Lec 26: Generalizations

1) $U_q(\mathfrak{sl}_n)$

2) Affine Kac-Moody & Hecke algebra

1) $U_q(\mathfrak{sl}_n)$

Consider the $(m-1)\times(m-1)$-matrix $A=(a_{ij})$ w. $a_{ij}=\begin{cases}2, & i=j\\ -1, & |i-j|=1\\ 0, & \text{else}\end{cases}$

We've seen in Sec 2 of Lec 25 that $\mathfrak{sl}_m$ is generated by e_i, h_i, f_i $(1\leqslant i\leqslant m-1)$ w. relations

- $[h_i,h_j]=0$
- $[h_i,e_j]=a_{ij}e_j$, $[h_i,f_j]=-a_{ij}f_j$
- $[e_i,f_j]=\delta_{ij}h_i$
- $ad(e_i)^{1-a_{ij}}e_j=0$ $(i\neq j)$
- $ad(f_i)^{1-a_{ij}}f_j=0$

Rem: $U(\mathfrak{sl}_m)$ is generated, as an associative algebra, by the same generators & same relations. The relation for e_i, e_j $(i\neq j)$ can be rewritten as

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}e_i^k e_j e_i^{1-a_{ij}-k}=0$$

The same for f's.

Definition: The $\mathbb{F}=\mathbb{C}(q)$-algebra $U_q(\mathfrak{sl}_n)$ is generated by elements K_i, K_i^{-1}, E_i, F_i w. relations:

- $K_iK_i^{-1}=K_i^{-1}K_i=1$

- $K_i K_j = K_j K_i$
- $K_i E_j K_i^{-1} = q^{a_{ij}} E_j$, $K_i F_j K_i^{-1} = q^{-a_{ij}} F_j$
- $E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$
- $\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_q E_i^k E_j E_i^{1-a_{ij}-k} = 0$ & same for F's $(i \neq j)$

where $\binom{\ell}{k}_q = \frac{[\ell]!}{[k]![\ell-k]!}$, $[\ell]! = \prod_{i=1}^{\ell} \frac{q^i - q^{-i}}{q - q^{-1}}$

So we get:

- $E_i E_j = E_j E_i$ if $|i-j| \neq 1$
- $E_i^2 E_j + E_j E_i = (q+q^{-1}) E_i E_j E_i$ if $|i-j| = 1$.

We equip $\mathcal{U}_q(\mathfrak{sl}_m)$ w. a Hopf algebra structure: we define Δ, η, S on E_i, K_i, F_i as for $\mathfrak{sl}_2$, e.g. $\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$

Rem: $\mathcal{U}_q(\mathfrak{sl}_m)$ has a tautological representation in $\mathbb{F}^n$ via $E_i \mapsto E_{i,i+1}$, $F_i \mapsto E_{i+1,i}$, $K_i \mapsto q E_{ii} + q^{-1} E_{i+1,i+1}$. It comes with an R-matrix $R_{\mathbb{F}^m, \mathbb{F}^m} \in End(\mathbb{F}^m \otimes \mathbb{F}^m)$ given by:

$$q^{-1} \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q^{-1} - q) \sum_{i<j} E_{ij} \otimes E_{ji}$$

generalizing the case of $m=2$.

Similarly to Sec 1.2 of Lec 25 one can produce a link invariant P_m satisfying the following skein-relation

$$q^m P_m(L_-) - q^{-m} P_m(L_+) = (q - q^{-1}) P_m(L_0)$$

It turns out that it is the specialization to $a = q^m$ of an invariant is $\mathbb{C}[a^{\pm 1}, q^{\pm 1}]$ called the HOMFLY-PT polynomial.

2) Affine Kac-Moody & Hecke algebras

2.1) $\hat{\mathfrak{sl}}_m$

We consider the following generalized Cartan $m\times m$- matrices.

$m=2$: $A=\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$m>2$: $A=\begin{pmatrix} 2 & -1 & & & -1 \\ -1 & \ddots & \ddots & 0 & \\ & \ddots & & & \\ 0 & & & \ddots & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$ (so for $m>2$ we have $a_{ij}=-1$ if $i-j\equiv\pm1 \mod m$)

We label the indices from 0 to $m-1$

We define the Lie algebra $\mathfrak{g}(A)$ using the Lie algebra versions of relations in Sec 1. We want to describe it explicitly.

As a first step, consider the following Lie algebra over $\mathbb{C}$: $\mathfrak{sl}_m(\mathbb{C}[t^{\pm1}])$ w. usual bracket of matrices. We consider the following elements: $e_i, f_i, h_i \in \mathfrak{sl}_m$ as before & $e_0=E_{m,1}t$, $f_0=E_{1,m}t^{-1}$, $h_0=E_{m,m}-E_{1,1}$.

