Lec 23: Hecke algebras, IV

0) Intro/overview
1) $U_q(\mathfrak{sl}_2)$

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In the previous three lectures we introduced various modifications of the Hecke algebra $\mathcal{H}_\kappa(W)$ ($\kappa\in\mathbb{C}$, $W=S_n$). We've seen the following connections w. other objects:

0) $\mathcal{H}_1(W)\simeq\mathbb{C}W$

1) $\mathcal{H}_{p^k}(W)=(\mathbb{C}[B(p^k)\backslash G(p^k)/B(p^k)],*)$, where $G(p^k)=GL_n(\mathbb{F}_{p^k})$ & $B(p^k)$ is the subgroup of upper triangular matrices.

2) The Kazhdan-Lusztig basis in $\mathcal{H}_v(W)$ controls character formulas for the irreducible $\mathfrak{sl}_n$-reps $L(\lambda)$ ($\lambda\in\Lambda$).

1) & 2) are related. Roughly, one relates "highest weight" representations of $\mathfrak{sl}_n$ to sheaves on the flag variety G/B that are constant on B-orbits (Beilinson-Bernstein localization or Soergel theory) & the uses Grothendieck's sheaf-function correspondence to get to functions on $B(p^k)\backslash G(p^k)/B(p^k)$.

Today we start to study a different appearance of Hecke algebras. We view Hecke algebras $\mathcal{H}_\kappa(W)$ as deformations of $\mathbb{C}S_n$. One of classical appearances of $\mathbb{C}S_n$ is in the Schur-Weyl duality. Namely, on $(\mathbb{C}^m)^{\otimes n}$ we have commuting actions of $\mathbb{C}S_n$ & $U(\mathfrak{sl}_m)$, where S_n acts by permuting the factors & $U(\mathfrak{sl}_m)$ acts by the usual action on tensor product, which makes sense b/c $U(\mathfrak{sl}_m)$ is a

Hopf algebra. There are results about properties of these commuting actions but we won't need them. Instead we are asking for an analog of Schur-Weyl duality, where on one side we have $\mathcal{H}_\varepsilon(W)$ w. $\varepsilon\neq 1$. Turns out that it exists: on the other side we'll have a "quantum group" $\mathcal{U}_q(\mathfrak{sl}_m)$. In this lecture we will treat the case $m=2$. And our task after this will be to discuss a connection between this construction & the Jones polynomial, a famous link invariant.

1) $\mathcal{U}_q(\mathfrak{sl}_2)$

1.0) Reminder on Hopf algebras

Recall (Sec 1 of Lec 10) that a Hopf algebra is an associative algebra A together w. algebra homomorphisms $\Delta: A\to A\otimes A$, coproduct, $S: A\to A^{op}$ (antipode), $\eta: A\to \mathbb{F}$ (counit, $\mathbb{F}$ is the base field) satisfying coproduct, antipode & counit axioms.

We won't repeat the axioms but let's mention a consequence. For two A-modules U, V we can equip the $A\otimes A$-module $U\otimes V$ with an A-module structure via $\Delta: A\to A\otimes A$. Coassociativity implies & in fact, is equivalent to the claim that $\forall$ A-modules U, V, W, the natural isomorphism

$$(U\otimes V)\otimes W \xrightarrow{\sim} U\otimes (V\otimes W)$$

is A-linear. On the other hand, the natural isomorphism

$$\sigma_{U,V}: U\otimes V \xrightarrow{\sim} V\otimes U,\ u\otimes v\mapsto v\otimes u$$

may fail to be A-linear, for this one needs to require that

$\Delta = \Delta^{op} := \sigma_{A,A} \circ \Delta.$

1.1) $U_q(\mathfrak{sl}_2)$ as an algebra

Consider the field of rational functions, $\mathbb{F} := \mathbb{C}(q) = \mathrm{Frac}\,\mathbb{C}[q]$. Define the $\mathbb{F}$-algebra $U_q(\mathfrak{sl}_2)$ by generators K, K^{-1}, E, F & relations

$$KK^{-1} = K^{-1}K = 1$$

$$KEK^{-1} = q^2 E,\quad KFK^{-1} = q^{-2} F$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

This algebra arises as a deformation of $U(\mathfrak{sl}_2)$ in the sense related to (but more complicated than) $\mathcal{H}_v(W)$ being a deformation of $\mathbb{C}W$ ($W = S_n$).

