

Lec 19: Representations of SL_n & SL'_n

1) Representations in char p

2) Flag variety

1) Representations in char p

In char 0, the fin. dim. SL'_n -irreps & rational SL_n -irreps are classified by the dominant weights (Lec 15) & the characters are given by the Weyl character formula (Lec 18). In this lecture we'll explain what happens when we work with SL_n -reps over an algebraically closed field of char p . Proofs and a more difficult case of Lie algebra representations will be addressed in bonus lecture.

Set $G = SL_n$, let B (resp. T) be the subgroup of all lower triangular (resp., diagonal) matrices in G . Here are results obtained in Lecs 7-9 for $n=2$ & their generalizations

1) Weight decompositions: any rational SL_2 -rep V decomposes as $V = \bigoplus_{n \in \mathbb{Z}} V_n$, where $V_n = \{v \in V \mid \text{diag} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} v = z^n v \ \forall z \in \mathbb{F} \setminus \{0\}\}$. This generalizes as follows. Any element $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in \Lambda$ ($\lambda_i \in \mathbb{Z}$) gives rise to an algebraic group homomorphism

$$\lambda: T \rightarrow \mathbb{G}_m^n, \text{diag}(t_1, \dots, t_n) \mapsto \prod_{i=1}^n t_i^{\lambda_i}$$

Then $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$, where $V_\lambda = \{v \in V \mid tv = \lambda(t)v \ \forall t \in T\}$

The proof basically repeats that for SL_2 (Sec 1.1 of Lec 9)

2) If V is a rational SL_n -representation, then $V_\mu \xrightarrow{\cong} V_{w\mu} \forall w \in W$ & $\mu \in \Lambda$. Indeed, let $M_w \in GL_n$ be the permutation matrix corresponding to $w \in S_n$. Then $M'_w = M_w \text{diag}(\text{sgn}(w), 1, \dots, 1) \in SL_n$ &

$$M'_w \text{diag}(t_1, \dots, t_n) M_w^{-1} = \text{diag}(t_{w(1)}, \dots, t_{w(n)})$$

An isomorphism $V_\mu \cong V_{w\mu}$ is given by $v \mapsto M'_w v$ (exercise).

In particular, every highest weight of a rational representation is dominant.

3) Dual Weyl module. For SL_2 we considered the representations $M(n) = \{\text{homogeneous deg } n \text{ polynomials in } x, y\}$ which we interpreted as $\text{Ind}_B^G \mathbb{F}_n = \{f \in \mathbb{F}[G] \mid f\left(\begin{pmatrix} z & 0 \\ u & z^{-1} \end{pmatrix} g\right) = z^n f(g) \forall g \in G, z \in \mathbb{F} \setminus \{0\}, u \in \mathbb{F}\}$

The latter description generalizes to SL_n : we can view λ as a homomorphism $\lambda: B \rightarrow \mathbb{C}_m^*$, $\begin{pmatrix} t_1 & & \\ & \ddots & \\ * & & t_n \end{pmatrix} \mapsto \prod t_i^{\lambda_i}$. Then we set

$M(\lambda) = \text{Ind}_B^G \mathbb{F}_\lambda$. Here are two important properties of $M(\lambda)$, $\lambda \in \Lambda_+$.

a) It contains a unique irreducible subrepresentation $L(\lambda)$ & every irreducible rational rep. with highest weight λ is isomorphic to $L(\lambda)$. The proof essentially mirrors the SL_2 case (Sec 0 of Lec 9).

b) For a rational representation V we can consider its formal character $\text{ch } V = \sum_{\lambda \in \Lambda} (\dim V_\lambda) e^\lambda$ (note that the actual character evaluated on T is $\text{tr}_V(\text{diag}(t_1, \dots, t_n)) = \sum_{\lambda} (\dim V_\lambda) t_1^{\lambda_1} \dots t_n^{\lambda_n}$). An important result is $\text{ch } M(\lambda)$ is still given by the Weyl character formula (cf. Sec 1.1 of Lec 18). For SL_2 , this follows directly from the explicit realization of $M(n)$, while the general case is more involved.

