Hecke algebra, part III

1) Kazhdan-Lusztig basic

1) For an indeterminate t we have defined (Lecture 21) the generic Hecke algebra $\mathcal{H}^{\mathbb{Z}}(W)$ (for $W=S_n$) over $\mathbb{Z}[t]$. In this lecture, we'll need a slight modification. Consider the homomorphism $\mathbb{Z}[t]\to\mathbb{Z}[v^{\pm 1}], t\mapsto v^{-2}$, and set $\mathcal{H}_v(W):=\mathbb{Z}[v^{\pm1}]\otimes_{\mathbb{Z}[t]}\mathcal{H}^{\mathbb{Z}}(W)$. For $w\in W$, define an element $H_w:=v^{-\ell(w)}\otimes T_w\in\mathcal{H}_v(W)$. These elements form a basis of $\mathcal{H}_v(W)$ called the standard basis. Note that the product on $\mathcal{H}_v(W)$ is uniquely recovered from

(1) $H_uH_w=H_{uw}$ if $\ell(uw)=\ell(u)+\ell(w)$ $(\Rightarrow H_w=H_{s_{i_1}}\dots H_{s_{i_\ell}}$ if $w=s_{i_1}\dots s_{i_\ell}$, $\ell:=\ell(w))$.

(2) $H_s^2=(v^{-1}-v)H_s+1 \Leftrightarrow (H_s+v)(H_s-v^{-1})=0 \Leftrightarrow T_s^2 = (t-1)T_s+t$

(1) & (2) imply (cf. Corollary in Sec 1.1 of Lec 21)

$$(3)\qquad H_sH_w=\begin{cases} H_{sw} & \text{if } \ell(sw)=\ell(w)+1\\ (v^{-1}-v)H_w+H_{sw}, & \text{else}\end{cases}$$

Our goal in this lecture is to produce a different basis of $\mathcal{H}_v(W)$, the Kazhdan-Lusztig basis.

1.1) Bar involution. Our first ingredient is a certain ring automorphism $\bar{\cdot}$ of $\mathcal{H}_v(W)$. Note that each H_s is invertible in $\mathcal{H}_v(W)$ ($(2)\Rightarrow H_s^{-1}=H_s+v-v^{-1}$) and hence each H_w is invertible thx to (1).

Proposition/definition: The map $x\mapsto\bar{x}$ given on $\mathbb{Z}$-basis v^kH_w by

$\overline{v^k H_w} = v^{-k} H_{w^{-1}}^{-1}$ is a ring automorphism called the bar involution.

Proof:

Similarly to Remark in Sec 1.1 of Lec 21, $\mathcal{H}_v(W)$ is generated/$\mathbb{Z}[v^{\pm 1}]$ by H_w's w. relations (1)&(2). So WTS (1) & (2) are preserved by $\overline{\bullet}$:

(1): $\overline{H_u H_w} = \overline{H_{uw}} = H^{-1}_{(uw)^{-1}} = H^{-1}_{w^{-1}u^{-1}} = [\ell((uw)^{-1}) = \ell(uw) = \ell(u)+\ell(w) = \ell(u^{-1})+\ell(w^{-1})]$

$[(1)] = (H_{w^{-1}} H_{u^{-1}})^{-1} = H^{-1}_{u^{-1}} H^{-1}_{w^{-1}} = \overline{H}_u \overline{H}_w.$ ✓

(2): $(\overline{H}_s + \overline{v})(\overline{H}_s - \overline{v^{-1}}) = (H_s^{-1} + v^{-1})(H_s^{-1} - v) = H_s^{-2}(1 + v^{-1} H_s)(1 - H_s v)$

$= -H_s^{-2}(H_s + v)(H_s - v^{-1}) = 0$ ✓

□

Remark: 1) $\overline{\bullet}$ is indeed an involution – exercise.

1.2) Kazhdan-Lusztig basis

Theorem (essentially Kazhdan & Lusztig '1979) $\exists!$ $\mathbb{Z}[v^{\pm 1}]$-basis C_w $(w \in W)$ of $\mathcal{H}_v(W)$ (Kazhdan-Lusztig basis) s.t.

(i) $C_w = \overline{C_w}$ $\forall\, w \in W$.

(ii) $C_w \in H_w + v\,\mathrm{Span}_{\mathbb{Z}[v]}(H_u \mid u \in W)$.

The following establishes the uniqueness part.

Lemma: Let C_u $(u \in W)$ be a KL basis. Pick $w \in W$ and let C'_w be an element satisfying (i) & (ii). Then $C'_w = C_w$.

