Lec 24: Hecke algebras, V

1) Quantum Schur-Weyl duality
2) Braids & links

1.0) Reminder

Let $\mathbb{F}=\mathbb{C}(q)$. Consider the Hecke algebra $\mathcal{H}_q(S_n)=\mathbb{F}\otimes_{\mathbb{C}[v^{\pm 1}]}\mathcal{H}_v(S_n)$. It has basis H_w ($w\in S_n$) & relations:

(1) $H_s H_w = H_{sw}$, $\forall s=s_i\ (=(i,i+1))$ if $\ell(sw)=\ell(w)+1$

(2) $(H_s-q^{-1})(H_s+q)=0$ $\forall s$.

We defined the following automorphism σ' of $\mathbb{F}^2\otimes\mathbb{F}^2$: in basis $e_1\otimes e_1, e_1\otimes e_2, e_2\otimes e_1, e_2\otimes e_2$ it's given by
$$\begin{pmatrix} q^{-1} & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & q^{-1}-q & 0\\ 0 & 0 & 0 & q^{-1}\end{pmatrix}$$

Equivalently,

(I) $\sigma'(e_i\otimes e_i)=q^{-1}e_i\otimes e_i$, $\sigma'(e_1\otimes e_2)=e_2\otimes e_1$, $\sigma'(e_2\otimes e_1)=e_1\otimes e_2+(q^{-1}-q)e_2\otimes e_1$

In Sec 1.3 of Lec 23 we have stated the following result. We write σ_i' for the endomorphism $id^{\otimes i-1}\otimes\sigma'\otimes id^{\otimes(n-i-1)}$ $(i=1,..,n-1)$ of $(\mathbb{F}^2)^{\otimes n}$.

Thm: $\exists!$ representation of $\mathcal{H}_q$ in $(\mathbb{F}^2)^{\otimes n}$ w. $H_i\mapsto\sigma_i'$

The uniqueness modulo existence follows b/c H_i's generate $\mathcal{H}_q$: if $w=s_{i_1}\dots s_{i_\ell}$ w. $\ell=\ell(w)$ (such are called reduced expressions of w) then $H_w=H_{i_1}..H_{i_\ell}$ by (1). So to establish the existence of the representation we must show that

(*) $\sigma'_w = \sigma'_{i_1} \dots \sigma'_{i_\ell}$ is independent of the choice of reduced expression $w = s_{i_1} \dots s_{i_\ell}$.

Once (*) is established, we recall that σ' (hence all σ'_i) satisfy (2). Since (1) & (2) are satisfied for the σ'_w's, Thm follows.

1.1) Braid relations.

To establish (*) we need to understand how different reduced expressions are related to each other. The answer is given by the Matsumoto theorem (1964). Note that

(i) $s_i s_j = s_j s_i$ for $|i-j|>1$ &

(ii) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

are reduced expressions (for (ii), of $(i, i+2)$)

Thm (Matsumoto) any two reduced expressions of the same element $w \in W (= S_n)$ are obtained from on another by a sequence of replacing $s_i s_j \leftrightarrow s_j s_i$ for $|i-j|>1$, $s_i s_{i+1} s_i \longleftrightarrow s_{i+1} s_i s_{i+1}$ ("braid moves")

Corollary: $H_q(S_n)$ is generated by $H_1, \dots H_{n-1}$ w. relations

(1′) $H_i H_j = H_j H_i$ if $|i-j|>1$.

(1″) $H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1}$.

(2) $(H_i - q^{-1})(H_i + q) = 0$.

And (*) will follow once we prove

Proposition: The endomorphisms σ_i'' $(i=1,\dots,n-1)$ satisfy (1') & (1'').

Sketch of proof:

(1') follows b/c σ_i', σ_j' act on disjoint pairs of tensor factors.

