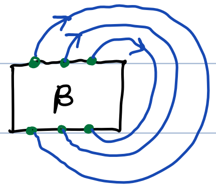
Lec 25: Hecke algebras, VI

1) Jones polynomial as Markov trace

2) Sl_n by generators & relations.

1) Jones polynomial as Markov trace

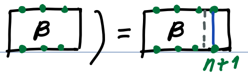
In Sec 2.2 of Lec 24 we've introduced the construction of braid closure that produces from a braid β an oriented link

$\bar{\beta}:=$ 

We stated Alexander's theorem that every oriented link can be presented as a braid closure but have seen that the corresponding braid is not unique. Our first task is understand this non-uniqueness.

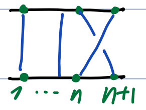
1.1) Markov moves

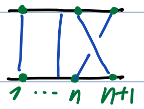
Let Br_n denote the braid group on n strands. Note that both algebraic & topological descriptions of Br_n yield a group homomorphism

$\iota_n: Br_n \to Br_{n+1}$, $\iota_n($  $)$

Proposition: 1) $\forall \beta_1, \beta_2 \in Br_n \Rightarrow \overline{\beta_1\beta_2} = \overline{\beta_2\beta_1}$

2) $\forall \beta \in Br_n \Rightarrow \bar{\beta} = \overline{\iota_n(\beta)\tau_n} = \overline{\iota_n(\beta)\tau_n^{-1}}$

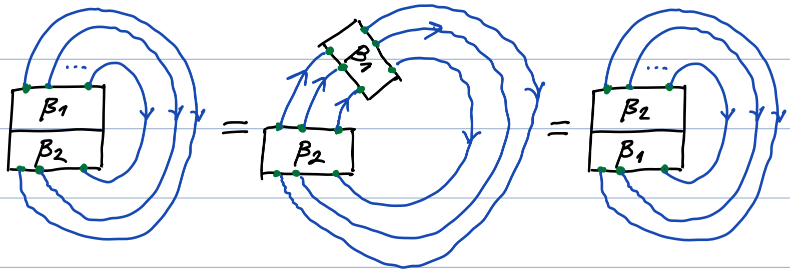
(Recall that $\tau_n =$ 

& $\tau_n^{-1} =$ 

)

Proof:

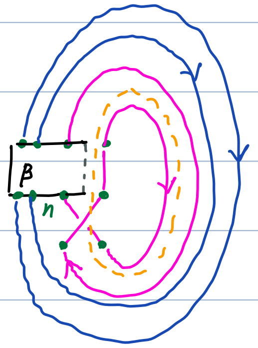
1) $\overline{\beta_1\beta_2} =$



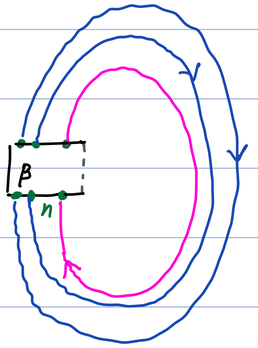
$= \overline{\beta_2\beta_1}$

2) We'll prove $\overline{\beta} = \overline{C_n(\beta)\tau_n}$, the other equality is similar

$\overline{C_n(\beta)\tau_n} =$



$=$ [isotopy contracting pink strand] $=$



$= \overline{\beta}$

□

By Markov moves we mean the following transformations of $\coprod_{n\geq 1} Br_n$:

(M1) $\beta_1\beta_2 \longleftrightarrow \beta_2\beta_1$ for $\beta_1, \beta_2 \in Br_n$

(M2) $\beta \longleftrightarrow C_n(\beta)T_n^{\pm 1}$ for $\beta \in Br_n$

Thm (A. Markov, 1935) For $\beta_1, \beta_2 \in \coprod_{n\geq 1} Br_n$ TFAE

(a) $\overline{\beta_1} = \overline{\beta_2}$

(b) β_1 is obtained from β_2 by a sequence of Markov moves (in both directions).

Example: $\tau_1\tau_2\tau_3^{-1}\tau_1^{-1}\tau_2^{-1} \xrightarrow{M1} \tau_1^{-1}\tau_2^{-1}\tau_1\tau_2\tau_3^{-1} \xrightarrow{M2} \tau_1^{-1}\tau_2^{-1}\tau_1\tau_2^{-1}$ give the same link

We are interested in $\mathbb{F}$-valued invariants of oriented links. Thx to Thm they correspond to functions $\coprod_{n\geqslant 1} Br_n \to \mathbb{F}$ invariant under (M1) & (M2). These are called Markov traces. The reason is for the name is that for any fin. dim. rep'n ρ_n of Br_n, the function $\beta \mapsto tr\, \rho_n(\beta)$ satisfies (M1).