Proof: Note that (ii) $\Leftrightarrow H_u \in C_u + v\,\mathrm{Span}_{\mathbb{Z}[v]}(C_x \mid x \in W) \Rightarrow$

$C'_w = \sum\limits_{u \in W} F_{wu}(v) C_u$ w. $F_{wu} \in \delta_{uw} + v\mathbb{Z}[v]$. Then $\overline{C}'_w = \sum F_{wu}(\overline{v}) \overline{C}_w =$

$[(i)] = \sum F_{wu}(v^{-1}) C_w \Rightarrow F_{wu}(v) = F_{wu}(v^{-1}) \Rightarrow F_{wu} = \delta_{wu}.$ □

Example: 1 & H_s+v satisfy (i) & (ii):

$$\overline{H_s+v} = [s=s^{-1}] = H_s^{-1}+v^{-1} = [H_s^{-1} = H_s+v-v^{-1}] = H_s+v.$$

So, we must have $C_1=1$, $C_s=H_s+v$.

1.3) Existence

We will prove the existence of a basis with stronger properties. We recall the notion that appeared in Sec 2.3 of Lec 19.

Definition: Define the **Bruhat order** $\preceq$ on W by $u\preceq w$ if $\exists$ transpositions $t_1,\dots t_k\in W$ s.t

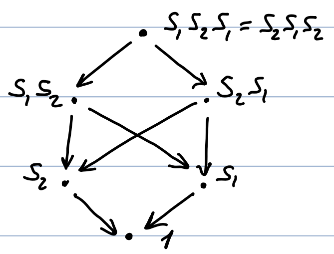
$$\ell(t_i\dots t_k w)<\ell(t_{i+1}\dots t_k w)\ \forall\, i=1\dots k$$

and $u=t_1\dots t_k w$. Note that this is indeed a partial order.

Exercise : 1) For $t=(i,j)$ w. $i<j$, $\ell(tw)<\ell(w) \Leftrightarrow w^{-1}(i)>w^{-1}(j)$.

2) 1 is the unique min. element, and $w_0 := \begin{pmatrix}1&2&\dots&n\\ n&\dots&&1\end{pmatrix}$ is the unique max. element.

Example: The Bruhat order on S_3 is described by the following directed graph, **the Bruhat graph**, ($u\preceq w$ if $\exists$ path $w\longrightarrow u$)



Proof of the existence part: We'll construct C_w satisfying (i) & (ii'): $C_w = H_w + \sum_{u \prec w} v p_{wu}(v) H_u$ w. $p_{wu}(v) \in \mathbb{Z}[v]$.

The construction is recursive: for $w \in W$ suppose we've constructed C_u satisfying (i) & (ii') $\forall\, u \prec w$. Set $\Lambda(\prec w) = \mathrm{Span}_{\mathbb{Z}[v]}(H_u \mid u \prec w)$ and define $\Lambda(\preceq w)$ analogously. (ii') becomes $C_w \in H_w + v\Lambda(\prec w)$.

Let $w = s_{i_1} \dots s_{i_\ell}$ w $\ell = \ell(w)$. Then for $s = s_{i_1}$ have $sw \prec w$. Consider $C_s C_{sw}$. Since $\bar{\cdot}$ is an algebra homomorphism, we get that $C_s C_{sw}$ satisfies (i). Let's see if it satisfies (ii').

$$C_s C_{sw} = (H_s + v)\Big(H_{sw} + \sum_{u \prec sw} v\, p_{sw,u}(v) H_u\Big) = [H_s H_{sw} = H_w] =$$
$$= H_w + \underbrace{vH_{sw} + v^2 \sum_{u \prec sw} p_{sw,u}(v) H_u}_{\in v\Lambda(\prec w)} + \underbrace{\sum_{u \prec sw} p_{sw,u}(v)\, v\, H_s H_u}_{\Sigma_1 + \Sigma_2}$$

We split the last sum into 2 parts: w. $\ell(su) > \ell(u)$ to be denoted by Σ_1 & w. $\ell(su) < \ell(u)$: Σ_2. The reason is (3) before Sec 1.1.

- $\ell(su) > \ell(u) \Rightarrow H_s H_u = H_{su}$. Note that $u \prec sw \prec w \Rightarrow su \prec w$. Namely, let transpositions $t_1 \dots t_k$ be s.t. $u = t_1 \dots t_k sw$ & $\ell(t_i \dots t_k sw) < \ell(t_{i+1} \dots sw)$. Consider 2 cases. First, let $s \neq t_i\ \forall\, i$. Then $su = st_1 s^{-1} st_2 s^{-1} \dots st_k s^{-1} w$ & $\ell(st_i s^{-1} \dots st_k s^{-1} w) \leq \ell(t_i s^{-1} \dots s^{-1} w) + 1 < \ell(st_{i+1} s^{-1} \dots st_k s^{-1} w) + 1 \Rightarrow$ $\ell(st_i s^{-1} \dots w) \leq \ell(st_{i+1} s^{-1} \dots w)$ & we must have $<$ b/c of different parities. This implies $su \prec w$.

