{ii}-E_{i+1,i+1}$; $N=\frac{n(n-1)}{2}$, $\beta_1,\dots,\beta_N$ - all positive roots.

PBW Thm: $\mathcal{U}\mathfrak{g}$ has basis $f^{\vec{k}}h^{\vec{\ell}}e^{\vec{m}}$ $(\vec{k},\vec{m}\in\mathbb{Z}^N_{\geqslant0}, \vec{\ell}\in\mathbb{Z}^{n-1}_{\geqslant0})$ (1)

$\mathfrak{g}$, hence $\mathfrak{h}$, acts on $\mathcal{U}(\mathfrak{g})$ by ad: $\operatorname{ad}(x)a:=[x,a]$ $(x\in\mathfrak{h}, a\in\mathcal{U}\mathfrak{g})$.

Exercise: (1) is a weight vector of weight $\sum_{j=1}^{N}(m_j-k_j)\beta_j$ (hint: $\forall x\in\mathfrak{h}$, $a,b\in\mathcal{U}(\mathfrak{g})$, have $[x,ab]=[x,a]b+a[x,b]$).

Now we define the map $\pi : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$ sending $a \in \mathcal{U}(\mathfrak{g})$ to the sum of all monomials in the expansion of $a$ in (1) that only have $h_i$'s.

Example: for $C = \frac{1}{2}h^2 + h + 2fe \in Z \subset \mathcal{U}(\mathfrak{sl}_2) \Rightarrow \pi(C) = \frac{1}{2}h^2 + h$.

Note that all monomials in the expansion of $a - \pi(a)$ must have $k_j > 0$, $m_{j'} > 0$ for some $j, j'$. If $a$ has weight $0$ ($\Leftrightarrow [\mathfrak{h}, a] = 0 \Leftarrow a \in Z$), then $\forall$ monomial in $a$ must have weight $0$. So, $\pi(a) \in \mathcal{U}(\mathfrak{h})$ satisfies

$$a = \pi(a) + \sum_{j=1}^{N} ? e_{\beta_j}. \qquad (2)$$

For $z \in Z$, set $HC_z := \pi(z)$. Note that $\mathfrak{h}$ is an abelian Lie algebra $\Rightarrow \mathcal{U}(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{F}[\mathfrak{h}^*]$. So we view $HC_z$ as a polynomial on $\mathfrak{h}^*$.

Proposition: 1) $\forall\, z \in Z$, $\lambda \in \mathfrak{h}^*$, $z$ acts on $\Delta(\lambda)$ & $L(\lambda)$ by $HC_z(\lambda)$.

2) $z \mapsto HC_z$ is an algebra homomorphism $Z \rightarrow \mathcal{U}(\mathfrak{h})$.

Proof: 1) Since $e_\beta v_\lambda = 0$ $\forall$ positive root $\beta$, (2) $\Rightarrow zv_\lambda = HC_z(\lambda) v_\lambda$; $\forall v \in \Delta(\lambda)$ $\exists\, a \in \mathcal{U}(\mathfrak{g}) \mid v = av_\lambda$. Have $zv = zav_\lambda = azv_\lambda = HC_z(\lambda) a v_\lambda = HC_z(\lambda) v$.
The claim for $L(\lambda)$ follows b/c $\Delta(\lambda) \twoheadrightarrow L(\lambda)$.
2) $z \mapsto HC_z$ is $\mathbb{F}$-linear by construction. By 1), $HC_{z_1 z_2}(\lambda) = HC_{z_1}(\lambda) HC_{z_2}(\lambda)$ $\forall\, \lambda \in \mathfrak{h}^*$, $z_1, z_2 \in Z$. Indeed both sides are the scalars by which $z_1 z_2$ acts on $\Delta(\lambda)$ computed in two different ways.
So $HC_{z_1 z_2} = HC_{z_1} HC_{z_2}$. $\square$

## 1.2) Harish-Chandra isomorphism.

For $i=1,\dots,n-1$, define $s_i\cdot\lambda = \lambda - (\langle\lambda,h_i\rangle+1)\alpha_i$ so that $s_i\cdot$ is an affine map $\mathfrak{h}^*\to\mathfrak{h}^*$.

