Lec 21: Hecke algebra, II

1) Generic Hecke algebra
2) Semisimple specializations

1.0) Recap: Let's recall some results from Lecture 20.

Let $G=GL_n(\mathbb{F}_q)$, $B\subset G$ be the subgroup of upper triangular matrices. We are interested in understanding the algebra $\operatorname{End}_G(\mathbb{C}[G/B])$. It's semisimple ($\simeq\bigoplus$ matrix algebras) b/c $\mathbb{C}[G/B]$ is a completely reducible $\mathbb{C}G$-module.

In Sec 2 of Lec 20, we have identified $\operatorname{End}_G(\mathbb{C}[G/B])$ w. the algebra $H(q):=(\mathbb{C}[B\backslash G/B],*)$. Using this, in Sec 3, we have produced a vector space basis $T_w\in \operatorname{End}_G(\mathbb{C}[G/B])$, $w\in W(:=S_n)$, where $T_1=1$.

We have also described the products of some basis elements. For $i\in\{1,2,...,n-1\}$, set $s_i=(i,i+1)$. We've seen that for $w\in W$, the length $\ell(w)$ coincides with $\min\{\ell \mid w=s_{i_1}...s_{i_\ell}\}$, e.g. $\ell(1)=0$, $\ell(w)=1\Leftrightarrow w=s_i$. This formula implies the classical claim that $s_1,...,s_{n-1}$ generate W.

The following was established in Sec 3 of Lec 20.

Proposition: 1) if $\ell(uw)=\ell(u)+\ell(w)$, then $T_uT_w=T_{uw}$.

2) For $s=s_i$ $(i=1,...,n-1)$, we have $T_s^2=(q-1)T_s+qT_1$.

1.1) Consequences.

Corollary:

1) $$T_sT_w=\begin{cases}T_{sw}, & \text{if } \ell(sw)=\ell(w)+1\\ qT_{sw}+(q-1)T_w, & \text{else}\end{cases}\qquad T_wT_s=\begin{cases}T_{ws}, & \text{if } \ell(ws)=\ell(w)+1\\ qT_{ws}+(q-1)T_w, & \text{else}\end{cases}$$

In particular, if $w = s_{i_1} \dots s_{i_\ell}$ w. $\ell = \ell(w)$, then $T_w = T_{s_{i_1}} \dots T_{s_{i_\ell}}$.

2) $\exists\, m_{uw}^v \in \mathbb{Z}[t]$ $(u, v, w \in W)$ s.t. $T_u T_w = \sum_{v \in W} m_{uw}^v(q) T_v$.

Proof:

1): The case $\ell(sw) = \ell(w)+1$ follows from 1) of Proposition. Note that $\ell(sw) \leq \ell(w)+1$ by the characterization of ℓ above. Since $s^2 = 1$, $\ell(w) \leq \ell(sw)+1$. Also $\operatorname{sgn}(w) = (-1)^{\ell(w)}$ so $\ell(w) \neq \ell(sw)$. Hence $\ell(sw) \neq \ell(w)+1$ means $\ell(w) = \ell(sw)+1 \Rightarrow T_w = T_s T_{sw}$. So $T_s T_w = T_s^2 T_{sw} =$ $[$2) of Prop'n: $T_s^2 = q + (q-1)T_s] = q T_{sw} + (q-1) T_s T_{sw} =$ [1) of Prop'n] $q T_{sw} + (q-1) T_w$.

The formula for $T_w T_s$ is proved similarly.

2): We write u as $s_{i_1} \dots s_{i_\ell}$ w. $\ell = \ell(u)$ so that $T_u T_w = [(*)] = T_{i_1} \dots T_{i_\ell} T_w$. We use 3) repeatedly: express $T_{i_\ell} T_w$, then multiply the summands by $T_{i_{\ell-1}}$, etc. In each step, the coefficients of T_v's are polynomials in q with integral coefficients. □

Rem: Let $\tilde{H}(q)$ denote the algebra generated by T_w $(w \in W)$ & relations in Proposition. Then $\tilde{H}(q) \xrightarrow{\sim} H(q)$. Indeed, since $H(q)$ is generated by T_w's, we have a natural surjection $\tilde{H}(q) \twoheadrightarrow H(q)$. And computation of 1) in the proof of Corollary shows that $\tilde{H}(q) = \operatorname{Span}_{\mathbb{C}}(T_w)$. Since $H(q)$ has basis T_w we conclude that $\tilde{H}(q) \xrightarrow{\sim} H(q)$.