Exercise: These elements generate $\mathfrak{sl}_m(\mathbb{C}[t^{\pm1}])$ & satisfy the defining relations of $\mathfrak{g}(A)$ yielding an epimorphism $\mathfrak{g}(A)\twoheadrightarrow\mathfrak{sl}_m(\mathbb{C}[t^{\pm1}])$

It turns out this is not an isomorphism. Namely we construct the Lie algebra $\hat{\mathfrak{sl}}_m$ (affine $\mathfrak{sl}_m$) as follows. As a vector space it's $\mathfrak{sl}_m(\mathbb{C}[t^{\pm1}])\oplus\mathbb{C}c$ with bracket defined by:

$[c,\xi]=0 \quad \forall\ \xi\in\hat{\mathfrak{sl}}_m$

$[xt^\ell, yt^k]=(xy-yx)t^{\ell+k}+\ell\delta_{\ell+k,0}\,tr(xy)c$

$\forall\ x,y\in\mathfrak{sl}_m,\ k,\ell\in\mathbb{Z}.$

For example: $[E_{m,1}t, E_{1,m}t^{-1}] = E_{mm} - E_{11} + c$

Now we consider elements e_i, h_i, f_i $(i=1,\dots,n-1)$, e_0, f_0 as before & modify h_0 to be $c + E_{mm} - E_{11}$.

Exercise cont'd: the new elements $e_i, f_i, h_i \in \hat{\mathfrak{sl}}_m$ satisfy the defining relations of $\mathfrak{g}(A)$.

Fact (a harder version of the corresponding claim for $\mathfrak{sl}_m$)
$\mathfrak{g}(A) \xrightarrow{\sim} \hat{\mathfrak{sl}}_m$.

2.2) Roots & Weyl group

Many structural features of $\mathfrak{sl}_m$ carry over to $\hat{\mathfrak{sl}}_m$. For example, for $\mathfrak{g} = \mathfrak{sl}_m$ we had the Cartan subalgebra $\mathfrak{h}$ & root spaces $\mathfrak{g}_\beta = \mathbb{C}E_{ij}$ w. $i \neq j$ & $\beta = \varepsilon_i - \varepsilon_j \in \mathfrak{h}^*$ (roots) so that $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\beta \mathfrak{g}_\beta$.

The similar decomposition holds for $\hat{\mathfrak{g}} = \hat{\mathfrak{sl}}_m$ w. important differences. Let $\mathfrak{h}^a = \mathrm{Span}_{\mathbb{C}}(h_0, \dots, h_{m-1})$, this is the Cartan subalgebra. Consider the space $\mathfrak{h}^{a,v}$ w. basis $\alpha_0, \dots \alpha_{m-1}$ & pairing $\mathfrak{h}^{a,v} \times \mathfrak{h}^a \to \mathbb{C}$ given by $\langle \alpha_i, h_j \rangle = a_{ij}$ $(i,j = 0,\dots,m-1)$. This pairing is <u>degenerate</u>: $\langle \alpha_i, c \rangle = 0 = \langle \delta, h_i \rangle$ $\forall i$, where $\delta = \alpha_0 + \dots + \alpha_{m-1}$. We still have a linear map $\mathfrak{h}^{a,v} \to \mathfrak{h}^{a,*}$, $\alpha \mapsto \langle \alpha, \cdot \rangle$

Definition: By roots of $\hat{\mathfrak{g}}$ we mean elements $\beta + k\delta \in \mathfrak{h}^{a,v}$, where β is a root for $\mathfrak{g}$ & $m \in \mathbb{Z}$, or $\ell\delta$ $(\ell \in \mathbb{Z} \setminus \{0\})$. The corresponding root spaces $\hat{\mathfrak{g}}_\alpha$ are $\mathbb{C}E_{ij}t^k$ (for $\alpha = \varepsilon_i - \varepsilon_j + k\delta$), $\mathfrak{h}t^\ell$ $(\alpha = \ell\delta)$ (so that unlike in the case of $\mathfrak{g}$), root spaces can have $\dim > 1$).

So we have $\hat{\mathfrak{g}} = \mathfrak{h}^a \oplus \bigoplus_\alpha \hat{\mathfrak{g}}_\alpha$ & $[x,y] = \langle \alpha, x\rangle y$ $\forall x \in \mathfrak{h}^a, y \in \hat{\mathfrak{g}}_\alpha$.
Next, we proceed to the Weyl group

Definition: For $i=0,\dots,m-1$, define $s_i \in GL(\mathfrak{h}^{a,*})$ by

$$s_i(\lambda) = \lambda - \lambda(h_i)\langle \alpha_i, \cdot\rangle.$$

The affine Weyl group W^a is, by definition, the subgroup of $GL(\mathfrak{h}^{a,*})$ generated by the elements s_i, $i=0,\dots,m-1$.

