Lec 18: representations of $\mathfrak{sl}_n$ & SL_n

1) Weyl character formula

1.0) Intro

Still, $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{F})$ (w. char $\mathbb{F}=0$, $\mathbb{F}=\overline{\mathbb{F}}$) & $\mathfrak{h}=\{diag(x_1,...x_n) \mid \sum x_i=0\}$

Recall that we have the weight lattice $\Lambda=\{\lambda\in\mathfrak{h}^* \mid \langle\lambda,h_i\rangle\in\mathbb{Z}\}$ containing the dominant weights, Λ_+, that classify the finite dimensional irreducible modules: $L(\lambda)$ w. $\lambda\in\Lambda_+$. A reasonable next question is to compute $\dim L(\lambda)$ or, even better, the dimensions of all weight spaces $L(\lambda)_\mu$ ($\mu\in\Lambda$). This is what characters are about

Notation: consider the group ring $\mathbb{Z}[\Lambda]$ of the weight lattice Λ. We write e^λ for the element of $\mathbb{Z}[\Lambda]$ corresponding to $\lambda\in\Lambda$. The Weyl group $W=S_n$ acts on $\mathfrak{h}^*$ preserving $\Lambda(=\{\sum\lambda_i\varepsilon_i \mid \lambda_i\in\mathbb{Z}\})$ and hence on $\mathbb{Z}[\Lambda]$: $we^\lambda=e^{w(\lambda)}$. This action is by ring automorphisms.

We also consider the completed version $\mathbb{Z}[[\Lambda]]$ consisting of all infinite linear combinations of e^λ, $\lambda\in\Lambda$.

1.1) Definition & key results

Definition: Let M be a representation of $\mathfrak{g}$ w. $M=\bigoplus_{\lambda\in\Lambda}M_\lambda$ w. $\dim M_\lambda<\infty$. Below we will call such a representation a weight module.

The (formal) character of M, $\operatorname{ch} M:=\sum_{\lambda\in\Lambda}(\dim M_\lambda)e^\lambda\in\mathbb{Z}[[\Lambda]]$.

Example: The Verma module $\Delta(\lambda)$ has weight basis $f^{\vec{k}}v_\lambda$ w. weights

$\lambda - \sum_{j=1}^{N} k_j\beta_j$ $(N = n(n-1)/2)$, $\beta_1, \dots \beta_N$ are the positive roots. So
$$\operatorname{ch}\Delta(\lambda) = \sum_{k_1,\dots k_N \geq 0} e^{\lambda - \Sigma k_j\beta_j} = e^{\lambda}\prod_{j=1}^{N}(1+e^{-\beta_j}+e^{-2\beta_j}+\dots) = e^{\lambda}\prod_{j=1}^{N}(1-e^{-\beta_j})^{-1}$$

Our goal is to compute $\operatorname{ch} L(\lambda)$ for $\lambda\in\Lambda^+$. Recall, Sec. 1.2 of Lec 16, $\rho = \sum_{i=1}^{n}(\frac{n+1}{2}-i)\varepsilon_i \Rightarrow \langle\rho, h_i\rangle = 1, \forall i$, so $\rho\in\Lambda$.

Thm (Weyl character formula): Let $\lambda\in\Lambda^+$. Then
$$\operatorname{ch} L(\lambda) = \frac{\sum_{w\in W}\operatorname{sgn}(w)e^{w(\lambda+\rho)}}{e^{\rho}\prod_{j=1}^{N}(1-e^{-\beta_j})} \left(= \sum_{w\in W}\operatorname{sgn}(w)\operatorname{ch}\Delta(w\cdot\lambda)\right)$$

Examples: 1) $L(0)$ is the trivial representation, and we recover $\operatorname{ch}\operatorname{triv} = e^0 (=1)$.

2) Let $n=2$. Then $\operatorname{ch} L(n) = [\dim L(n)_i = 1$ for $i = n, n-2, \dots, -n$ and 0, else$] = e^n + e^{n-2} + \dots + e^{-n} = \frac{e^{n+1}-e^{-(n+1)}}{e-e^{-1}}$. Since ρ is identified with $\langle\rho, h\rangle = 1$, & $\alpha\leftrightarrow 2$, this agrees with the theorem.

Exercise: For $\mathfrak{g}$-representations M, M' as in the definition above, and a finite dimensional $\mathfrak{g}$-representation V, we have
$$\operatorname{ch}(M\oplus M') = \operatorname{ch}(M)+\operatorname{ch}(M'), \quad \operatorname{ch}(V\otimes M) = \operatorname{ch}(V)\operatorname{ch}(M).$$

Rem: As other results in this course, Thm holds for all simple Lie algebras. For $\mathfrak{sl}_n$ specifically, the r.h.s. can be interpreted as the Jacobi-Trudy formula for Schur polynomials.