1.2) Jones polynomial

We are now going to construct a Markov trace. The main ingredient is Sec 1 of Lec 24. There we produced a representation ρ_n of Br_n in $(\mathbb{F}^2)^{\otimes n}$ (factoring through $\mathcal{H}_q(S_n)$):

$$\tau_i \longmapsto \sigma_i' = id^{\otimes i-1} \otimes \sigma' \otimes id, \text{ where } \sigma' = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q^{-1}-q & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

By construction in Sec 1.3 of Lec 23, ρ_n commutes w. action of $\mathcal{U}_q(\mathfrak{sl}_2)$, in particular, w. K acting by $\begin{pmatrix} q & 0 \\ 0 & q^{-1}\end{pmatrix}^{\otimes n}$. But $(tr\,\rho_n)_{n\geqslant 1}$ doesn't satisfy (M2) so it's not a Markov trace, we need to tweak it. The first ingredient is the following general construction.

Let V be a finite dimensional vector space. We identify $End(V)^{\otimes n} \xrightarrow{\sim} End(V^{\otimes n})$ via $[\varphi_1 \otimes \ldots \otimes \varphi_n](v_1 \otimes \ldots \otimes v_n) = \varphi_1(v_1) \otimes \ldots \otimes \varphi_n(v_n)$.
This gives a map $tr\ End(V^{\otimes n+1}) \longrightarrow End(V^{\otimes n})$,

$$\varphi_1 \otimes \ldots \otimes \varphi_{n+1} \mapsto tr(\varphi_{n+1})\, \varphi_1 \otimes \ldots \otimes \varphi_n$$

E.g. for $V = \mathbb{F}^m$ we can identify

$$(V^{\otimes n})^{\oplus m} \xrightarrow{\sim} V^{\otimes n+1}, \qquad (w_1, \ldots w_m) \mapsto \sum_{i=1}^m w_i \otimes e_i$$

& view an element of $End(V^{\otimes n+1})$ as $(B_{ij})_{i,j=1}^m$ w $B_{ij} \in End(V^{\otimes n})$.

Then tr_n sends this element to $\sum_{i=1}^{n} B_{ii}$.

The following describes basic properties of tr_{n+1}:

Exercise: a) $tr_{n+1}([A\otimes id_V]\circ B) = A\circ tr_{n+1}(B)\ \forall A\in End(V^{\otimes n}), B\in End(V^{\otimes n})$

6) $\forall B\in End(V^{\otimes 2}) \Rightarrow tr_{n+1}(id_V^{\otimes n-1}\otimes B) = id_V^{\otimes n-1}\otimes tr_2(B)$.

c) $tr_{V^{\otimes n+1}} = tr_{V^{\otimes n}}\circ tr_{n+1}$

Now pick $\xi\in GL(V^{\otimes 2})$ $\quad(\rightsquigarrow \xi_i = id_V^{\otimes i-1}\otimes\xi\otimes id_V^{\otimes n-i-1}\in GL(V^{\otimes n}))$ & $k\in End(V)\ (\rightsquigarrow k^{\otimes n}\in End(V^{\otimes n}))$

Proposition: Suppose that

1) $\xi_1\xi_2\xi_1 = \xi_2\xi_1\xi_2\ (\in End(V^{\otimes 3})) \rightsquigarrow$ rep'n $\rho_n: Br_n \to GL(V^{\otimes n}),\ \tau_i\mapsto\xi_i$

2) $\xi k^{\otimes 2} = k^{\otimes 2}\xi\ \ (\in End(V^{\otimes 2}))$

3) $k = tr_2(\xi k^{\otimes 2}) = tr_2(\xi^{-1}k^{\otimes 2})\ (\in End(V))$

Then $\beta\mapsto tr(\rho_n(\beta)k^{\otimes n})$ (makes sense thx to 1)) is a Markov trace.

Proof: (M1): 2) $\Rightarrow \xi_i$ & $k^{\otimes n}$ commute $\Rightarrow \rho_n(\beta)$ & $k^{\otimes n}$ commute $\forall\beta$
$\Rightarrow tr(\rho_n(\beta_1)\rho_n(\beta_2)k^{\otimes n}) = tr(\rho_n(\beta_1)k^{\otimes n}\rho_n(\beta_2)) = tr(\rho_n(\beta_2)\rho_n(\beta_1)k^{\otimes n})$

(M2): $tr_{V^{\otimes n+1}}(\rho_{n+1}(\iota_n(\beta)\tau_n)k^{\otimes n+1}) =$
$tr_{V^{\otimes n+1}}([\rho_n(\beta)\otimes id_V]\xi_n k^{\otimes n+1}]) = tr_{V^{\otimes n+1}}([\rho_n(\beta)k^{\otimes n-1}\otimes id_V][id_V^{\otimes n-1}\otimes(\xi k^{\otimes 2})]$
$= [a) + c)$ of Exercise$] = tr_{V^{\otimes n}}(\rho_n(\beta)k^{\otimes n-1}tr_{n+1}(id_V^{\otimes n-1}\otimes(\xi k^{\otimes 2})))$
$= [6)$ of Exercise $+ 3)] = tr_{V^{\otimes n}}([\rho_n(\beta)k^{\otimes n-1}][id_{V^{\otimes n-1}}\otimes k] = tr_{V^{\otimes n}}(\rho_n(\beta)k^{\otimes n})$

