Lec 17: Representations of $\mathfrak{sl}_n$ & SL_n, IV

0) Recap
1) Chevalley restriction theorem.
2) Proof of HC isomorphism.

0) Recap

Let $\mathbb{F}$ be an alg. closed field of char 0.

Let $\mathfrak{g}=\mathfrak{sl}_n$, $\mathfrak{h}\subset\mathfrak{g}$ is the subalgebra of diagonal matrices & $W(\simeq S_n)$ be the Weyl group acting on $\mathfrak{h}$ & $\mathfrak{h}^*$ by permuting the entries.

In Sec 1.1 of Lec 16 we've constructed a linear map $\pi: U(\mathfrak{g})\to U(\mathfrak{h})$. We interpret $U(\mathfrak{h})$ as $\mathbb{F}[\mathfrak{h}^*]$ & equip w. a "shifted" W-action:

$$w\cdot\lambda = w(\lambda+\rho)-\rho \quad (\lambda\in\mathfrak{h}^*), \text{ where } \rho=\frac{1}{2}\sum_{i=1}^{n}(n+1-2i)\varepsilon_i.$$

Last time we stated (& today we'll prove)

Thm: The restriction of π to the center, Z, of $U(\mathfrak{g})$ is injective & the image is $\mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}$

1) Chevalley restriction theorem.

1.1) Connection to group invariants

Let $G=SL_n$. Recall (Sec 1.0 in Lec 8) that the adjoint representation of G in $\mathfrak{g}$ is by Lie algebra automorphisms & so gives rise to a representation of G in $U(\mathfrak{g})$ by algebra automorphisms.

Lemma: $Z\subset U(\mathfrak{g})$ coincides with the subalgebra of invariants $U(\mathfrak{g})^G$.

Proof:

Recall that we have an algebra filtration $\mathcal{U}(\mathfrak{g})_{\leq n} = \mathrm{Span}_{\mathbb{F}}(x_1 \dots x_i \mid x_j \in \mathfrak{g} \ \&\ i \leq n\}$; G preserves this filtration. The tangent map for the G-action is the $\mathfrak{g}$-action $\mathrm{ad}(x): a \mapsto xa - ax: \mathfrak{g} \to \mathrm{End}(\mathcal{U}(\mathfrak{g})_{\leq n})$. Indeed, let $\eta: \mathfrak{g}^{\otimes i} \to \mathcal{U}(\mathfrak{g})_{\leq n}$, $x_1 \otimes \dots \otimes x_i \mapsto x_1 \dots x_i$ $(i \leq n)$, it's G-linear. For the tangent action we have $x.(x_1 \dots x_i) = \eta(x.(x_1 \otimes \dots \otimes x_i)) = \eta(\sum_{j=1}^{i} x_1 \otimes \dots \otimes [x, x_j] \otimes \dots x_i)$
$= \sum_{j=1}^{i} x_1 \dots [x, x_j] \dots x_i = [\,[x, x_j] = xx_j - x_j x \rightsquigarrow \text{cancellations}] = x x_1 \dots x_i - x_1 \dots x_i x$.

Note that since $\mathfrak{g}$ generates $\mathcal{U}(\mathfrak{g})$, $\mathcal{Z} = \{z \in \mathcal{U}(\mathfrak{g}) \mid xz = zx \ \forall x \in \mathfrak{g}\}$

So now we reduce to the following:

Exercise: Let V be a rational G-representation. If $\mathrm{char}\, \mathbb{F} = 0 \Rightarrow V^G = V^{\mathfrak{g}} := \{v \in V \mid xv = 0 \ \forall x \in \mathfrak{g}\}$. Hint: $V^G \xleftarrow{\sim} \mathrm{Hom}_G(\mathbb{F}_{triv}, V)$ & $V^{\mathfrak{g}} \xleftarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(\mathbb{F}_{triv}, V)$ & G is irreducible; then see Sec 2 in Lec 5. □

1.2) From $\mathcal{U}(\mathfrak{g})^G$ to $\mathbb{F}[\mathfrak{g}]^G$

Recall that the filtration on $\mathcal{U}(\mathfrak{g})$ comes from $T(\mathfrak{g}) \twoheadrightarrow \mathcal{U}(\mathfrak{g})$ & while $S(\mathfrak{g})$ is a graded quotient of $T(\mathfrak{g})$ & this yields a natural (hence G-linear) epimorphism $S(\mathfrak{g}) \twoheadrightarrow \mathrm{gr}\, \mathcal{U}(\mathfrak{g})$ (Lec 11) & it's iso (conceptual form of the PBW theorem; Sec 1 of Lec 12).

If $z \in \mathcal{U}(\mathfrak{g})_{\leq n}$ is G-invariant, then so is $z + \mathcal{U}(\mathfrak{g})_{\leq n-1}$. So we get the inclusion $\mathrm{gr}[\mathcal{U}(\mathfrak{g})^G] \hookrightarrow S(\mathfrak{g})^G$, $z + \mathcal{Z}_{\leq n-1} \mapsto z + \mathcal{U}(\mathfrak{g})_{\leq n-1}$ (on n-th graded component). Below we'll see it's an isomorphism.