Second, let $\exists\ i$ s.t. $t_i = s$. Pick i to be minimal possible. Set $w' = t_{i+1} \dots t_k w$ & $sw' \prec w'$. By Case 1, $su \prec w'$ & since $w' \prec w \Rightarrow su \prec w$.

So $\Sigma_1 = \sum_{u\prec sw} v\, p_{sw,u}(v) H_{su} \in [su \prec w]\ v\Lambda(\prec w)$

• $\ell(su)<\ell(u) \Rightarrow vH_sH_u = (1-v^2)H_u + vH_{su}$. So Σ_2 becomes

$$\sum_{su\prec u\prec sw} p_{sw,u}(v)[(1-v^2)H_u+vH_{su}] \in \sum_{su\prec u\prec sw} p_{sw,u}(0)H_u + v\Lambda(\prec sw) =$$

$$[C_u-H_u\in v\Lambda(\prec u);\ \Lambda(\prec u), \Lambda(\prec sw)\subset\Lambda(\prec w)] \subset \sum_{su\prec u\prec sw} p_{sw,u}(0)C_u + v\Lambda(\prec w).$$

Conclusion $C_sC_{sw} \in H_w + \sum_{su\prec u\prec sw} p_{sw,u}(0)C_u + v\Lambda(\prec w)$

So $C_w := C_sC_{sw} - \sum p_{sw,u}(0)C_u$ satisfies (i) and (ii') & we are done. □

Example: Let's compute the elements C_w for $W=S_3$. We already know (Example in Sec 1.2) $C_1 = 1$, $C_{s_1} = H_{s_1}+v$, $C_{s_2} = H_{s_2}+v$.

$$C_{s_1}C_{s_2} = (H_{s_1}+v)(H_{s_2}+v) = H_{s_1s_2} + v(H_{s_1}+H_{s_2}) + v^2 = C_{s_1s_2},$$

$$H_{s_2s_1} + v(H_{s_2}+H_{s_1}) + v^2 = C_{s_2s_1}$$

$$C_{s_2}C_{s_1s_2} = (H_{s_2}+v)(H_{s_1s_2}+v(H_{s_1}+H_{s_2})+v^2) = H_{s_2s_1s_2} + vH_{s_2s_1} + \overbrace{vH_{s_2}^2}^{\text{needs treatment}} + \cancel{v^2H_{s_2}}$$

$$+ vH_{s_1s_2} + v^2(H_{s_1}+H_{s_2}) + v^3 = [vH_{s_2}^2 = (1\cancel{-v^2)H_{s_2}} + v] =$$

$$= \underbrace{(H_{s_2s_1s_2} + v(H_{s_1s_2}+H_{s_2s_1}) + v^2(H_{s_1}+H_{s_2}) + v^3)}_{\overset{\parallel}{C_{s_2s_1s_2}}} + \underbrace{(H_{s_2}+v)}_{\overset{\parallel}{C_{s_2}}}$$

1.4) Kazhdan-Lusztig conjecture.

Definition: For $u,w\in W$, let the Kazhdan-Lusztig polynomial $c_{wu}(v)$ be defined by $C_w = \sum_{u\in W} c_{wu}(v)H_u$ (so that $c_{ww}=1$, $c_{wu}\neq 0 \Rightarrow u\preceq w$ and for $u\prec w$ we have $c_{wu}(v) = vp_{wu}(v)$).

Let $\lambda\in\Lambda_+=\{\sum_{i=1}^{n}\lambda_i\varepsilon_i \mid \lambda_i-\lambda_{i+1}\in\mathbb{Z}_{\geq 0}\}$, $\rho=\sum_{i=1}^{n}\frac{n+1-2i}{2}\varepsilon_i$. Recall the element $w_0\in S_n$ ($w_0(i)=n+1-i$). We have $w_0\rho=-\rho$ so $w_0\cdot\lambda=w_0(\lambda+\rho)-\rho$ $=w_0\lambda-2\rho=:\lambda^-$.