The other equality is analogous. $\square$

Example: Set $V=\mathbb{F}^2$, $R=K=\begin{pmatrix} q & 0 \\ 0 & q^{-1}\end{pmatrix}$, $\xi = q^2\sigma' = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q^2 & 0 \\ 0 & q^2 & q-q^3 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$

Then conditions of Proposition are satisfied:

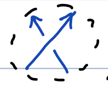
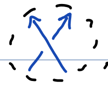
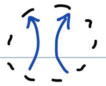
e.g.

$$tr_2(\xi R^{\otimes 2}) = tr_2\left[\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & q^2 & 0 \\ 0 & q^2 & q-q^3 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}\begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-2} \end{pmatrix}\right] = \begin{pmatrix} q^3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} q-q^3 & 0 \\ 0 & q^{-1} \end{pmatrix} = R$$

$$tr_2(\xi^{-1} R^{\otimes 2}) = tr_2\left[\begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & q^{-1}-q^{-3} & q^{-2} & 0 \\ 0 & q^{-2} & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}\begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-2} \end{pmatrix}\right] = \begin{pmatrix} q & 0 \\ 0 & q^{-1}-q^{-3} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & q^{-3} \end{pmatrix} = R$$

Rem: We've used that we have a collection of representations of Br_n but not that they factor through $\mathcal{H}_q(n)$. The latter is important: it allows to compute the link invariant from the link diagram itself.

Recall $(\sigma'-q^{-1})(\sigma'+q)=0 \Leftrightarrow (\xi-q)(\xi+q^3)=0 \Leftrightarrow \xi^2+(q^3-q)\xi-q^4=0$ $\Leftrightarrow q^2\xi^{-1}-q^{-2}\xi=(q-q^{-1})$. This results in the following relation for the corresponding link invariant (up to normalization, this is the Jones polynomial) to be denoted by $L\mapsto P(L)$. Let L_+, L_- & L_0 be links with the same diagrams outside a small circle, where they look as follows:

L_+: , L_-: , L_0: 

Then it turns out that

$(*)$ $\quad q^2P(L_-)-q^{-2}P(L_+)=(q-q^{-1})P(L_0)$ ("skein relation")

value on the unknot determines the invariant uniquely & allow to compute it from the diagram.

2) $\mathfrak{sl}_n$ by generators & relations.

Our next goal is to define $U_q(\mathfrak{sl}_n)$. The construction for $\mathfrak{sl}_2$ mirrored relations between e,h,f. So we'll start w. presentation of $\mathfrak{sl}_n$ by generators & relations.

Recall elements $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$ & $h_i = E_{ii} - E_{i+1,i+1}$ $(i=1,\dots n-1)$ in $\mathfrak{sl}_n$.

Exercise: 1) These elements generate $\mathfrak{sl}_n$ (hint: $E_{i,j} = [E_{i,j-1}, E_{j-1,j}]$ for $i<j$)

2) Consider the "Cartan matrix" $A=(a_{ij})$ w. $a_{ii}=2$, $a_{ij}=-1$ if $|i-j|=1$ & $a_{ij}=0$ else $(1\le i,j\le n-1)$. Then

- $[h_i, h_j]=0$, $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$
- $[e_i, f_j] = \delta_{ij} h_i$
- $\mathrm{ad}(e_i)^{1-a_{ij}} e_j = 0$, $\mathrm{ad}(f_i)^{1-a_{ij}} f_j = 0$ $(i\neq j)$

Fact (morally similar to presentation of $L(\lambda)$ by generators & relations): 2) gives defining relations for $\mathfrak{sl}_n$

Remark: This presentation can be generalized. Namely, let $A = (a_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$ satisfy the following conditions:

- $a_{ii}=2$ $\forall i$
- $a_{ij}\le 0$ if $i\neq j$
- $\exists$ $d_1,\dots d_n \in \mathbb{Z}_{>0}$ s.t. $\mathrm{diag}(d_1,\dots d_n)A$ is symmetric

Then one can define the Kac-Moody Lie algebra $\mathfrak{g}(A)$ by generators e_i, h_i, f_i $(i=1,..n)$ & relations as in the exercise. All finite dimensional simple Lie algebras arise in this way. In the next lecture we'll briefly discuss an infinite dimensional example, the affine Lie algebra $\hat{\mathfrak{sl}}_n$.