**Proposition:** $\forall z\in Z$, $\lambda\in\mathfrak{h}^*$ have $HC_z(\lambda)=HC_z(s_i\cdot\lambda)$.

Proof: Case 1: $\langle\lambda,h_i\rangle\in\mathbb{Z}_{\geq 0}$. By Sec 1.1 of Lec 15, $\exists$ nonzero $U(\mathfrak{g})$-linear homomorphism $\Delta(s_i\cdot\lambda)\to\Delta(\lambda)$ $\Rightarrow$ scalars of actions of $z\in U(\mathfrak{g})$ on $\Delta(s_i\cdot\lambda)$, $\Delta(\lambda)$ coincide. By Prop 1, $HC_z(\lambda)=HC_z(s_i\cdot\lambda)$.
Case 2: general. The locus $\{\lambda\in\mathfrak{h}^*\mid\langle\lambda,h_i\rangle\in\mathbb{Z}_{\geq 0}\}$ is a countable union of hyperplanes: $\{\lambda\in\mathfrak{h}^*\mid\langle\lambda,h_i\rangle=m\}$ for $m\in\mathbb{Z}_{\geq 0}$. Any polynomial vanishing on such locus is identically 0. Apply this to the polynomial $\lambda\mapsto HC_z(\lambda)-HC_z(s_i\cdot\lambda)$ & finish the proof. $\square$

**Example:** For $\mathfrak{sl}_2$: $\mathfrak{h}\simeq\mathbb{C}$ w. $h\leftrightarrow 1\rightsquigarrow\mathfrak{h}^*\simeq\mathbb{C}$ w. $\alpha=2$, $\rho=1$, $s\cdot\lambda=-\lambda-2$. Since $HC_C=\frac{1}{2}h^2+h$, we get $HC_C(\lambda)=\frac{1}{2}\lambda^2+\lambda=HC_C(-\lambda-2)$.

In fact, $\lambda\mapsto s_i\cdot\lambda$ extends to an action of the Weyl group $W(=S_n)$ on $\mathfrak{h}^*$. Set $\rho=\frac{1}{2}\sum_{i<j}(\epsilon_i-\epsilon_j)=\sum_{i=1}^{n}\left(\frac{n+1}{2}-i\right)\epsilon_i\in\mathfrak{h}^*$ so that $\langle\rho,h_i\rangle=1$
$\Rightarrow s_i\rho=\rho-\alpha_i$. Then $s_i(\lambda+\rho)-\rho=\lambda+\rho-\langle\lambda+\rho,h_i\rangle\alpha_i-\rho=\lambda-(\langle\lambda,h_i\rangle+1)\alpha_i=s_i\cdot\lambda$
Here $\alpha_i=\epsilon_i-\epsilon_{i+1}$ is a simple root.

**Definition:** The shifted action of $W$ on $\mathfrak{h}^*$ is given by $w\cdot\lambda:=w(\lambda+\rho)-\rho$.

Consider the subalgebra $\mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}=\{f\in\mathbb{F}[\mathfrak{h}^*]\mid f(w\cdot\lambda)=f(\lambda),\ \forall\lambda\in\mathfrak{h}^*, w\in W\}$ of invariant polynomials. Since the elements $s_i$ generate $W$, Proposition above implies $HC_z \in \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$ $\forall z \in Z$. The following will be proved next time.

**Thm** (Harish-Chandra) $z \mapsto HC_z : Z \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$.

**Corollary**: For $\lambda, \mu \in \mathfrak{h}^*$ TFAE

(1) $\lambda \in W\cdot\mu$.

(2) $HC_z(\lambda) = HC_z(\mu)$, $\forall z \in Z$.

Proof: (1) $\Rightarrow$ (2) is a direct consequence of the theorem. (2) $\Rightarrow$ (1) becomes: if $f(\lambda) = f(\mu)$ $\forall f \in \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$, then $\lambda \in W\cdot\mu$. This is exercise (hint: find a polynomial $f$ that is 1 on $W\cdot\lambda$, 0 on $W\cdot\mu$ and average w.r.t. $W$-action: $f \mapsto \frac{1}{|W|}\sum_{w\in W} f(w\cdot ?)$).