1.2) The generic Hecke algebra and its specializations

Definition: The **generic Hecke algebra** (a.k.a. Iwahori-Hecke algebra) is the free $\mathbb{Z}[t]$-module $H^{\mathbb{Z}}(W)$ w. basis $T_w, w\in W$, and product

$$T_uT_w=\sum_{v\in W}\underbrace{m_{uw}^{v}(t)}_{\text{from 2) of Corollary.}}T_v.$$

Lemma: This is an associative algebra w. unit T_1.

Proof: Associativity can be checked on basis elements, where it's a collection of quadratic equations on the entries of the multiplication table – $m_{uw}^{v}\in\mathbb{Z}[t]$. These equations hold after specializing t to <u>any</u> prime power q, by 2) of Corollary. So they hold for the m_{uw}^{v}, hence $H^{\mathbb{Z}}(W)$ is associative. The claim that T_1 is a unit is an exercise. $\square$

We write $H(W)$ for $\mathbb{C}\otimes_{\mathbb{Z}}H^{\mathbb{Z}}(W)$. For $\kappa\in\mathbb{C}$, we write $H_\kappa(W)$ for $H(W)/(t-\kappa)H(W)$. This is a $\mathbb{C}$-algebra w. basis $T_w, w\in W$ & product $T_uT_w=\sum_{v\in W}m_{uw}^{v}(\kappa)T_v$.

Example: 1) For $\kappa=q$, a prime power, $H_q(W)=H(q)$, a semisimple algebra.

2) Let $\kappa=1$. By 3) of Corollary, $T_sT_w=T_{sw}\Rightarrow T_uT_w=T_{uw}$, $\forall u,w\in W$.
$\Rightarrow H_1(W)=\mathbb{C}W$, a semisimple algebra.

2) Semisimple specializations

It turns out that 1) & 2) in the example above already imply

$\mathcal{H}_q(W) \simeq \mathcal{H}_1(W) = \mathbb{C}W$ as a special case of the following theorem.

Theorem (Tits deformation principle): Let $\mathbb{F}$ be an algebraically closed field and A be an associative unital $\mathbb{F}[t]$-algebra that is a free finite rank $\mathbb{F}[t]$-module. Let $\alpha, \beta \in \mathbb{F}$ be such that A_α, A_β are semisimple. Then $A_\alpha \simeq A_\beta$.

This accomplishes the last goal stated in the previous lecture and finishes our treatment of the representations theory of $GL_n(\mathbb{F}_q)$.

Remark: $\mathcal{H}_k(W)$ is semisimple $\Leftrightarrow k \neq \sqrt[m]{1}$ w. $m \leq n$. In this case there's an explicit construction of $\mathcal{H}_k(W) \xrightarrow{\sim} \mathbb{C}S_n$. It's possible to construct an isomorphism with the third algebra, a "cyclotomic KLR (Khovanov-Lauda-Rouquier) algebra" that arises in the study of representations of Lie algebras in categories. See Kleshchev, arXiv: 0909.4844.

2.1) Structure of completion

Our proof of Thm will consist of two steps. The first, which we will carry in the lecture describes the base change of A to a "formal neighborhood" of α in $\mathbb{F}$.

Consider the inclusion $\mathbb{F}[t] \hookrightarrow \mathbb{F}[[t-\alpha]]$ via the expansion of polynomials in $t-\alpha$. It turns $\mathbb{F}[[t-\alpha]]$ into an $\mathbb{F}[t]$-algebra.

Proposition: $\mathbb{F}[[t-\alpha]] \otimes_{\mathbb{F}[t]} A \simeq \mathbb{F}[[t-\alpha]] \otimes_{\mathbb{F}} A_\alpha$, an $\mathbb{F}[[t-\alpha]]$-algebra iso.

Before we prove this, we'll explain how to think about the l.h.s.

For an open disc $\{z \mid |z-\alpha|<\varepsilon\} \subset \mathbb{C}$, the reasonable function algebra on it to consider is that of all holomorphic functions, denote it by $\mathcal{O}_\varepsilon$. For $\mathbb{F}=\mathbb{C}$, we can view $\mathcal{O}_\varepsilon \otimes_{\mathbb{F}[t]} A$ as the restriction of the bundle of algebras A from $\mathbb{C}$ to the disc. And then for β w. $|\beta-\alpha|<\varepsilon$, $A_\beta = \mathbb{C} \otimes_{\mathcal{O}_\varepsilon} (\mathcal{O}_\varepsilon \otimes_{\mathbb{F}[t]} A)$, where the homomorphism $\mathcal{O}_\varepsilon \to \mathbb{C}$ is the evaluation at β. So a version of Proposition w. $\mathcal{O}_\varepsilon$ instead of $\mathbb{F}[[t-\alpha]]$ would yield $A_\beta \simeq A_\alpha$ if $|\beta-\alpha|<\varepsilon$

Note that $\mathcal{O}_\varepsilon$ consists of all powers series in $z-\alpha$ that converge when $|z-\alpha|<\varepsilon$. In this way, we may informally view $\mathbb{F}[[t-\alpha]]$ as the algebra of functions on the disc around α so small that all power series converge, in other words the algebra of functions on the "formal disc."