Side remark: to make the claim about deformation more formal one can consider the algebra $U_\hbar(\mathfrak{sl}_2)/\mathbb{C}[[\hbar]]$, the quotient of $\underbrace{\mathbb{C}\langle e,h,f\rangle}_{\text{free algebra}}[[\hbar]] / ([h,e]=2e,\ [h,f]=-2f,\ [e,f]=(e^{\hbar h}-e^{-\hbar h})/(e^{\hbar}-e^{-\hbar}))$

where $e^{\hbar h} = \sum_{i=0}^{\infty} \frac{1}{i!} h^n \hbar^n$. Its quotient by $\hbar = 0$ is $U(\mathfrak{sl}_2)$. On the other hand, we have a homomorphism $U_q(\mathfrak{sl}_2) \to U_\hbar(\mathfrak{sl}_2)$: $q \mapsto e^{\hbar}$, $E \mapsto e$, $F \mapsto f$, $K \mapsto e^{\hbar h}$. In a way $U_\hbar(\mathfrak{sl}_2)$ is a "completion" of $U_q(\mathfrak{sl}_2)$.

We'll need a representation of $U_q(\mathfrak{sl}_2)$ in $\mathbb{F}^2$:

$E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $K \mapsto \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$

(to check it's a representation is an easy but important exercise).

1.2) $U_q(\mathfrak{sl}_2)$ as a Hopf algebra

Since Δ, S, η are algebra homomorphisms, it's enough to define them on generators.

Proposition: The following maps extend to a coproduct, antipode & counit:

$$\Delta(E) = E\otimes 1 + K\otimes E,\quad \Delta(F) = F\otimes K^{-1} + 1\otimes F,\quad \Delta(K) = K\otimes K$$

$$S(E) = -K^{-1}E,\quad S(F) = -FK,\quad S(K) = K^{-1}$$

$$\eta(E) = \eta(F) = 0,\quad \eta(K) = 1.$$

Sketch of proof:

This involves checking a bunch of equalities: relations between the images of generators to show that Δ, S, η give an algebra homomorphism & then check coassociativity, counit & antipode axioms on generators. We'll do two of these checks:

- $[\Delta(E), \Delta(F)] = \dfrac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}$:

$$[E\otimes 1 + K\otimes E, F\otimes K^{-1} + 1\otimes F] = [E,F]\otimes K^{-1} + \overbrace{[E\otimes 1, 1\otimes F]}^{=0} + [K\otimes E, F\otimes K^{-1}]$$
$$+ K\otimes [E,F] = \frac{K-K^{-1}}{q-q^{-1}}\otimes K^{-1} + \underbrace{KF\otimes EK^{-1} - FK\otimes K^{-1}E}_{=0} + K\otimes \frac{K-K^{-1}}{q-q^{-1}} =$$
$$\frac{K\otimes K - K^{-1}\otimes K^{-1}}{q-q^{-1}}.$$

- $(\Delta\otimes id)(\Delta(E)) = \Delta(E)\otimes 1 + \Delta(K)\otimes E = E\otimes 1\otimes 1 + K\otimes E\otimes 1 + K\otimes K\otimes E$

$\quad\parallel$

$(id\otimes\Delta)(\Delta(E)) = E\otimes\Delta(1) + K\otimes\Delta(E) = E\otimes 1\otimes 1 + K\otimes E\otimes 1 + K\otimes K\otimes E$ $\square$

Important remark: $U_q(\mathfrak{sl}_2)$ is NOT cocommutative:

$$\Delta(E) = E\otimes 1 + K\otimes E \neq 1\otimes E + E\otimes K = \Delta^{op}(E).$$

1.3) R-matrix for $\mathbb{F}^2$

Let U, V be fin. dim. $U_q(\mathfrak{sl}_2)$-modules. Since $\Delta \neq \Delta^{op}$, the natural isomorphism $\sigma_{U,V}: U\otimes V \xrightarrow{\sim} V\otimes U$ is not $U_q(\mathfrak{sl}_2)$-linear. It turns out that there's a distinguished isomorphism $R_{V,U}: V\otimes U \to V\otimes U$ s.t. $\sigma'_{U,V} := R_{V,U}\circ \sigma_{U,V}$ is $U_q(\mathfrak{sl}_2)$-linear.

We are only going to consider a special case when $U, V = \mathbb{F}^2$. We consider the basis $e_1\otimes e_1, e_1\otimes e_2, e_2\otimes e_1, e_2\otimes e_2$ and define $R_{\mathbb{F}^2,\mathbb{F}^2}$ to be the following matrix:

$$\begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1}-q & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix} \Rightarrow \sigma'_{\mathbb{F}^2,\mathbb{F}^2} = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q^{-1}-q & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

Lemma: $\sigma'_{\mathbb{F}^2,\mathbb{F}^2}$ is a $U_q(\mathfrak{sl}_2)$-linear automorphism of $\mathbb{F}^2\otimes\mathbb{F}^2$.