4) Steinberg tensor product theorem. For $n=2$, we take $n \in \mathbb{Z}_0$ & write its p -adic expansion: $n = n_0 + n_1 p + \dots + n_k p^k$ ($n_i \in \{0, 1, \dots, p-1\}$). Then $L(n) = L(n_0) \otimes L(n_1)^{(1)} \otimes \dots \otimes L(n_k)^{(k)}$, where $\cdot^{(i)}$ is the i -fold Frobenius twist. The proof is based on the fact that $L(n_0)$ continues to be irreducible over \mathfrak{g} .

This generalizes to SL_n . Set $\Lambda'_+ = \{\lambda \in \Lambda \mid \langle \lambda, h_i \rangle \in \{0, 1, \dots, p-1\} \forall i=1, \dots, p-1\}$. An arbitrary element $\lambda \in \Lambda_+$ is uniquely written as

$$\lambda = \lambda_0 + p\lambda_1 + \dots + p^k \lambda_k \quad (\lambda_i \in \Lambda'_+)$$

Then one has $L(\lambda) = L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \dots \otimes L(\lambda_k)^{(k)}$. The proof is still based on the observation that $L(\lambda_0)$ remains irreducible over \mathfrak{g} , which is considerably more subtle than for $n=2$.

5) For $n=2$, we can completely compute $\text{ch } L(\lambda)$ since we know that $L(\lambda_0) = M(\lambda_0) \forall \lambda_0 \in \Lambda'_+$ ($= \{0, 1, \dots, p-1\}$). This no longer the case already for $n=3$. The character formulas are known for $p \gg 0$ thanks to the work of a lot of people from 1990 to 2020. The answer is given via Kazhdan-Lusztig combinatorics but unlike what we've seen in Lec 18 the Weyl group in question is not S_n but the affine Weyl group $W \ltimes \Lambda_r$, where $\Lambda_r = \text{Span}_{\mathbb{Z}}(\text{roots}) \subset \Lambda$ is the root lattice.

The case when p is not enormous (compared to n) is open despite a lot of progress in the last 15 years or so.

2) Flag variety

2.1) Definition

Let F be a field & V be a vector space of $\dim = n$. By a **complete flag** in V one means a collection of subspaces

$$V^\circ = (\{0\} \subsetneq V^1 \subsetneq V^2 \subsetneq \dots \subsetneq V^n = V)$$

where $\dim V_i = i$ (automatically). The set of complete flags is denoted by \mathcal{FL}_n . We say that a basis v_1, \dots, v_n of V is **compatible** w. V° if $V^i = \text{Span}(v_1, \dots, v_i) \forall i$.

Here are some observations about the structure of \mathcal{FL}_n

1) $GL_n(F)$ acts transitively on \mathcal{FL}_n (for two flags V_1° & V_2° choose bases compatible w. them, then the transition matrix sends V_1° to V_2° . The stabilizer of V° in $GL_n(F)$ is the subgroup of all matrices in GL_n that are upper triangular w.r.t. any basis compatible w. V° .

Exercise: $SL_n(F)$ also acts transitively on \mathcal{FL}_n .

2) If $F = \mathbb{R}$ or \mathbb{C} , then \mathcal{FL}_n is a compact manifold (real or complex). And if F is an algebraically closed field, then \mathcal{FL}_n is naturally a projective variety (= zeroes of homogeneous polynomial equations in a projective space)

2.2) Schubert stratification

We write G for GL_n & B for the subgroup of all upper triangular matrices. For $w \in S_n$, let M_w denote the corresponding permuta-

tion matrix. The following is a consequence of the Gauss elimination algorithm. We write BwB for $BM_wB = \{g, M_w g_i \mid g_i \in B\}$.

Proposition 1: $G = \coprod_{w \in W} BwB$

Definition: The Schubert cell labelled by w in $Fl_n = G/B$ is BwB/B

Here's a description of BwB/B . We write U (resp. U^-) for the subgroup of upper (resp. lower) triangular matrices w. 1 on the diagonal. Set $U_w := U \cap M_w U^- M_w^{-1}$. By the construction, this subgroup consists of all matrices (a_{ij}) s.t. $a_{ii} = 1 \forall i$, $a_{ij} = 0$ if $i > j$ or $w^{-1}(i) < w^{-1}(j)$. The entries a_{ij} w. $i < j$ & $w^{-1}(i) > w^{-1}(j)$ are arbitrary so $U_w \simeq \mathbb{F}^{\ell(w)}$, where $\ell(w) = |\{(i, j) \mid i < j \text{ \& } w^{-1}(i) > w^{-1}(j)\}|$ is the number of inversions in the permutation w^{-1} . This is called the length of w .