If we knew an analog of Proposition on the actual small discs ($\mathbb{F}=\mathbb{C}$) we would argue as follows. One can show that $\{\beta \in \mathbb{F} \mid A_\beta$ is not semisimple$\}$ is finite (premium exercise: prove this using that a finite dimensional algebra $B/\mathbb{F}$ is semisimple $\Leftrightarrow$ the form $(\cdot,\cdot)$ on $B \otimes B^{op}$ given by $(x,y)=tr_B(xy)$ is non-degenerate). So over $\mathbb{C}$, we would connect α & β by a path avoiding that finite set. We would use a "nonformal" version to show that every γ on the path has a neighborhood s.t. $A_\gamma \simeq A_{\gamma'}$, $\forall \gamma'$ in that neighborhood. Finishing the proof is then an exercise.

We'll formally deduce Thm from Proposition in a bonus section.

2.2) Proof of Proposition

We start with "lifting of idempotents."

Lemma 1: Let $\mathbb{F}$ be a field, A an $\mathbb{F}[[t]]$-algebra that is a free finite rank $\mathbb{F}[[t]]$-module. Set $A_0 = A/tA$. Suppose $e_0 \in A_0$ is an idempotent, i.e $e_0^2 = e_0$. Then $\exists\, e \in A$ s.t. $e + tA = e_0$ & $e^2 = e$.

Proof: We lift "order by order": suppose $e_{k-1} \in A/t^kA$ satisfies $e_{k-1}^2 = e_{k-1}$. We claim $\exists\, e_k \in A/t^{k+1}A$ mapping to e_{k-1} & $e_k^2 = e_k$. Note that $A_0 \xrightarrow[\sim]{t^k\cdot} t^kA/t^{k+1}A$ b/c A is free over $\mathbb{F}[[t]]$. Fix some lift $\bar{e}_{k-1}$ of e_{k-1} in $A/t^{k+1}A$ so that $\bar{e}_{k-1} - \bar{e}_{k-1}^2 = t^k a$ for $a \in A_0$. We look for e_k in the form $\bar{e}_{k-1} + t^k b$. Then $(\bar{e}_{k-1} + t^k b)^2 = \bar{e}_{k-1}^2 + t^k(e_0 b + b e_0)$ should be equal to $\bar{e}_{k-1} + t^k b \Leftrightarrow a + e_0 b + b e_0 = b$. Note that $\bar{e}_{k-1} - \bar{e}_{k-1}^2 = t^k a \Rightarrow$ [exercise] $t^k e_0 a = t^k a e_0 \Leftrightarrow e_0 a = a e_0$. We take $b = (1-e_0) a (1-e_0) - e_0 a e_0$. It satisfies $a + e_0 b + b e_0 = b$.

There is a unique element $e \in A$ s.t. $e + t^{k+1}A = e_k, \forall k$. It satisfies the required conditions. $\square$

Proposition 2: Suppose that in the previous proposition, A_0 is the direct sum of matrix algebras. Then we have an algebra isomorphism $A \to A_0 \otimes \mathbb{F}[[t]]$.

Proof: Let $A_0 = \bigoplus_{i=1}^{k} \mathrm{End}_{\mathbb{F}}(V^i)$. Pick a primitive (i.e. rk 1) idempotent $e_0^i \in \mathrm{End}(V^i)$ and lift it to $e^i \in A$. We get A-modules Ae^i