Recall that $S(\mathfrak{g}) \simeq \mathbb{F}[\mathfrak{g}^*]$, a canonical identification, hence G-linear. The following allows to identify $\mathbb{F}[\mathfrak{g}^*]$ w. $\mathbb{F}[\mathfrak{g}]$.

Lemma: The symmetric bilinear form $(x, y) \mapsto \mathrm{tr}(xy)$ on $\mathfrak{g}$ is G-invariant & nondegenerate.

Proof: the G-invariance is an exercise. The form is non-degenerate on $\mathfrak{gl}_n$ (E_{ji} $(1\le i,j\le n)$ is a dual basis to E_{ij}). Since char $\mathbb{F}$ doesn't divide p, $(id, id)=n\neq 0$. Since $\mathfrak{sl}_n = id^\perp$, $(\cdot,\cdot)$ is nondegenerate on $\mathfrak{g}$. $\square$

To summarize, $gr[\mathcal{U}(\mathfrak{g})^G] \hookrightarrow S(\mathfrak{g})^G \xrightarrow[tr]{\sim} \mathbb{F}[\mathfrak{g}]^G$. In the next section, we determine $\mathbb{F}[\mathfrak{g}]^G$.

1.3) The restriction isomorphism

Let $\iota: \mathfrak{h}\hookrightarrow\mathfrak{g} \rightsquigarrow \iota^*: \mathbb{F}[\mathfrak{g}]\to\mathbb{F}[\mathfrak{h}]$, the restriction. For example, consider the coefficients F_i of char. polynomial.

$$\det(x-\lambda\cdot id) = (-\lambda)^n + \sum_{i=0}^{n-2}(-\lambda)^i F_i(x)$$

We have $\iota^*(F_i)=\sigma_i$, the ith elementary symmetric polynomial restricted to $\mathfrak{h}$. Note that $F_i\in\mathbb{F}[\mathfrak{g}]^G$.

Thm (Chevalley restriction thm for $\mathfrak{sl}_n$):
$\iota^*: \mathbb{F}[\mathfrak{g}]^G \to \mathbb{F}[\mathfrak{h}]$ is injective w. image $\mathbb{F}[\mathfrak{h}]^W$ (for the usual action of W on $\mathfrak{h}$).

Proof: Note that SL_n-orbits in $\mathfrak{sl}_n$ = conjugacy classes of matrices ($\forall A\in GL_n\ \exists a\in\mathbb{F}\setminus\{0\} \mid aA\in SL_n$) so $\forall F\in\mathbb{F}[\mathfrak{g}]^G$ is constant on conjugacy classes.

- Injectivity: If $F\in\mathbb{F}[\mathfrak{g}]^G$ vanishes on $\mathfrak{h}$ (= diagonal matrices), then it vanishes on all diagonalizable matrices. This subset is dense in the Zariski topology (exercise - look at matrices w. distinct e-values).

• $\mathrm{im}\,\iota^* \subset \mathbb{F}[\mathfrak{h}]^W = \{f \in \mathbb{F}[\mathfrak{h}^*]^W$ constant on W-orbits$\}$. Notice that matrices $\mathrm{diag}(x_1, \dots, x_n)$ & $\mathrm{diag}(x_{\sigma(1)}, \dots x_{\sigma(n)})$ are conjugate (via a permutation matrix) $\forall \sigma \in S_n$

• $\mathrm{im}\,\iota^* \supset \mathbb{F}[\mathfrak{h}]^W$: $\sigma_i \in \mathrm{im}\,\iota^*$ (see above) & $\sigma_2, \dots \sigma_n$ generate $\mathbb{F}[\mathfrak{h}]^W$ (see Sec 1.3 of Lec 16). $\square$

2) Proof of HC isomorphism.

2.1) Commutative diagram

Recall that $\pi: U(\mathfrak{g}) \to U(\mathfrak{h})$ was defined by

$$\pi(f^{\vec{k}} h^{\vec{\ell}} e^{\vec{m}}) = \begin{cases} h^{\vec{\ell}} & \text{if } \vec{k} = \vec{m} = 0 \\ 0, & \text{else} \end{cases}$$

In particular, it's filtered & under the identification $\mathrm{gr}\, U(\mathfrak{g}) \xleftarrow{\sim} S(\mathfrak{g})$ & $\mathrm{gr}\, U(\mathfrak{h}) \xleftarrow{\sim} S(\mathfrak{h})$ its associated graded map $\mathrm{gr}\,\pi$ coincides w. the map $\pi_0: S(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ defined by the same formula.

