Lec 2: algebraic groups & Lie algebras II

0) Overview

1) Tangent spaces in Algebraic geometry.

2) Structures on the tangent space at 1 of an algebraic group.

0) Overview

As before, $\mathbb{F}$ denotes an algebraically closed field.

Algebraic (or Lie) groups are non-linear objects (defined by non-linear equations). A basic paradigm to study such objects is linearization. In our case we seek to replace an algebraic group G with the tangent space T_eG, where $e\in G$ is the unit. A discussion of such tangent spaces is the 1st topic of this lecture.

Generally, tangent spaces are just vector spaces/$\mathbb{F}$. However, T_eG comes with additional structure(s). The first (and essentially the only in the case when char $\mathbb{F}=0$) was observed already by Sophus Lie (1870's): the Lie bracket. When char $\mathbb{F}=p$, there's one more structure - the restricted pth power map discovered by Jacobson (1930's). In this lecture we will start our discussion of these structures.

1) Tangent spaces in Algebraic geometry.

1.1) Basics on tangent spaces

Let X be an affine algebraic variety/$\mathbb{F}$ & $A=\mathbb{F}[X]$ denote the algebra of polynomial functions on X. Pick $\alpha\in X$.

Definition: 1) An α-derivation of A is an $\mathbb{F}$-linear map $\xi: A \to \mathbb{F}$ satisfying the following version of Leibniz identity:

(L) $$\xi(fg) = f(\alpha)\xi(g) + g(\alpha)\xi(f).$$

2) Note that the α-derivations form a vector subspace in the space A^* of linear functions $A \to \mathbb{F}$. The space of α-derivations is denoted by $T_\alpha X$ and is called the tangent space of X at α.

Here is a calculation of $T_\alpha X$. Let $A = \mathbb{F}[x_1, \ldots x_n]/(f_1, \ldots, f_m)$ & write $\bar{x}_i$ for $x_i + (f_1, \ldots f_m) \in A$, these are generators of A. If $\xi \in T_\alpha X$ & $g \in \mathbb{F}[x_1, \ldots x_n]$, then the usual rule of differentiation applies:

(1) $$\xi(g[\bar{x}_1, \ldots \bar{x}_n]) = \sum_{i=1}^n [\partial_i g](\alpha)\xi(\bar{x}_i), \text{ where } \partial_i := \frac{\partial}{\partial x_i}.$$

So we get a map

$$\varphi: T_\alpha X \to \mathbb{F}^n,\quad \xi \mapsto \xi((\bar{x}_i))_{i=1}^n.$$

By (1), φ is injective, also it's linear. Now plug $g = f_j$ to (1). The l.h.s is $\delta(0) = 0$ hence the image of φ is contained in

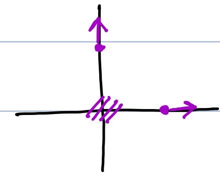
(2) $$\{(a_1, \ldots a_n) \mid \sum_{i=1}^n [\partial_i f_j](\alpha) a_i = 0\ \forall j\} \subset \mathbb{F}^n.$$

In fact, φ is an isomorphism $T_\alpha X \xrightarrow{\sim} (2)$: $\forall (a_1, \ldots a_n) \in (2)$, the map $g(\bar{x}_1, \ldots \bar{x}_n) \mapsto \sum_i [\partial_i g](\alpha) a_i$ is well-defined (doesn't depend on the choice of g) and is automatically an α-derivation (exercise).

An identification of $T_\alpha X$ w. (2) is an algebro-geometric version of a computation of the tangent space of an embedded submanifold in $\mathbb{R}^n$.

Example: Let $X = \{(x_1, x_2) \in \mathbb{F}^2 \mid x_1 x_2 = 0\}$. Then for $f = x_1 x_2$, we have $(\partial_1 f, \partial_2 f) = (x_2, x_1)$. So

$$T_\alpha X = \begin{cases} \{(*,0)\} & \text{if } \alpha_1 \neq 0 \ (\Rightarrow \alpha_2 = 0) \\ \{(0,*)\} & \text{if } \alpha_2 \neq 0 \ (\Rightarrow \alpha_1 = 0 \\ \mathbb{F}^2 & \text{if } \alpha_1 = \alpha_2 = 0 \end{cases}$$