and hence an algebra homomomorphism $A \to \tilde{A} := \bigoplus_{i=1}^{k} \mathrm{End}_{\mathbb{F}[[t]]}(Ae^i)$ Note that, for a fin. gen'd $\mathbb{F}[[t]]$-module being free is equivalent to being torsion-free. Hence $Ae^i (\subset A)$ is free over $\mathbb{F}[[t]]$. Moreover, $Ae^i/tAe^i \simeq V^i$. So, it's enough to show that $A \to \tilde{A}$ is an isomorphism. Modulo t, this homomorphism gives $A_0 \to \bigoplus_{i=1}^{k} \mathrm{End}_{\mathbb{F}}(V^i)$, an isomorphism. In particular, $A \twoheadrightarrow \tilde{A}$, by the Nakayama lemma. Next $\tilde{A}$ is a free $\mathbb{F}[[t]]$-module. So, as the epimorphism of $\mathbb{F}[[t]]$-modules, $A \twoheadrightarrow \tilde{A}$ splits: $A \simeq_{\mathbb{F}[[t]]} \tilde{A} \oplus K$. Recalling that $A/tA \simeq \tilde{A}/t\tilde{A}$, we see that $K/tK = 0$ thus getting $K=0$. □

2.3) Proof of Theorem in Sec 1.2.

The rest of the proof is some algebro-geometric manipulation. Let V be a finite dimensional vector space over an algebraically closed field $\mathbb{F}$. The set of all associative bilinear products $V \times V \to V$ is a closed subvariety in $\mathrm{Hom}_{\mathbb{F}}(V \otimes V, V)$. Denote it by X. The group $GL(V)$ acts on X and the orbits are isomorphism classes of algebras.

We now produce a polynomial map $\mu: \mathbb{F} \to X$. Choose a basis in the free $\mathbb{F}[t]$-module A, say $v_1, \ldots v_n$. The map μ is the multiplication table of A in this basis, i.e. $\mu(\gamma)$ is the multiplication table of $A := A/(t-\gamma)A$ for $\gamma \in \mathbb{F}$.

Let Y° denote the orbit corresponding to the isomorphism class of A_α. Let Y denote its closure in the Zariski topology. A basic fact is that Y° is Zariski open in Y.

We know $\mu(\alpha)\in Y^\circ$ and it's enough to show $\operatorname{im}\mu\subset Y$. Then $\mu^{-1}(Y^\circ)$ is Zariski open in $\mathbb{F}$ and we use that $\mathbb{F}$ is an irreducible variety to conclude that $\mu(\alpha),\mu(\beta)\in Y_\circ \Rightarrow A_\alpha\sim A_\beta$, an isomorphism of associative algebras.

Pick $f\in\mathbb{F}[\mathrm{Hom}_{\mathbb{F}}(V\otimes V,V)]$ w. $f|_Y=0$. We need to check $\mu^*(f)=0$. For this, we need to show that the image of $\mu^*(f)$ in $\mathbb{F}[[t-\alpha]]$ is zero (b/c $\mathbb{F}[t]\hookrightarrow\mathbb{F}[[t-\alpha]]$). This image is f evaluated at the multiplication table of $\mathbb{F}[[t-\alpha]]\otimes_{\mathbb{F}[t]}A$ in the basis $1\otimes v_i$. Since we have an algebra isomorphism $\mathbb{F}[[t-\alpha]]\otimes_{\mathbb{F}[t]}A\simeq\mathbb{F}[[t-\alpha]]\otimes_{\mathbb{F}}A$, we see that this multiplication table is obtained from that of A_α by applying an element of $GL_n(\mathbb{F}[[t-\alpha]])$. In other words, $\exists\ g(t)\in GL_n(\mathbb{F}[[t-\alpha]])$ s.t. the multiplication table of $\mathbb{F}[[t-\alpha]]\otimes_{\mathbb{F}[t]}A$ is $g(t)\mu(\alpha)$. Our claim is that $f(g(t)\mu(\alpha))=0$.

On the other hand we know that $f(g\mu(\alpha))=0\ \forall g\in GL_n(\mathbb{F})$. We can view $g\mapsto f(g\mu(\alpha))$ as a polynomial in the matrix coefficients of g (and the inverse of the determinant). It vanishes. But $f(g(t)\mu(\alpha))$ is the same polynomial (but now in the matrix coefficients & $\det^{-1}$ for $g(t)$). It has to vanish. It follows that $\mu^*(f)=0$ and completes the proof. $\square$

Rem: Here is the intuition behind the proof. We want to show $\mu(\mathbb{F})\subset Y$. The isomorphism $\mathbb{F}[[t-\alpha]]\otimes_{\mathbb{F}[t]}A\xrightarrow{\sim}\mathbb{F}[[t-\alpha]]\otimes_{\mathbb{F}}A_\alpha$ can be interpreted as saying that the image of the "formal neighborhood" of α under μ in X lies in Y°. This implies that $\mu(\mathbb{F})\subset Y$.