Lec 20: Hecke algebras, I

1) Introduction.
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1) Introduction

We now switch to a different kind of representation theory: of the finite group $GL_n(\mathbb{F}_q)$ over $\mathbb{C}$. One reason to care about $GL_n(\mathbb{F}_q)$ is that it's close to being simple (roughly, in the same way as S_n is close to the simple alternating group A_n). In more detail, $PSL_n(\mathbb{F}_q):=SL_n(\mathbb{F}_q)/\text{center}$ is simple if $(n,q)\neq(2,2),(2,3)$. It should be noted though that that understanding irreps of $PSL_n(\mathbb{F}_q)$ (or $SL_n(\mathbb{F}_q)$) from those of $GL_n(\mathbb{F}_q)$ is more complicated compared to S_n vs A_n.

We are not going to study all $GL_n(\mathbb{F}_q)$-irreps. Set $G=GL_n(\mathbb{F}_q)$ and let B be the subgroup of all upper triangular matrices.

Definition/lemma: for a G-irrep V (over $\mathbb{C}$), TFAE

(a) $V^B\neq 0$

(b) V appears as a direct summand in $\mathbb{C}[G/B]\,(=\{f\in\mathbb{C}[G]\mid f(g'b)=f(g'),\ \forall g'\in G, b\in B\}$ w. G-action via $[g.f](g')=f(g^{-1}g'))$

We say that V is a unipotent principal series representation.

Proof: Observe that $\mathbb{C}[G/B]\xrightarrow{\sim}\mathrm{Ind}_B^G\mathbb{C}$. The latter was defined as $\{f\in\mathbb{C}[G]\mid f(bg')=f(g')\ \forall g'\in G, b\in B\}$ w. action $g.f(g')=f(g'g)$ & the isomorphism is induced by $g \mapsto g^{-1}: G \to G$. Use Frobenius reciprocity:

(1) $$\mathrm{Hom}_G(\mathrm{Ind}_B^G \mathrm{triv}, V) \xleftarrow{\sim} V^B$$

(see Sec 3.5 in basic rep. theory write-up) For an irrep V,
$\mathrm{Hom}_G(\mathrm{Ind}_B^G \mathrm{triv}, V) \neq 0 \Leftrightarrow V$ occurs as a summand of $\mathrm{Ind}_B^G \mathrm{triv}$. $\square$

2) $\mathrm{End}_G(\mathbb{C}[G/H])$ & convolution

Here G is a finite group & $H \subset G$ is a subgroup. The base field is $\mathbb{C}$.

2.1) Motivation

Our goal in this section is to prove

Proposition: Let V be a finite dimensional G-representation. There is a natural bijection between:

a) The irreducible G-representations occurring in V.

b) The irreducible modules over $\mathrm{End}_G(V)$

Proof: Reference: Sec 2.3 in basic Rep. theory write-up.

Let $U_1, \dots U_k$ be the G-irreps occurring in V so that
$$V = \bigoplus_{i=1}^k U_i \otimes M_i, \quad M_i = \mathrm{Hom}_G(U_i, V)$$
A consequence of the Schur lemma is that $\mathrm{End}_G(V) = \bigoplus_{i=1}^k \mathrm{End}(M_i)$ & the irreducibles are $M_1, \dots M_k$. The bijection we need is $V_i \mapsto M_i$ $\square$

Our short term (this lecture) goals are

1) Describe a basis in $\mathrm{End}_G(\mathbb{C}[G/H])$
2) Explain how the basis elements multiply.
3) Make this explicit for $G = GL_n(\mathbb{F}_q)$ & $H = B$.

Using 3) we'll show that $\mathrm{End}_G(\mathbb{C}[G/B]) \xrightarrow{\sim} \mathbb{C}S_n$ (Lec 21). Thx to Proposition this will describe the principal series unipotent reps.