Remarks: i) One can completely describe $T_\alpha X$ as a subspace of A^*. Let $\mathfrak{m}_\alpha = \{f \in A \mid f(\alpha) = 0\}$, the maximal ideal of α. Thx to (L), for $\xi \in T_\alpha X$ we have $\xi(1) = 0$ (use $1 = 1 \cdot 1$) & $\xi(fg) = 0$ $\forall f, g \in \mathfrak{m}_\alpha$ $\Rightarrow \delta(\mathfrak{m}_\alpha^2) = 0$. Conversely, let $\xi \in A^*$ be s.t $\xi(1) = 0$ & $\xi(\mathfrak{m}_\alpha^2) = 0$. We claim that ξ satisfies (L) ($\Leftrightarrow \xi \in T_\alpha X$). For this note that $\xi(1) = 0 \Rightarrow$ (L) holds if f or $g \in \mathbb{F}$ & $\xi(\mathfrak{m}_\alpha^2) = 0 \Rightarrow$ (L) for $f, g \in \mathfrak{m}_\alpha$. Now we use that $\forall f \in A \Rightarrow f = f(\alpha) + (f - f(\alpha))$ & $f - f(\alpha) \in \mathfrak{m}_\alpha$ & similarly for g. So

$$(3) \quad T_\alpha X = \{\xi \in A^* \mid \xi(1) = 0 \ \& \ \xi(\mathfrak{m}_\alpha^2) = 0\}$$

ii) (2) shows $T_\alpha \mathbb{F}^n = \mathbb{F}^n$ $\forall \alpha \in \mathbb{F}^n$. This identification, in fact, is independent of the choice of coordinates. Namely, let V be a vector space, $\alpha \in V$ & $v \in V$. We assign $\xi_v \in T_\alpha V$ (the derivative in the direction of v) by: $\xi_v(f) = \frac{1}{t}\underbrace{(f(\alpha + tv) - f(\alpha))}_{\in t\mathbb{F}[V][t]}\Big|_{t=0}$. This gives an isomorphism $V \to T_\alpha V$, $v \mapsto \xi_v$ (exercise).

1.2) Tangent maps

Let $\varphi: X \to Y$ be a morphism of affine varieties $\rightsquigarrow$ the pullback homomorphism $\varphi^*: \mathbb{F}[Y] \to \mathbb{F}[X]$, hence the dual map $\varphi_*: \mathbb{F}[X]^* \longrightarrow \mathbb{F}[Y]^*$, $\langle \varphi_*(\delta), g \rangle = \langle \delta, \varphi^*(g) \rangle$ $(g \in \mathbb{F}[Y], \delta \in \mathbb{F}[X]^*)$

Lemma: $\varphi_*: \mathbb{F}[X]^* \to \mathbb{F}[Y]^*$ restricts to $T_\alpha X \to T_{\varphi(\alpha)} Y$.

Proof: Let $\xi \in T_\alpha X$.

Thx to (3), it suffices to prove $[\varphi_*(\xi)](1) = 0$ (this is exercise) & $[\varphi_*(\xi)](\mathfrak{m}^2_{\varphi(\alpha)}) = 0$. But $[\varphi_*(\xi)](\mathfrak{m}^2_{\varphi(\alpha)}) = \xi(\varphi^*(\mathfrak{m}^2_{\varphi(\alpha)}))$. We claim that $\varphi^*(\mathfrak{m}^k_{\varphi(\alpha)}) \subset \mathfrak{m}^k_\alpha$ $\forall$ $k > 0$. For $k=1$: $g \in \mathfrak{m}_{\varphi(\alpha)} \Leftrightarrow g(\varphi(\alpha)) = 0 \Leftrightarrow$ $[\varphi^* g](\alpha) = 0 \Leftrightarrow \varphi^* g \in \mathfrak{m}_\alpha$ & the containment follows. The case $k > 1$ reduces to $k=1$ b/c φ^* is an algebra homomorphism. $\square$

Definition: This map $T_\alpha X \to T_\beta Y$ is called the tangent map of φ at α and is denoted $T_\alpha \varphi$.

Important exercise 2 (functoriality a.k.a. chain rule). If $\psi: Y \to Z$ be another mophism, then $T_\alpha(\psi \circ \varphi) = T_{\varphi(\alpha)} \psi \circ T_\alpha \varphi$.

Now we discuss properties of tangent maps in special cases.

I) Let $X \subset Y$ be Zariski closed & $\iota: X \hookrightarrow Y$ denote the inclusion. Then $\forall$ $\alpha \in X \Rightarrow T_\alpha \iota: T_\alpha X \to T_\alpha Y$ is injective. This is because $\iota^*: \mathbb{F}[Y] \to \mathbb{F}[X]$ is surjective (Remark in Sec 1 of Lec 1), hence $? \circ \iota^*: \mathbb{F}[X]^* \to \mathbb{F}[Y]^*$ is injective $\Rightarrow$ its restriction $T_\alpha \iota$ is so.