Recall that to $\mu\in\Lambda$ we can assign the following representations of $\mathfrak{sl}_n$: the Verma module $\Delta(\mu)$ & its irreducible quotient $L(\mu)$. For $\mu=w\cdot\lambda$, the only irreducibles that can occur in $\ker[\Delta(w\cdot\lambda)\twoheadrightarrow L(w\cdot\lambda)]$ are $L(u\cdot\lambda)$ w. $u\cdot\lambda<w\cdot\lambda$, Sec 1.2 in Lec 18. If we know their multiplities, we can express the (unknown) ch $L(w\cdot\lambda)$ via (known) ch $\Delta(u\cdot\lambda)$ w. $u\cdot\lambda\leqslant w\cdot\lambda$.

Thm (Kazhdan-Lusztig conjecture (1979) proved by Beilinson-Bernstein & Brylinski-Kashiwara (1981), reproved a number of times afterwards).

- The multiplicity of $L(u\cdot\lambda)$ in $\Delta(w\cdot\lambda)$ is $c_{u,w}(1)$ $\Rightarrow$

$$ch(\Delta(w\cdot\lambda))=\sum_{u\succeq w}c_{u,w}(1)\,ch(L(u\cdot\lambda)).$$

- $ch\,L(w\cdot\lambda^-)=\sum_{u\preceq w}(-1)^{\ell(w)-\ell(u)}c_{w,u}(1)\,ch(\Delta(u\cdot\lambda^-))$.

Note that the upper triangularity in the theorem is different from what we had before, it's stronger, as $u\prec w\Rightarrow u\cdot\lambda>w\cdot\lambda$ (exercise).

1.5) Properties of KL polynomials.

Kazhdan-Lusztig bases/polynomials are remarkable objects that were extensively studied since they were discovered. Yet, much is still unknown. Here's a brief account of some developments.

- **Positivity**: Theorem in Sec 1.4, in particular, means that $c_{u,w}(1)\geq 0$ $\forall\, u,w\in W$. More is true: $c_{u,w}\in\mathbb{Z}_{\geq 0}[v]$. This is completely not obvious from the construction in Sec 1.3 – or any other combinatorial construction. The claim that $c_{u,w}\in\mathbb{Z}_{\geq 0}[v]$ was proved by Kazhdan and Lusztig in 1980: they checked that the coefficients of $c_{u,w}$ are the dimensions of stalks of $IC(\overline{BuB}/B,\mathbb{Q})$ on BwB. A connection to the IC's (Intersection complexes) is an important ingredient of the classical proofs of the theorem.

No enumerative meanings of the coefficients of $c_{u,w}$ (or of $c_{u,w}(1)$) is known (in general) – and none is expected to exist. Still KL combinatorics has deep connections to the classical enumerative combinatorics (for example, via the theory of "cells").

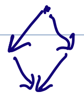
- **Other restrictions**: one has $c_{w,u}(v)=v^{\overbrace{\ell(w)-\ell(u)}^{\mathbb{Z}_{\geq 0} \text{ if } w\succ u}}P_{w,u}(v^2)$ for $P_{w,u}$, a polynomial w. integral coefficients. One can trace this from the definition. Another restriction is that the $P_{w,u}(0)=1$. And that's it: any polynomial w. non-negative integer coefficients & constant term 1 arises as $P_{w,u}$ for some $w,u\in S_n$ for some n, P. Polo, "Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups", Representation theory, 1999.

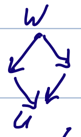
- **Kazhdan-Lusztig inversion formula**: Theorem in Sec. 1.4 implies
$$\sum_{y\in W}(-1)^{\ell(w)-\ell(y)}c_{u,y}(1)\,c_{yw_0,ww_0}(1)=\delta_{u,w}\quad\forall\, u,w\in W.$$
In fact, KL'79, we have $\sum_{y\in W}(-1)^{\ell(w)-\ell(y)}c_{u,y}\,c_{yw_0,ww_0}=\delta_{u,w}$.

This is a combinatorial shadow of a deep representation theoretic fact: the principal block of category $\mathcal{O}$ is Koszul self-dual. We'll mention some more on this later.

- Combinatorial invariance conjecture

A fundamental issue w. computing $c_{w,u}$ is that to compute them one needs to start with $w=1$ and do induction on the Bruhat order. At the same time, there's a lot of evidence suggesting that $c_{w,u}$ depends not on w,u themselves but on the interval between w and u in the Bruhat graph (the full subgraph, whose vertices are all vertices on a path from w to u). For example, the interval between $s_1s_2s_1$ and s_2 in the Bruhat graph looks like



and each time the interval between w and u is  we should have $c_{w,u}=v^2$. The general conjecture is known as the "combinatorial invariance conjecture", see arXiv:2111.15161 for recent developments and more details.