2.2) Convolution

Our goal here is to give an alternative description of $\mathrm{End}_G(\mathbb{C}[G/H])$. We have a vector space identification

$\mathrm{End}_G(\mathbb{C}[G/H]) = \mathrm{Hom}_G(\mathbb{C}[G/H], \mathbb{C}[G/H]) = [(1)$ w. $V = \mathbb{C}[G/H]] = \mathbb{C}[G/H]^H$
$(= \mathbb{C}[G]^{H\times H} = \mathbb{C}[\underbrace{H\backslash G/H}_{\text{double cosets}}])$

This already gives a basis in $\mathrm{End}_G(\mathbb{C}[G/H])$: for $\mathbb{O} \in H\backslash G/H$, we write $\delta_{\mathbb{O}}$ for the characteristic function of $\mathbb{O}$ (taking value 1 on $\mathbb{O}$ & 0 elsewhere). These elements form a basis in $\mathbb{C}[H\backslash G/H]$. Our next question is how to interpret an element $f \in \mathbb{C}[H\backslash G/H]$ as an endomorphism. We write $\mathbb{C}[G]^{H_\ell}, \mathbb{C}[G]^{H_r}$ for subalgebras of functions on G invariant for the actions of H on the left & the right, so that $\mathbb{C}[G]^{H_r} = \mathbb{C}[G/H]$, $\mathbb{C}[G]^{H_\ell} = \mathbb{C}[H\backslash G]$.

Definition: For $F \in \mathbb{C}[G]^{H_r}, f \in \mathbb{C}[G]^{H_\ell}$ define their convolution $F*f \in \mathbb{C}[G]$ by
$$[F*f](g') = \frac{1}{|H|}\sum_{(g_1,g_2)\mid g_1g_2=g'} F(g_1)f(g_2)$$

Consider the representation of G on $\mathbb{C}[G]$ coming from left translation $f \mapsto g.f$ w. $[g.f](g') = f(g^{-1}g')$. Note that

(2) $$(g.F)*f = g.(F*f):$$

$$[(g.F)*f](g') = \frac{1}{|H|}\sum_{g_1g_2=g'} \underbrace{[g.F](g_1)}_{F(g^{-1}g_1)} f(g_2) = [g_1' = g^{-1}g_1] = \frac{1}{|H|}\sum_{g_1'g_2 = g^{-1}g'} F(g_1')F(g_2)$$
$= [g.(F*f)](g')$.

Similarly, we can consider the representation $f \mapsto f.g$ of G in $\mathbb{C}[G]$ from right translations. Then

(3) $$[F*f].g = F*(f.g)$$

This computation shows

a) $F \in \mathbb{C}[G/H]$, $f \in \mathbb{C}[H\backslash G/H] \Rightarrow F*f \in \mathbb{C}[G/H]$ (apply (3) to $g \in H$)

b) $?*f: \mathbb{C}[G/H] \to \mathbb{C}[G/H]$ is G-linear (apply (2))

c) $f_1, f_2 \in \mathbb{C}[H\backslash G/H] \Rightarrow f_1 * f_2 \in \mathbb{C}[H\backslash G/H]$

Also note that for $F \in \mathbb{C}[G/H]$, $f_1, f_2 \in \mathbb{C}[H\backslash G/H]$

(4) $$[(F*f_1)*f_2](g') = \frac{1}{|H|^2} \sum_{g_0 g_1 g_2 = g'} F(g_0) f_1(g_1) f_2(g_2) = [F*(f_1 * f_2)](g').$$

(5) $F * \delta_{1H} = F$ (exercise), $\delta_{1H} * f = f$

These show

d) $(\mathbb{C}[H\backslash G/H], *)$ is an associative algebra w. unit δ_{1H}.

e) $\mathbb{C}[G/H]$ is a right module over this algebra

Important exercise: For $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}$, $H \times H$-orbits in G:

(6) $$\delta_{\mathcal{O}_1} * \delta_{\mathcal{O}_2}(\mathcal{O}) = \frac{1}{|H|} \left|\{(g_1, g_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \mid g_1 g_2 = g\}\right| \quad \forall g \in \mathcal{O}.$$

Lemma: The map $f \mapsto ?*f: \mathbb{C}[H\backslash G/H]^{op} \to \mathrm{End}_G(\mathbb{C}[G/H])$ is an isomorphism of algebras ("op" means opposite order of product)

Proof: (4) & (5) show we have an algebra homomorphism, while $\delta_{1H} * f = f$ shows it's injective. Finally, we've seen that the two algebras have the same dimensions. $\square$

3) The case of $B \subset G = GL_n(\mathbb{F}_q)$

Recall (Sec 2 of Lec 19) that for $w \in W$:

- We write BwB for $BM_wB \subset G$, where M_w is the permutation matrix
- $\ell(w) := |\{(i,j) \mid i<j \ \&\ w^{-1}(i) > w^{-1}(j)\}|$

We write $\mathcal{H}(q)$ for $(\mathbb{C}[B \backslash G/B], *)$ & T_w for $\delta_{BwB} \in \mathcal{H}(q)$. In particular, T_1 is the unit in associative algebra $\mathcal{H}(q)$.