Proof:

It's enough to show that $\sigma'_{\mathbb{F}^2,\mathbb{F}^2}$ intertwines $\Delta(E), \Delta(K), \Delta(F)$. We'll do this for $\Delta(E)$; $\Delta(F)$ is analogous & $\Delta(K)$ is easier. Recall that $\Delta(E) = E\otimes 1 + K\otimes E$, E is represented in $\mathbb{F}^2$ by $\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$ & K by $\begin{pmatrix} q & 0 \\ 0 & q^{-1}\end{pmatrix}$. So $\Delta(E)$ is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & q & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It's a straightforward check that this matrix commutes w. $\sigma'_{\mathbb{F}^2,\mathbb{F}^2}$ $\square$

Important exercise: $(\sigma'_{\mathbb{F}^2,\mathbb{F}^2} - q^{-1})(\sigma'_{\mathbb{F}^2,\mathbb{F}^2} + q) = 0$, which is precisely relation (2) for $H_s \in \mathcal{H}_v(S_n)$ (w $v=q$).

Now we ready to state a version of the quantum Schur-Weyl duality for $U_q(\mathfrak{sl}_2)$. Note that $(\mathbb{F}^2)^{\otimes n}$ is a $U_q(\mathfrak{sl}_2)$-module & it has $n-1$ automorphisms $\breve{\sigma}_i'$ $(i=1,..n-1)$, $\breve{\sigma}_i' := id^{\otimes(i-1)} \otimes \breve{\sigma}'_{\mathbb{F}^2,\mathbb{F}^2} \otimes id^{\otimes n-i-1}$ (so that $\breve{\sigma}'_{\mathbb{F}^2,\mathbb{F}^2}$ acts on factors # $i, i+1$).

Thm: $\exists!$ homomorphism from $\mathcal{H}_q(S_n) = \mathbb{C}(q) \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathcal{H}_v(S_n)$ to $End_{U_q(\mathfrak{sl}_2)}((\mathbb{F}^2)^{\otimes n})$ w. $H_i (= H_{s_i}) \mapsto \breve{\sigma}_i'$

We will sketch the proof in the next lecture.

1.4) More general R-matrices

Here we sketch a construction of $R_{U,V}$ for general fin. dim. $U_q(\mathfrak{sl}_2)$-modules U, V. Cf. Lec 6, the case of usual $\mathfrak{sl}_2$.

1) K acts diagonalizably on U w. eigenvalues in $\tilde{\Lambda} := \{\pm q^n | n \in \mathbb{Z}\}$

2) Let $U_{\pm q^n}$ be the corresponding e-space. Then $EU_{\pm q^n} \subset U_{\pm q^{n+2}}$, $FU_{\pm q^n} \subset U_{\pm q^{n-2}}$. In particular, E, F act by nilpotent operators.

We construct $R_{U,V}$ as the composition of two operators. First, thanks to 2), the following infinite sum gives a well-defined operator on $U \otimes V$:

(1) $\Theta := \sum_{n=0}^{\infty} a_n F^n \otimes E^n$ $(a_n \in \mathbb{F})$

Set $[n]! = \prod_{i=1}^{n} \frac{q^i - q^{-i}}{q - q^{-1}}$ ("q-factorial")

One can show (exercise) that for $a_n = (-1)^n q^{-n(n-1)/2} \frac{(q-q^{-1})^n}{[n]!}$ we have

(2) $\Delta(a) \circ \Theta = \Theta \circ \Delta'(a)$ on $U \otimes V$ $\forall a \in U_q(\mathfrak{sl}_2)$,

where the coproduct Δ' is given by (switch K & K^{-1} in formulas for Δ)

$\Delta'(E) = E\otimes 1 + K^{-1}\otimes E$

$\Delta'(F) = F\otimes K + 1\otimes F$

$\Delta'(K) = K\otimes K$

Then take any map $f: \tilde{\Lambda}\times\tilde{\Lambda} \to \mathbb{F}\setminus\{0\}$ s.t.

$f(\lambda,\mu) = \lambda f(\lambda, q^2\mu) = \mu f(q^2\lambda,\mu) \quad \forall \lambda,\mu\in\tilde{\Lambda}$

Let $\tilde{f}: U\otimes V \to U\otimes V$ be the linear operator given on $U_\lambda\otimes U_\mu$ by scalar $f(\lambda,\mu)$. Then (exercise)

(3) $\Delta'(a)\tilde{f} = \tilde{f}\Delta^{op}(a) \quad \forall a\in U_q(\mathfrak{sl}_2)$ on $U\otimes V$.

Combining (2) & (3) we see that for $R := \Theta\circ\tilde{f}$ we get

$R := \Theta\circ\tilde{f}$ satisfies

(4) $\Delta(a)\circ R = R\circ\Delta^{op}(a) \quad \forall a\in U_q(\mathfrak{sl}_2)$

Note that R is invertible

Rem: If $f(q,q) = q^{-1}$ ($\Rightarrow f(q^{-1},q^{-1}) = q^{-1}, f(q,q^{-1}) = f(q^{-1},q) = 1$) we recover $R_{\mathbb{F}^2,\mathbb{F}^2}$ (exercise).