Here's the description of W^a. Let $\Lambda_r := \mathrm{Span}_{\mathbb{Z}}(\alpha_1,\dots\alpha_{m-1}) \subset \mathfrak{h}^{*,a}$ (the root lattice of $\mathfrak{g}$). We define homomorphisms $W, \Lambda_r \to GL(\mathfrak{h}^{a,*})$ as follows: W acts on $\mathfrak{h}^{a,*}$ as the subgroup generated by $s_1,\dots s_{m-1}$, while Λ_r acts by:

$$t_\mu . \lambda = \lambda + \lambda(c)\langle \mu, \cdot\rangle \quad (\mu \in \Lambda_r \hookrightarrow \mathfrak{h}^{a,\vee},\ \lambda \in \mathfrak{h}^{a,*}).$$

Exercise: 1) These actions combine into an action of $W \ltimes \Lambda_r$ & the corresponding homomorphism $W \ltimes \Lambda_r \to GL(\mathfrak{h}^{a,*})$ is injective

2) The image of $W \ltimes \Lambda_r$ in $GL(\mathfrak{h}^{a,*})$ coincides w. W^a (hints: $s_{\varepsilon_1-\varepsilon_m} s_{\alpha_0} = t_{\varepsilon_1-\varepsilon_m}$; W^a is generated by W & s_{α_0}, while $W \ltimes \Lambda_r$ is generated W & $t_{\varepsilon_1-\varepsilon_m}$).

2.3) Affine Hecke algebra

It turns out that W^a together w. generators $s_0,\dots s_{n-1}$ has formal properties similar to those of W w. generators $s_1,\dots s_{n-1}$ (both are "Coxeter systems"). Namely, to $x \in W$ we can assign its length $\ell(x) = \min\{\ell \in \mathbb{Z}_{\geq 0} \mid x = s_{i_1}\dots s_{i_\ell}$ w. $i_j \in \{0,1,\dots,m-1\}\}$. In particular, we can talk about the reduced expressions of x. When $m=2$, every element has unique reduced expression, while for $m>2$ any two reduced expressions of the same element are related by braid moves that in this case look as follows:

$$s_i s_j = s_j s_i \text{ if } i-j \neq \pm 1 \bmod m$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ (numbered mod } m).$$

In particular, we can form the generic Hecke algebra $\mathcal{H}_v(W^a)$ completely analogously to $\mathcal{H}_v(W)$. It comes with a standard basis $H_x, x\in W^a$. It makes sense to speak about the Kazhdan-Lusztig basis C_x $(x\in W^a)$ and hence about affine Kazhdan-Lusztig polynomials. These polynomials carry the following representation-theoretic info:

1) As was mentioned in Lec 19, they allow to compute the characters of irreducible SL_n-reps $L(\lambda)$ w. $0\leq \lambda_i-\lambda_{i+1}\leq p-1$ over an algebraically closed char p field if $p\gg 0$.

2) More directly, they allow to compute characters of irreducible highest weight modules for $\hat{\mathfrak{g}}$.

Rem: Let q be a prime power. We've seen (this is how to Hecke algebra appeared in our story) that $\mathbb{C}\otimes_{\mathbb{Z}[v^{\pm 1}]}\mathcal{H}_v(W)$ w. $v\mapsto q^{-1/2}$ controls a part of the representation theory of $\mathbb{C}GL_n(\mathbb{F}_q)$ (we can also work w. $\mathbb{C}SL_n(\mathbb{F}_q)$). $\mathbb{C}\otimes_{\mathbb{Z}[v^{\pm 1}]}\mathcal{H}_v(W^a)$ have a similar interpretation. The group of interest is $G:=SL_n(\mathbb{F}_q((t)))$ (we can also replace $\mathbb{F}_q((t))$ w. any "non-Archimedian local field" e.g. $\mathbb{Q}_p$)

The group G is infinite so we need to be careful about a kind of representations we consider (we need "smooth representations" suitably compatible w. a topology on $SL_n(\mathbb{F}_q((t)))$. Namely, consider the Iwahori subgroup $I \subset SL_n(\mathbb{F}_q[[t]])$, the preimage of $B \subset SL_n(\mathbb{F}_q)$ under the homomorphism $SL_n(\mathbb{F}_q[[t]]) \twoheadrightarrow SL_n(\mathbb{F}_q)$ of evaluating at 0. The set of double cosets $I \backslash G / I$ is identified w. W^a. The space of functions $I\backslash G/I \to \mathbb{C}$ that are 0 outside of fin. many points acquires a convolution product, under which it's identified w. $\mathbb{C} \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathcal{H}_v(W^a)$.