## 1.3) Algebra $\mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$

Consider the affine isomorphism $\tau: \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}^*$, $\lambda \mapsto \lambda + \rho$ so that $\tau(w\cdot\lambda) = w\,\tau(\lambda)$. So $\tau$ gives rise to an isomorphism $\tau: \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)} \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}^*]^W$. Let's describe the target. Embed $\mathfrak{h}^* \hookrightarrow \mathbb{F}^n$ via $\varepsilon_i \mapsto e_i - \frac{1}{n}(e_1 + \ldots + e_n)$ w. image $= \{(x_1 \ldots x_n) \mid x_1 + \ldots + x_n = 0\}$. Let $\sigma_i \in \mathbb{F}[\mathbb{F}^n]$ be the $i$th elementary symmetric polynomial: $\sigma_i(x_1, \ldots x_n) = \sum_{j_1 < \ldots < j_i} x_{j_1} \ldots x_{j_i}$.

**Lemma**: $\mathbb{F}[\mathfrak{h}^*]^W$ is the algebra of polynomials in $\sigma_2, \ldots, \sigma_n$.

Proof: exercise (hint: show $\mathbb{F}[\mathfrak{h}^*]^W = \mathbb{F}[\mathbb{F}^n]^W/(\sigma_1)$ & use the fundamental thm about symmetric polynomials to describe $\mathbb{F}[\mathbb{F}^n]^W$. $\square$

Exercise: $\mathcal{Z} \subset \mathcal{U}(\mathfrak{sl}_2)$ is generated by $C$.

## 1.4) Application: complete reducibility.

Thm: Every finite dimensional representation of $\mathfrak{g}$ is completely reducible.

Lem: Let $\lambda \neq \mu \in \Lambda_+$. Then $HC_z(\lambda) \neq HC_z(\mu)$ for some $z \in \mathcal{Z}$.

Proof: The entries of $\lambda+\rho$, $\mu+\rho$ are still decreasing so $\lambda \in W\cdot\mu$ ($\Leftrightarrow$ $\lambda+\rho \in W(\mu+\rho)$ for the usual action $\Leftrightarrow$ $\lambda+\rho$ is obtained from $\mu+\rho$ by permutation) implies $\lambda=\mu$. So, thx to Corollary in Sec 1.2, $\exists\, z \in \mathcal{Z}$ acting on $L(\lambda), L(\mu)$ by different scalars. $\square$

Sketch of proof of Thm: Let $\chi: \mathcal{Z} \to \mathbb{F}$ be a homomorphism. E.g. for $\lambda \in \mathfrak{h}^*$ we can consider $\chi_\lambda: z \mapsto HC_z(\lambda)$. For a $\mathfrak{g}$-rep. $V$ & $\chi: \mathcal{Z} \to \mathbb{F}$ set $V^\chi = \{v \in V \mid \forall\, z \in \mathcal{Z}\ \exists\, n_z \in \mathbb{Z}_{>0}$ s.t. $(z-\chi(z))^{n_z} v = 0\}$ – the intersection of generalized e-spaces. Note that if $z_2 \dots z_n \in \mathcal{Z}$ correspond to $\sigma_2 \dots \sigma_n \in \mathbb{F}[\mathfrak{h}^*]^W$, then $V^\chi$ is the intersection of the generalized e-spaces $V_{\chi(z_i)}(z_i)$, $i=2,\dots n$. Note that since $\mathcal{Z}$ is the center, $V^\chi$ is a $\mathfrak{g}$-subrep $\forall\, \chi$.

Exercise: If $V$ is finite dimensional, then

1) $V = \bigoplus_{\chi: \mathcal{Z} \to \mathbb{F}} V^\chi$

2) $V^\chi = 0$ unless $\chi = \chi_\lambda$ for some $\lambda \in \Lambda_+$.

So it's enough to show that $V$ is completely reducible if $V = V^{\chi_\lambda}$.

First observe that if $v \in V_\mu$ satisfies $e_\beta v = 0$ $\forall$ positive roots $\beta \Rightarrow \mu = \lambda$. if $v \in V_\mu$, then $zv = HC_z(\mu)v$ $\forall z \in Z$. This implies $HC_z(\mu) = HC_z(\lambda)$ $\forall z \in Z$ and, by Lemma, $\lambda = \mu$.