II) Let $f \in \mathbb{F}[X]$ be s.t. $f(\alpha) \neq 0$. Let $j: X_f \hookrightarrow X$ be the inclusion. We claim that $T_\alpha j: T_\alpha(X_f) \xrightarrow{\sim} T_\alpha X$. Indeed, $\forall$ $\xi \in T_\alpha X$ has unique preimage given by the "quotient rule":

$$f^{-n}g \mapsto f(\alpha)^{-2n}\left(\xi(g)f(\alpha)^n - g(\alpha)\xi(f^n)\right).$$

The proof is an exercise.

Finally, consider the following situation: let $\varphi = (f_1, \ldots, f_m): \mathbb{F}^n \to \mathbb{F}^m$ be a morphism. Let $\alpha \in \mathbb{F}^n$ & $\beta = \varphi(\alpha)$. We want to compare $\ker T_\alpha \varphi = \{(a_1, \ldots a_n) \mid \sum_i [\partial_i f_j](\alpha) \cdot a_i = 0\}$ & $T_\alpha \varphi^{-1}(\beta)$. One can show (exercise) that $T_\alpha \varphi^{-1}(\beta) \subset \ker T_\alpha \varphi$ but the inclusion can be strict (e.g. take $\varphi(x) = x^2$ & $\alpha = 0$ – a note for experts: we view $\varphi^{-1}(\alpha)$ as a subvariety not a subscheme). We need a sufficient condition for $T_\alpha \varphi^{-1}(\beta) = \ker T_\alpha \varphi$ as the r.h.s. is easier to compute. The following may be viewed as an Algebro-geometric version of the implicit function (regular value) theorem.

Fact (see Sec 5.5 in [H] for a related statement)
If $T_\alpha \varphi$ is surjective, then $T_\alpha \varphi^{-1}(\beta) = \ker T_\alpha \varphi$

1.3) Spaces $T_e G$ for classical groups

Recall the classical groups $G = GL(V), SL(V), O(V,B), Sp(V,\omega)$, where V is a finite dimensional vector space over $\mathbb{F}$ & B, ω are nondegenerate symmetric & skew-symmetric forms on V (that are only considered if $\operatorname{char} \mathbb{F} \neq 2$). See Section 2.1 in Lec 1. Our goal now is to compute the tangent spaces $T_e G$.

Example 0: To compute $T_e G$ for $G = GL(V)$ we set $X = \operatorname{End}(V)$, the vector space of endomorphisms of V & $f = \det \in \mathbb{F}[X]$. By

Remark in Sec 1.1, $T_e X$ is naturally identified with $\mathrm{End}(V)$ & by II) in Sec 1.2, $T_e G \xrightarrow{\sim} T_e X = \mathrm{End}(V)$. In a basis: $T_e GL_n(\mathbb{F}) = \mathrm{Mat}_n(\mathbb{F})$. We will write $\mathfrak{gl}(V)$ (or $\mathfrak{gl}_n(\mathbb{F})$) for $T_e G$.

For other three groups G, we have $G \subset GL(V)$, Zariski closed, so $T_e G \subset T_e GL(V) = \mathfrak{gl}(V)$ is a subspace, I) in Sec 1.2. To determine these subspaces, we use Fact in Sec 1.2.

Example 1 $SL(V) \subset \mathrm{End}(V)$ is $\varphi^{-1}(1)$ for $\varphi = \det: \mathrm{End}(V) \to \mathbb{F}$. To compute $T_e\varphi$ it's convenient to use a basis in V to identify $\mathrm{End}(V)$ w. $\mathrm{Mat}_n(\mathbb{F})$. The map $T_e\varphi$ sends $(a_{ij}) \in T_e \mathrm{Mat}_n(\mathbb{F}) = \mathrm{Mat}_n(\mathbb{F})$ to the linear (in a_{ij}) term in $\det((\delta_{ij} + a_{ij}))$, where δ_{ij} = Kronecker symbol. Using the initial formula for det we see that this term is $\sum_{i=1}^n a_{ii} = \mathrm{tr}((a_{ij}))$. The map $\mathrm{tr}: \mathrm{Mat}_n(\mathbb{F}) \to \mathbb{F}$ is surjective, so $T_e G = T_e \varphi^{-1}(1) = [\text{Fact}] = \ker \mathrm{tr}$. This subspace is denoted by $\mathfrak{sl}_n(\mathbb{F})$ (or $\mathfrak{sl}(V)$ in the basis-free setting).