(1''): it's enough to assume that $n=3$. Then we check the equalities $\sigma_1''\circ\sigma_2''\circ\sigma_1''(e_i\otimes e_j\otimes e_k)=\sigma_2''\circ\sigma_1''\circ\sigma_2''(e_i\otimes e_j\otimes e_k)$ $\forall\, i,j,k\in\{1,2\}$. E.g. let $i=k=2, j=1$. We use (I)

$\sigma_1''\circ\sigma_2''\circ\sigma_1''(e_2\otimes e_1\otimes e_2)=\sigma_1''\circ\sigma_2''(e_1\otimes e_2\otimes e_2+(q^{-1}-q)e_2\otimes e_1\otimes e_2)=\sigma_1''(q^{-1}e_1\otimes e_2\otimes e_2+(q^{-1}-q)e_2\otimes e_2\otimes e_1)=q^{-1}e_2\otimes e_1\otimes e_2+(q^{-2}-1)e_2\otimes e_2\otimes e_1$

$\sigma_2''\circ\sigma_1''\circ\sigma_2''(e_2\otimes e_1\otimes e_2)=\sigma_2''\circ\sigma_1''(e_2\otimes e_2\otimes e_1)=\sigma_2''(q^{-1}e_2\otimes e_2\otimes e_1)=q^{-1}e_2\otimes e_1\otimes e_2+(q^{-2}-1)e_2\otimes e_2\otimes e_1$. $\square$

1.2) Quantum Yang-Baxter equation

Let U,V,W be fin. dimensional $U_q(\mathfrak{sl}_2)$-modules. Recall that we constructed the R-matrix $R_{U,V}: U\otimes V\longrightarrow U\otimes V$ (Sec 1.4 of Lec 23). It gives rise to three automorphisms of $U\otimes V\otimes W$:

$R_{12}:=R_{U,V}\otimes id_W$, $R_{23}=id_U\otimes R_{V,W}$, R_{13} coming from $R_{U,W}$ (and id_V).

It turns out that we have the following (Quantum Yang-Baxter equation, QYBE):

(ii) $\quad R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}$

Now let $U=V=W$ & set $\sigma''=R\circ\sigma$. Then (ii)$\Leftrightarrow$(1'') for σ_1'', σ_2''.

QYBE originates in Statistical Mechanics

2) Braids & links

Define the braid group on n strands (Br_n) as generated by elements $\tau_1, \dots \tau_{n-1}$ with relations (1') & (1''). So

$$\mathcal{H}_q(S_n) = \mathbb{F}Br_n / ((\tau_i - q^{-1})(\tau_i + q) = 0 \mid i = 1, \dots, n-1)$$

A crucial observation is that Br_n has a topological interpretation.

2.1) Braids, topologically.

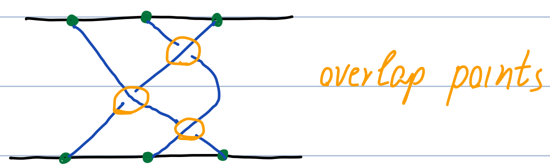
Consider the space $\mathbb{R}^2 \times [0,1]$ w. marked points $(i,0,j)$ w. $i = 1, \dots n$, $j = 0,1$. A braid is a collection of n non-intersecting strands in $\mathbb{R}^2 \times [0,1]$ connecting the points $(i,0,0)$ to $(\sigma(i), 0, 1)$ for some permutation $\sigma \in S_n$ in such a way that

(*) Every plane $\{(x,y,t)\}$ w. $0 \leq t \leq 1$ intersects every strand exactly once.