Now let $v \in V_\lambda \rightsquigarrow \varphi: \Delta(\lambda) \to V$ w. $v_\lambda \mapsto v$. Then $V' := \Delta(\lambda)/\ker\varphi$ is finite dimensional. We claim $V' \cong L(\lambda)$. Indeed, since every highest weight of $\ker[V' \twoheadrightarrow L(\lambda)] \subset V$ has to be $\lambda$ by the previous paragraph. On the other hand, all weights of $\ker[\Delta(\lambda) \twoheadrightarrow L(\lambda)]$ & hence of $\ker[V' \twoheadrightarrow L(\lambda)]$ are $< \lambda$. Combining these two observations show $V' \cong L(\lambda)$.

Now the proof repeats the case of $\mathfrak{sl}_2$ (Sec 2 of Lec 6) & is left as an exercise. $\square$

**Corollary:** Every nonzero finite dimensional quotient of a Verma module is irreducible. In particular, $\tilde{L}(\lambda) \xrightarrow{\sim} L(\lambda)$ ($\tilde{L}(\lambda)$ was defined in Lec 15).

Proof: exercise.

**Rem:** We don't need the full power of HC isomorphism to prove the complete reducibility – there are more elementary proofs. We will use the theorem when we compute the character of $L(\lambda)$, $\lambda \in \Lambda_+$.

## 2) Proof, started.

### 2.1) $\mathcal{Z}$ vs $\mathcal{U}(\mathfrak{g})^G$.

To establish the HC isomorphism, we'll need an alternative description of $\mathcal{Z}$. Let $G$ be a connected algebraic group w. Lie algebra $\mathfrak{g}$. Recall, Sec 1.2 of Lec 10, that $G$ acts on $\mathcal{U}(\mathfrak{g})$ by algebra automorphisms $\rightsquigarrow$ the subalgebra $\mathcal{U}(\mathfrak{g})^G \subset \mathcal{U}(\mathfrak{g})$ of invariants.

**Lemma:** $\mathcal{Z} = \mathcal{U}(\mathfrak{g})^G$.

*Proof:* $\mathcal{Z} = \{a \in \mathcal{U}(\mathfrak{g}) \mid ad(x)a = 0 \ \forall x \in \mathfrak{g}\}$. We write $\mathbb{F}$ for the trivial representation of $\mathfrak{g}$ or of $G$. Then

$$\begin{tikzcd}
\mathcal{Z} \arrow[d, "\sim"'] & \ni \varphi(1) \arrow[d, leftrightarrow] \\
\mathrm{Hom}_{\mathfrak{g}}(\mathbb{F}, \mathcal{U}(\mathfrak{g})) \arrow[d, equal] & \ni \varphi \\
\mathrm{Hom}_G(\mathbb{F}, \mathcal{U}(\mathfrak{g})) \arrow[d, "\sim"'] & \ni \varphi \arrow[d, leftrightarrow] \\
\mathcal{U}(\mathfrak{g})^G & \ni \varphi(1)
\end{tikzcd}$$

By Thm 2 in Sec 1.3 of Lec 7.

$\square$

## 3) Complements.

Here are some details for proving Theorem in Sec 1.4.

- Decomposition into "infinitesimal blocks": Let $V$ be a $\mathfrak{g}$-representation (not necessarily finite dimensional). Let $\chi: \mathcal{Z} \to \mathbb{F}$ be an algebra homomorphism. Set

$$V^\chi = \{ v \in V \mid \forall z \in Z \ \exists m > 0 \text{ s.t. } (z - \chi(z))^m v = 0 \}$$

This is a $U(\mathfrak{g})$-submodule in $V$. If $V$ is finite dimensional, then $V = \bigoplus_\chi V^\chi$. Moreover, $z$ acts by $\chi(z)$ on every irreducible constituent of $V^\chi$. It follows that $\chi(z) = HC_z(\lambda)$ for some $\lambda \in \Lambda^+$ whenever $V^\chi \neq \{0\}$. Moreover, by the observation in the proof of the theorem in Sec 1.4, in this case $L(\lambda)$ is the unique irreducible constituent of $V^\chi$.

So assume $V = V^\chi \Leftrightarrow V$ is filtered by $L(\lambda)$ w. $\chi(z) = HC_z(\lambda)$. Then $L(\lambda) \otimes V_\lambda \xrightarrow{\sim} V$, the proof repeats that in Sec 1.3 of Lec 9. Details are left as an exercise.