Example 2: $G = O_n(\mathbb{F})$. Note that AA^T is symmetric $\forall A \in \mathrm{Mat}_n(\mathbb{F})$. So we can consider $\varphi: A \to AA^T: \mathrm{Mat}_n(\mathbb{F}) \longrightarrow \{\text{symmetric matrices}\}$ $\Rightarrow G = \varphi^{-1}(I)$. The map $T_e\varphi$ sends $\xi \in \mathrm{Mat}_n(\mathbb{F}) = (T_e \mathrm{Mat}_n(\mathbb{F}))$ to the linear in ξ term of $(I+\xi)(I+\xi^T)$, which is $\xi + \xi^T$. Since $\mathrm{char}\,\mathbb{F} \neq 2$, the map $\xi \mapsto \xi + \xi^T$ is surjective (exercise – restrict to upper triangular matrices). So $T_e G = \ker T_e\varphi = \{\xi \in \mathrm{Mat}_n(\mathbb{F}) \mid \xi^T = -\xi\}$ to be denoted by $\mathfrak{so}_n(\mathbb{F})$.

We also could (and should) view this basis-free. We get $T_e O(V,B)=\{\xi\in\mathfrak{gl}(V) \mid B(\xi u,v)+B(u,\xi v)=0\ \forall u,v\in V\}=:\mathfrak{so}(V,B)$ – the space of operators skew-symmetric w.r.t. B.

Exercise: $T_e Sp(V,\omega)=\{\xi\in\mathfrak{gl}(V) \mid \omega(\xi u,v)+\omega(u,\xi v)=0\ \forall u,v\in V\}$ $=:\mathfrak{sp}(V,\omega)$.

2) Structures on the tangent space at 1 of an algebraic group

For $\xi,\eta\in\mathfrak{gl}_n(\mathbb{F})$ we write $[\xi,\eta]:=\xi\eta-\eta\xi$. For an algebraic subgroup $G\subset GL_n(\mathbb{F})$ we write $\mathfrak{g}$ for T_eG, as mentioned in Sec 1.3 this is a subspace in $\mathfrak{gl}_n(\mathbb{F})$.

The following theorem is the main result for this lecture.

Thm: 1) $\mathfrak{g}$ is closed under $[\cdot,\cdot]$.

1') If char $\mathbb{F}=p>0$, then $\mathfrak{g}$ is also closed under $\xi\mapsto\xi^p$.

Let $H\subset GL_m$ be another algebraic group & $\Phi: G\to H$ be an algebraic group homomorphism & $\varphi=T_e\Phi$. Then

2) $[\varphi(\xi),\varphi(\eta)]=\varphi([\xi,\eta])\ \forall \xi,\eta\in\mathfrak{g}$.

2') If char $\mathbb{F}=p>0$, then $\varphi(\xi^p)=\varphi(\xi)^p\ \forall \xi\in\mathfrak{g}$.

Example (of 1), 1')). Let $G=SL_n(\mathbb{F})\subset GL_n(\mathbb{F})$ so that $\mathfrak{g}=\mathfrak{sl}_n(\mathbb{F})$ $=\{\xi\in Mat_n(\mathbb{F}) \mid tr\,\xi=0\}$. It's closed under $[\cdot,\cdot]$ b/c $tr(\xi\eta)=$ $=tr(\eta\xi)\ \forall \xi,\eta\in Mat_n(\mathbb{F})$ (exercise – on matrix multiplication). To show that $\mathfrak{g}$ is closed under $\bullet^p$ when char $\mathbb{F}=p$, it's sufficient to prove $tr(\xi^p) = [tr(\xi)]^p$ $\forall \xi \in Mat_n(\mathbb{F})$. For this we use the fact that for a matrix A w. e-values $\lambda_1, \dots \lambda_n$ (w. multiplicities) we have $tr(A) = \sum_i \lambda_i$. Now let $\lambda_1, \dots \lambda_n$ be e-values of ξ so that $\lambda_1^p, \dots \lambda_n^p$ are e-values of ξ^p. So

$$[tr(\xi)]^p = (\sum_i \lambda_i)^p = [\text{char } \mathbb{F} = p \Rightarrow \lambda \mapsto \lambda^p \text{ is field homomorphism}] = \sum_i \lambda_i^p = tr(\xi^p).$$

Exercise: Prove (1) & (2) for $\mathfrak{so}$ & $\mathfrak{sp}$.

Note that (2) & (2') show, in particular, that the operations $[\cdot, \cdot]$ and $\cdot^p$ are well-defined on $\mathfrak{g}$, i.e. do not depend on the embedding $G \hookrightarrow GL_n(\mathbb{F})$. Moreover, we have the following result that we will (partially) prove later in the course.

Fact: Every affine algebraic group is isomorphic to an algebraic subgroup of GL_n

Thanks to this Fact, we don't need G (& H) to be embedded into $GL_?$.