1.2) Characters of $L(\lambda)$'s vs characters of $\Delta(\mu)$'s

Recall that $W \curvearrowright \Lambda$ via $w\cdot\lambda = w(\lambda+\rho)-\rho$ & Λ is equipped w. partial order: $\lambda \leq \mu$ if $\lambda-\mu \in \mathrm{Span}_{\mathbb{Z}_{\geq 0}}$(positive roots). The goal of this section is to prove

Proposition: $\lambda\in\Lambda \Rightarrow \mathrm{ch}\, L(\lambda) = \mathrm{ch}\,\Delta(\lambda) + \sum\limits_{\mu\in W\cdot\lambda,\, \mu<\lambda} a_\mu\, \mathrm{ch}\,\Delta(\mu)$ $(a_\mu \in \mathbb{Z})$

Proof:

Step 1: Let M be a weight $\mathfrak{g}$-module & $N\subset M$ be a submodule. Then $N, M/N$ are weight modules &

$$\dim M_\mu = \dim N_\mu + \dim (M/N)_\mu \quad \forall \mu\in\Lambda \qquad (1)$$

(exercise; cf. the proof of Proposition 2 in Sec 1.3 of Lec 6).

Step 2: consider a sequence $0=N^0 \subsetneq N^1 \subsetneq \ldots \subsetneq N^\ell = \Delta(\mu)$ of subreps. We claim that

$$\ell \leq \sum_{\nu\in W\cdot\mu,\, \nu\leq\mu} \dim \Delta(\mu)_\nu \qquad (2)$$

Set $L^i := N^i/N^{i-1}\ (\neq 0)$, $i=1,\ldots,\ell$. By (1), $\forall \nu\in\Lambda \Rightarrow$

$$\dim\Delta(\mu)_\nu = \sum_{i=1}^{\ell} L^i_\nu. \qquad (3)$$

So (2) will follow if we show that $\forall i\ \exists w_i \in W \mid L^i_{w_i\cdot\mu} \neq 0$, automatically $w_i\cdot\mu \leq \mu$. Since $L^i_\nu \neq 0 \Rightarrow \Delta(\mu)_\nu \neq 0 \Rightarrow \nu\leq\mu$, L^i has a highest weight, say $\nu_i \rightsquigarrow$ nonzero $\Delta(\nu_i) \xrightarrow{\varphi_i} L^i$. Note that the center Z of $U(\mathfrak{g})$ acts on $\Delta(\nu_i), \Delta(\mu)$ via $\chi_{\nu_i}, \chi_\mu : Z \to \mathbb{F}$, respectively (w. $\chi_\mu(z) = HC_z(\mu)$). Since $\varphi_i \neq 0 \Rightarrow \chi_{\nu_i} = \chi_\mu$. By Corollary in Sec 1.2 of Lec 6, $\exists w_i \mid \nu_i = w_i\cdot\mu$. This proves (2).

Step 3: Take $0=N^0 \subsetneq N^1 \subsetneq \ldots \subsetneq N^\ell = \Delta(\mu)$ w. maximal possible ℓ (a.k.a. Jordan-Hölder filtration), it exists by Step 2. Automatically, each L^i is irreducible. And thx to nonzero $\varphi_i: \Delta(w_i\cdot\mu)\to L^i \Rightarrow$ $L^i \simeq L(w_i\cdot\mu)$.

Step 4: $(3) \Rightarrow \operatorname{ch}\Delta(\mu) = \sum_{i=1}^{\ell} \operatorname{ch} L(w_i\cdot\mu)$ w. $w_i\cdot\mu \leqslant \mu$. Since $\dim \Delta(\mu)_\mu = 1$, we have $w_i\cdot\mu=\mu$ for exactly one i & $w_j\cdot\mu<\mu\ \forall j\neq i$. So we get

$$\operatorname{ch}\Delta(\mu) = \operatorname{ch} L(\mu) + \sum_{\nu\in W\cdot\mu,\, \nu<\mu} b_\nu \operatorname{ch} L(\nu) \quad (b_\nu\in\mathbb{Z}_{\geqslant 0}) \qquad (4)$$

Step 5: Let $\mu_1,\ldots\mu_k$ be an ordering of all elements of $W\cdot\mu$ s.t. $\mu_i\leqslant\mu_j \Rightarrow i\geqslant j$ (at least one such exists). Applying (4) for all μ_i, we see that $(\operatorname{ch}\Delta(\mu_i))_{i=1}^{k}$ is obtained from $(\operatorname{ch} L(\mu_i))_{i=1}^{k}$ by applying an upper triangular matrix w. 1's on diagonal. Inverting this matrix, we arrive at an expression of $(\operatorname{ch} L(\mu_i))_{i=1}^{k}$ via $(\operatorname{ch}\Delta(\mu_i))_{i=1}^{k}$ yielding the claim of the proposition. □

1.3) Proof of Weyl character formula.