We consider the braids up to isotopy: we allow to move strands avoiding intersections of strands and preserving (*)

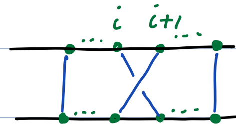
We depict braids by braid diagrams: we project them along $(x,y,t) \mapsto (x,t)$. We always assume that:

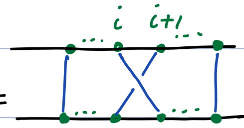
- there are no triple overlap points
- all overlap points have distinct t-coordinate:



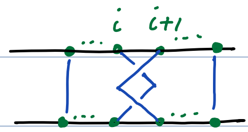
Let Br_n^{top} denote the set of all braids w. n strands (up to isotopy as above). It turns out it has a group structure. First, it has a product via vertical stacking:

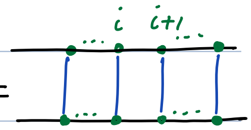
$\beta_1 * \beta_2 = \begin{array}{|c|}\hline \beta_1 \\ \hline \beta_2 \\ \hline \end{array}$ it's associative

E.g. consider the braids $\tau_i =$ 

, $\tau_i^{-1} =$ 

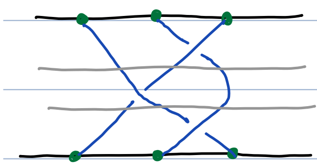
$(i = 1, \ldots n-1)$. Then

$\tau_i \tau_i^{-1} =$ 

$= \left[\begin{array}{l}\text{isotopy straightening} \\ \text{strands}\end{array}\right] =$ 

the right hand side is known as the trivial braid to be denoted by e. Similarly, $\tau_i^{-1}\tau_i = e$.

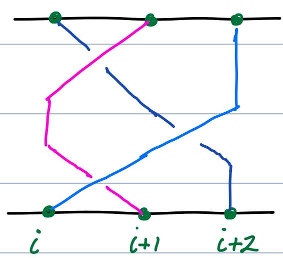
Note that e is the unit for the product (exercise: use isotopy compressing the diagram of e). So Br_n^{top} becomes a monoid. To see it's actually a group notice that Br_n^{top} is generated by τ_i, τ_i^{-1} $(i = 1, \ldots n-1)$. In order to show that we slice a braid diagram so there's just one overlap in each piece

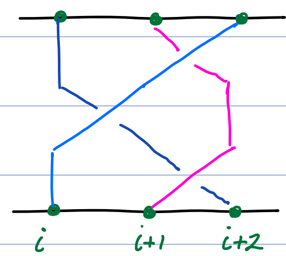


& notice that the braid diagrams w. exactly one overlap are $\tau_i^{\pm 1}$.

Since the generators of a monoid are invertible, any element is, so Br_n^{top} is indeed a group.

Now we observe that $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_i$ in Br_n^{top}:

$\tau_i \tau_{i+1} \tau_i =$ 

$=$ 

$= \tau_{i+1} \tau_i \tau_{i+1}$

isotopy slides the strands

$\tau_i \tau_j = \tau_j \tau_i$ in Br_n^{top} (exercise – the braid diagrams are planar isotopic).

So we get an epimorphism $Br_n \twoheadrightarrow Br_n^{top}$ sending τ_i to τ_i

Thm (E. Artin, 1947) This is iso.

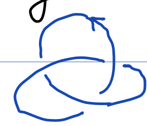
2.2) Links.

A *link* in $\mathbb{R}^3$ is a (smooth) embedding of $\mathbb{S}^1 \sqcup \ldots \sqcup \mathbb{S}^1$ (into $\mathbb{R}^3$). All our links are oriented. Links with one component are called knots.

As with braids, we can represent links by link diagrams via projecting them to $\mathbb{R}^2$:

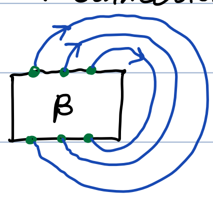
Hopf link

or

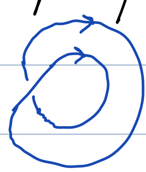


trefoil (knot)

A connection with braids is via *braid closure*:



For example $\beta = \tau_1$ w. $n=2$ gives the unknot (trivially embedded $\mathbb{S}^1$):



. The same is true for τ_1^{-1}

Thm (Alexander, 1928) Every link can be obtained as the closure of a braid.

This braid is not unique but one can describe when two braids give the same link leading to an approach to link invariants via braids. We will explain this in the next lecture.