Proposition: We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{gr}[U(\mathfrak{g})^G] & \xrightarrow{\mathrm{gr}\,\pi} & \mathrm{gr}[\mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)}] \\ \text{Sec 1.2} \rightarrow \big\downarrow & & \big\downarrow \wr \\ S(\mathfrak{g})^G & \xrightarrow[\pi_0]{\sim} & S(\mathfrak{h})^W \end{array} \qquad (1)$$

Proof:

Step 1 (right verticular map): Recall $\tau: \mathfrak{h}^* \to \mathfrak{h}^*$, $\lambda \mapsto \lambda + \rho \rightsquigarrow$ $\tau: \mathbb{F}[\mathfrak{h}^*] \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}^*]$. Note that $\mathrm{gr}\,\tau = id$ & recall $\tau: \mathbb{F}[\mathfrak{h}^*]^{(W,\cdot)} \xrightarrow{\sim} \mathbb{F}[\mathfrak{h}^*]^W$ as subalgebras in $\mathbb{F}[\mathfrak{h}^*]$. This gives a right vertical map, it's an iso.

Step 2 (bottom horizontal map): Recall that we identify $\mathfrak{g} \simeq \mathfrak{g}^*$ & $\mathfrak{h} \simeq \mathfrak{h}^*$ using trace pairing. This yields the following commutative diagram

$$\begin{tikzcd} \mathbb{F}[\mathfrak{g}] \arrow[r, "\iota^*"'] \arrow[d, "s"', "\wr"] & \mathbb{F}[\mathfrak{h}] \arrow[d, "s"', "\wr"] \\ S(\mathfrak{g}) \arrow[r, "\pi_0"] & S(\mathfrak{h}) \end{tikzcd}$$

Indeed under tr-pairing, we have $\iota^* = \pi_0$ on $\mathfrak{g} \simeq \mathfrak{g}^*$ (if x is diagonal & y has 0 entries on the diagonal $\Rightarrow tr(xy)=0$). Now we apply the Chevalley restriction thm to see that π_0 restricts to an isomorphism $S(\mathfrak{g})^G \xrightarrow{\sim} S(\mathfrak{h})^W$.

Step 3 (commutativity). By the construction of π, π_0, the following is commutative

$$\begin{tikzcd} gr[\mathcal{U}(\mathfrak{g})^G] \arrow[r, hook] \arrow[d] & gr\,\mathcal{U}(\mathfrak{g}) \arrow[r, "gr\,\pi"] \arrow[d] & gr\,F[\mathfrak{h}^*] \arrow[d, "s"', "\wr"] \\ S(\mathfrak{g})^G \arrow[r, hook] & S(\mathfrak{g}) \arrow[r, "\sim", "\pi_0"'] & S(\mathfrak{h}) \end{tikzcd}$$

The images of composed horizontal maps lie in, respectively, $gr[F[\mathfrak{h}^*]^{(W,\cdot)}]$ & $S(\mathfrak{h})^W$. This establishes the required diagram.

2.2) Symmetrization.

To show π is an isomorphism it suffices to show $gr\,\pi$ is (cf. Prob 3 in HW 2). Thx to Proposition in Sec 2.1, it suffices to show that $gr[\mathcal{U}(\mathfrak{g})^G] \hookrightarrow S(\mathfrak{g})^G$ is surjective. For this we will produce its right inverse.

Consider a map $\mathfrak{g}^k \longrightarrow U(\mathfrak{g})$, $(x_1 \ldots x_k) \mapsto \frac{1}{k!}\sum_{\sigma \in S_k} x_{\sigma(1)} \cdots x_{\sigma(k)}$, multi-linear, G-equivariant and symmetric (the image stays the same if we permute the arguments) so gives a G-linear map sym: $S^k(\mathfrak{g}) \to U(\mathfrak{g})$, w. $x_1 \ldots x_k \mapsto \frac{1}{k!}\sum_{\sigma \in S_k} x_{\sigma(1)} \cdots x_{\sigma(k)}$. Extend it to $S(\mathfrak{g})$ by linearity. Since sym is G-linear, it restricts to $S(\mathfrak{g})^G \longrightarrow U(\mathfrak{g})^G$. And it gives a right inverse to $gr[U(\mathfrak{g})^G] \hookrightarrow S(\mathfrak{g})^G$ thx to the following:

Exercise: $\mathrm{sym}(x_1 \ldots x_k) \in x_1 \ldots x_k + U(\mathfrak{g})_{\leq k-1}$ $\forall x_i \in \mathfrak{g}$

This finishes the proof of HC isomorphism.

And now we can explain how Casimir $C \in U(\mathfrak{sl}_2)$ arises

Exercise: Show that under $\mathbb{F}[\mathfrak{g}]^G \xrightarrow{\sim} S(\mathfrak{g})^G$, $F(x) = tr(x^2)$ goes to $\frac{1}{2}h^2 + 2fe$, which under sym: $S(\mathfrak{g})^G \xrightarrow{\sim} U(\mathfrak{g})^G$ goes to $\frac{1}{2}h^2 + h + 2fe = C$.