Recall (Proposition in Sec 1.2 of Lec 15): $\forall$ fin. dim. $\mathfrak{g}$-rep. $V \Rightarrow$

$$\dim V_\mu = \dim V_{w\mu} \quad \forall\, w\in W, \mu\in\Lambda$$

We apply this to $V=L(\lambda)$ w. $\lambda\in\Lambda_+$ to get

$$\operatorname{ch} L(\lambda) \in \mathbb{Z}[\Lambda]^W \qquad (5)$$

(for the usual action of W on Λ). Now the Weyl character formula follows from (5), Proposition in Sec 1.2 & the next claim

Lemma: Let $\lambda\in\Lambda^+$ & $\theta\in\mathbb{Z}[\Lambda]^W$ be s.t.

$$\theta = \operatorname{ch}\Delta(\lambda) + \sum_{\mu\in W\cdot\lambda\setminus\{\lambda\}} a_\mu \operatorname{ch}\Delta(\mu).$$

Then

$$\theta = \frac{\sum_{w\in W} sgn(w) e^{w(\lambda+\rho)}}{e^{\rho}\prod_{j=1}^{N}(1-e^{-\beta_j})}$$

Proof:

Recall (Example in Sec 1.1) that

$$ch\, \Delta(\mu) = e^{\mu}\prod_{j=1}^{N}(1-e^{-\beta_j})^{-1}$$

where $\beta_1, \dots \beta_N$ are all positive roots

Step 1: we claim that $\eta := e^{\rho}\prod_{j=1}^{N}(1-e^{-\beta_j})$ satisfies $w\eta = sgn(w)\eta$ $\forall w\in W$. This will follow if we show $s_i\eta = -\eta$ $\forall i$.

Notice that $s_i = (i, i+1)$ sends a root to a root, that $s_i\alpha_i = -\alpha_i$ ($\alpha_i = \varepsilon_i - \varepsilon_{i+1}$) & if $\beta\neq\alpha_i \Rightarrow s_i\beta$ is a positive root. So

$$s_i\prod_{j=1}^{N}(1-e^{\beta_j}) = s_i\left((1-e^{-\alpha_i})\prod_{\beta\neq\alpha_i}(1-e^{-\beta})\right) = (1-e^{\alpha_i})\prod_{\beta\neq\alpha_i}(1-e^{-\beta})$$

Recall (Sec 1.2 of Lec 16) that $s_i\rho = \rho - \alpha_i$. So

$$s_i\eta = e^{\rho-\alpha_i}(1-e^{\alpha_i})\prod_{\beta\neq\alpha_i}(1-e^{-\beta}) = e^{\rho}(e^{-\alpha_i}-1)\prod_{\beta\neq\alpha_i}(1-e^{-\beta}) = -\eta$$

Step 2: Note that $\mathbb{Z}[\Lambda]$ (Laurent polynomials) is a domain. It's enough to show $\eta\theta = \sum_{w\in W} sgn(w)e^{w(\lambda+\rho)}$. Note that θ is W-invainvariant & η is W-sgn-invariant. So $\eta\theta$ is W-sgn-invariant. On the other hand, $\eta\theta = \eta(ch\,\Delta(\lambda) + \sum_{w\neq 1} a_w\, ch\,\Delta(w\cdot\lambda)) =$ [Example in Sec 1.1] $= e^{\rho}(e^{\lambda} + \sum_{w\neq 1} a_w e^{w\cdot\lambda}) = e^{\lambda+\rho} + \sum_{w\neq 1} a_w e^{w(\lambda+\rho)}$. And since $\eta\theta$ is W-sgn-invariant, this forces $a_w = sgn(w)$ proving the lemma. □

1.4) ch $L(\lambda)$ for λ non-dominant.

One can ask about computing the characters of $L(\lambda)$ w. general $\lambda\in\Lambda$, equivalently about computing the coefficients

a_w in $\mathrm{ch}\, L(\lambda) = \mathrm{ch}\, \Delta(\lambda) + \sum_{\mu \in W\cdot\lambda,\, \mu<\lambda} a_w\, \mathrm{ch}\, \Delta(\mu)$. Here are remarkable features of this question.

1) The answer for a_w is known but it's not expressed using elementary combinatorics, which is the case for dominant λ. Instead one needs "Kazhdan-Lusztig combinatorics." Moreover, in most cases in Representation theory, where characters don't have elementary description but can be computed, the answer is expressed in a similar way.

2) This looks like a completely algebraic question. Yet both the conjectural answer (Kazhdan-Lusztig '79) & "20th century proofs" (closely related proofs due to Beilinson-Bernstein & Brylinski-Kashiwara 81; and a later quite different proof by Soergel 90) require geometry (of the flag variety). Kazhdan-Lusztig, Beilinson-Bernstein & Brylinski-Kashiwara works are among the pioneering papers in Geometric Representation theory (initiated by Deligne-Lusztig & Springer in 76). Only in 2012 Elias & Williamson found an elementary (but not easy) proof based on "Hodge theory w/o geometry."