Recall also that in Sec 2.2 of Lec 19 we proved:

(7) $|BwB/B| = q^{\ell(w)} \Leftrightarrow |BwB| = q^{\ell(w)}|B|$

Proposition: 1) if $\ell(uw) = \ell(u) + \ell(w)$, then $T_u T_w = T_{uw}$.

2) For $s = (i, i+1)$ $(i = 1, \dots, n-1)$, we have $T_s^2 = (q-1)T_s + qT_1$.

Proof: Consider the map $BuB \times BwB \xrightarrow{\pi} G$, $(x,y) \mapsto xy$. The group B acts on $BuB \times BwB$ by $b.(x,y) = (xb^{-1}, by)$. This action is free & each fiber of π is a union of orbits. By (6) in Sec 2.2

$(*)$ $T_u T_w = \sum_{v \in W} m_{uw}^v T_v$, where $m_{uw}^v = \#$ of B-orbits in $\pi^{-1}(z)$, $z \in BvB$.

1): Note that $BuwB \subset \operatorname{im}\pi$. By Fact 2, $|BuwB| = q^{\ell(uw)}|B|$, $|BuB \times BwB| = |B|^2 q^{\ell(u)} q^{\ell(w)} = |B|^2 q^{\ell(uw)} = |B| \cdot |BuwB|$. Since each fiber of π is a union of free B-orbits, we get $BuwB = \operatorname{im}\pi$ and each fiber is a single B-orbit. Our claim follows from $(*)$.

2): Consider the subgroup $P_s = \begin{array}{cc} \begin{matrix} \quad\ \ i \\ \quad\ \downarrow \end{matrix} & \\ \begin{pmatrix} * & & & \\ & * & & * \\ & * & * & \\ 0 & & & \ddots \ * \end{pmatrix} & \begin{matrix} \\ \\ \leftarrow i+1 \\ \\ \end{matrix} \end{array}$ so that $P_s = BsB \sqcup B$, Indeed, P is $B\times B$-stable so $P_s = \bigsqcup BwB$, where the union is taken over $w\in W$ s.t. $M_w \in P_s \Leftrightarrow w = 1$ or s.

By $(*)$, $T_s^2 = m_{ss}^s T_s + m_{ss}^1 T_1$. First of all, $m_{ss}^1 = \frac{1}{|B|}|\pi^{-1}(1)| = [\pi^{-1}(1) = (g, g^{-1}),\ g \Leftrightarrow g^{-1}\in BsB] = \frac{1}{|B|}|BsB| = q$. Next,

$$|BsB\times BsB| = |\pi^{-1}(BsB)| + |\pi^{-1}(B)| = m_{ss}^s |BsB||B| + m_{ss}^1 |B||B| \Longrightarrow$$

$$q^2 = m_{ss}^s q + q \Rightarrow m_{ss}^s = q-1.$$ divide by $|B|^2$ $\square$

Exercise: i) for $s = (i, i+1)$, we have $\ell(ws) = \begin{cases} \ell(w)+1, & w^{-1}(i) < w^{-1}(i+1) \\ \ell(w)-1, & \text{else} \end{cases}$

ii) We write s_i for $(i, i+1)$. Deduce from i) that $\ell(w)$ = minimal ℓ s.t. w can be presented as $s_{i_1}\dots s_{i_\ell}$ for some $i_1,\dots,i_\ell \in \{1,2,\dots n-1\}$

iii) Use ii) & 1) of Proposition to show $T_w = T_{i_1}\dots T_{i_\ell}$ (w. $w = s_{i_1}\dots s_{i_\ell}$ & $\ell = \ell(w)$), where $T_i := T_{s_i}$. In particular, $T_1,\dots,T_{n-1}$ generate $H(q)$.