Example: $n=3, w=(2,3,1), \ell(w)=2, U_w = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

Proposition 2: We have an isomorphism $U_w \xrightarrow{\sim} BwB/B$ given by $u \mapsto uM_wB/B$.

Proof (in the case $\mathbb{F} = \mathbb{F}_q$, the only one we are going to use)

Set $U^w := U \cap M_w U M_w^{-1}$. In particular, $U^w, U_w \subset U$ are subgroups & $U^w \cap U_w \subset M_w U M_w^{-1} \cap M_w U^- M_w^{-1} = [U \cap U^- = \{e\}] = \{e\}$. Note that $B =$

$T \times U$ & $TM_w B/B = [M_w \text{ normalizes } T] = M_w T B/B = M_w B/B$. So the action of U on BwB/B is transitive (b/c B acts transitively). The stabilizer of $M_w B/B \in BwB/B$ in U is $U \cap M_w B M_w^{-1} = [e\text{-val's}=1] = U \cap M_w U M_w^{-1} = U^w$. Since $U_w \cap U^w = \{e\}$, the map $U_w \rightarrow U/U_w = BwB/B$, $u \mapsto uM_w B/B$ is injective. On the other hand, the cardinalities are the same: for U_w it's $q^{\ell(w)}$, while for U/U_w it is $|U|/|U_w| = q^{N_1}/q^{N_2}$, where

$$N_1 = \frac{n(n-1)}{2} = |\{(i,j) \mid i < j\}| \quad \& \quad N_2 = |\{(i,j) \mid i < j \ \& \ w^{-1}(i) < w^{-1}(j)\}|$$

so $N_1 - N_2 = \ell(w)$. So any injective map $U_w \rightarrow U/U_w$ must be a bijection. □

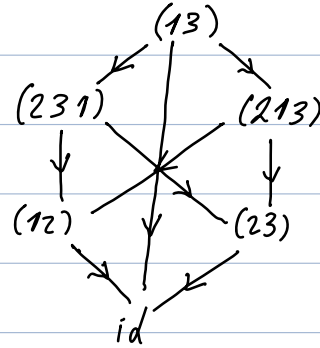
2.3) Geometry of $\mathcal{F}\ell_n$

Assume the base field is \mathbb{C} . In the last lecture we computed the character of the irreducible $\mathcal{S}\ell_n$ -representation $\mathcal{L}(\lambda)$, $\lambda \in \Lambda_+$.

Computing the characters of $\mathcal{L}(\lambda)$ with $\lambda \in \Lambda$ possibly not dominant reduces to certain questions about Algebraic geometry of $\mathcal{F}\ell_n$. We'll mention some of these questions here w/o explaining the reduction.

What we look at are the Schubert varieties $\overline{BwB/B}$, the closures of Schubert cells in Zariski topology. One can describe it as a subset as follows. We equip $W = S_n$ w. partial order (Bruhat order) by declaring that $u \preceq w$ if \exists transpositions $t_1 = (i_1, j_1), \dots, t_k = (i_k, j_k) \in S_n$ s.t. $u = t_k \dots t_1 w$ & $\ell(t_1 \dots t_k w) < \ell(t_{i_1} \dots t_{i_k} w)$ $\forall i=1, \dots, k$. Then $\overline{BwB/B} = \bigsqcup_{u \preceq w} BuB/B$.

Example (of order) Let $n=3$, then the relation $w \rightarrow tw$ for a transposition t looks like this



The computation of characters of $L(\lambda)$'s reduces to the study of singularities of the Schubert varieties (in some aspects). The cases when $\overline{BwB/B}$ is smooth (every point has a neighborhood in the usual topology that looks like a ball in $\mathbb{C}^{\ell(w)}$) corresponds to easy character formulas. One such case: $w = \begin{pmatrix} 1 & 2 & \dots & n \\ n & \dots & 1 \end{pmatrix}$, the longest element. Here $\overline{BwB/B} = G/B$, smooth thx to the transitive G -action. It corresponds to $L(\lambda)$ w. dominant λ , where the character is given by the Weyl character formula.