Lec 4: algebraic groups & Lie algebras IV

0) Completion of proof of Thm from Lec 2
1) Lie algebras & their representations
2) Universal enveloping algebras

0) Completion of proof of Thm from Lec 2

Consider an algebraic subgroup $G \subset GL_n$ & $\mathfrak{g} := T_e G \subset T_e GL_n = \mathfrak{gl}_n$.
Our first goal is to finish the proof of:

Thm: 1) $\mathfrak{g}$ is closed under $[\cdot,\cdot]$: $\xi, \eta \in \mathfrak{g} \Rightarrow [\xi,\eta] = \xi\eta - \eta\xi \in \mathfrak{g}$.
1') If char $\mathbb{F} = p > 0$, then $\mathfrak{g}$ is also closed under $\xi \mapsto \xi^p$.
Let $H \subset GL_m$ be another algebraic subgroup & $\Phi: G \to H$ be an algebraic group homomorphism & $\varphi = T_e \Phi$. Then
2) $[\varphi(\xi), \varphi(\eta)] = \varphi([\xi,\eta])\ \forall \xi, \eta \in \mathfrak{g}$.
2') If char $\mathbb{F} = p > 0$, then $\varphi(\xi^p) = \varphi(\xi)^p\ \forall \xi \in \mathfrak{g}$.

Most of the work on this was done in Lec 3. Namely to an algebraic group G we assigned an associative algebra $(\mathcal{D}(G), *)$ (the distribution algebra) s.t. $\mathfrak{g} = T_e G \subset \mathcal{D}(G) \subset \mathbb{F}[G]^*$ & for algebraic group homomorphism $\Phi: G \to H$ we get an algebra homomorphism $\Phi_*: \mathcal{D}(G) \to \mathcal{D}(H)$, the restriction of $\Phi_*: \mathbb{F}[G]^* \to \mathbb{F}[H]^*$, and hence an extension of $T_e\Phi: T_e G \to T_e H$ (Sec 1.2 in Lec 2 & Sec 1.2 of Lec 3) yielding the commutative diagram

$$\begin{array}{ccc} T_e G & \xrightarrow{T_e\Phi} & T_e H \\ \cap & & \cap \\ \mathcal{D}(G) & \xrightarrow{\Phi_*} & \mathcal{D}(H) \end{array} \qquad (a)$$

Also in Sec 1.3 of Lec 3 we proved the following:

(6) $\begin{cases} \text{For } \xi,\eta\in\mathfrak{gl}_n\subset D(GL_n), \text{ we have } \xi*\eta-\eta*\xi=[\xi,\eta]\ (\in\mathfrak{gl}_n) \text{ \& in char.} \\ p,\ \xi^{*p}=\xi^p\ (\in\mathfrak{gl}_n). \end{cases}$

Now we get to proving the theorem. Let $\iota: G\hookrightarrow GL_n$ denote the inclusion of an algebraic subgroup $\Rightarrow \iota^*: \mathbb{F}[GL_n]\twoheadrightarrow\mathbb{F}[G] \Rightarrow \iota_*: \mathbb{F}[G]^* \hookrightarrow \mathbb{F}[GL_n]^*$. Then (a) applied to $H=GL_n$ & $\varphi=\iota$ gives

(a') $$\begin{array}{ccc} \mathfrak{g} & \xhookrightarrow{\iota_*} & \mathfrak{gl}_n \\ \cap & & \cap \\ D(G) & \xhookrightarrow[\iota_*]{} & D(GL_n) \end{array}$$

Proof of Theorem

Claim:

1) $\mathfrak{g}=D(G)\cap\mathfrak{gl}_n$ (inside $D(GL_n)$).

2) $\xi,\eta\in\mathfrak{g}\subset D(G)\Rightarrow \xi*\eta-\eta*\xi$ & (in char p) $\xi^{*p}\in\mathfrak{g}$.

Proof of Claim:

1) $\subset$ follows from (a').

To prove $\supset$ pick $\xi\in D(G)\cap\mathfrak{gl}_n\subset\mathbb{F}[G]^*\cap\mathfrak{gl}_m$ & let $A=\mathbb{F}[G], \tilde{A}=\mathbb{F}[GL_n]$, $\mathfrak{m}\subset A$, $\tilde{\mathfrak{m}}\subset\tilde{A}$ are max'l ideals of e (so that $\iota^*(\tilde{\mathfrak{m}})=\mathfrak{m}$). We view ξ as an e-derivation $\tilde{A}\to\mathbb{F}$ (as $\xi\in\mathfrak{gl}_n$). The inclusion $\xi\in\mathbb{F}[G]^*$ (i.e. $\iota_*\mathbb{F}[G]^*$) just means that ξ factors through ι^* (thx to $\langle\iota_*\delta,f\rangle=$ $=\langle\delta,\iota^*(f)\rangle$. So we need to show that every e-derivation of $\tilde{A}$ that factors through ι^* is an e-derivation of A (= element of $\mathfrak{g}$). The check is as follows:

$\xi(\tilde{\mathfrak{m}}^2)=\xi(1_{\tilde{A}})=0 \Rightarrow [\mathfrak{m}^2=\iota^*(\tilde{\mathfrak{m}}^2)$ b/c ι^* is surjective; $1_A=\iota^*(1_{\tilde{A}})]$

$\xi(\mathfrak{m}^2)=\xi(1_A)=0 \Rightarrow \xi\in\mathfrak{g}$. $\square$ of 1).

2) follows from 1) & 6) $\square$ of Claim.

Now 1), 1') follow from Claim & 2), 2') follow from 2) of Claim & a) $\square$

1) Lie algebras & their representations

So, for an algebraic group G, the tangent space $\mathfrak{g}=T_eG$ comes w. a natural bracket operation & in case when the base field has characteristic p, the pth power map.

A natural question is to study $\mathfrak{g}$ with the structures appearing in the theorem axiomatizing their formal properties. If we are dealing only with $[\cdot,\cdot]$ we arrive at the notion of a Lie algebra (going back to Sophus Lie). We will study Lie algebras & their representations extensively in this course. If we incorporate the pth power map - that only makes sense in char p - we'll get the notion of a "restricted Lie algebra" that will also be used in this course although we are not going to formally define it - the axioms are quite complicated.

1.1) Definitions & basic examples.

Let $\mathbb{F}$ be a field.

Definition 1: • A Lie algebra over $\mathbb{F}$ is an $\mathbb{F}$-vector space $\mathfrak{g}$ equipped w. a bilinear map $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ (Lie bracket or commutator). satisfying the following two properties:

- Skew-symmetry: $[x,x]=0\ \forall x\in\mathfrak{g}$ ($\Rightarrow [x,y]=-[y,x]\ \forall x,y\in\mathfrak{g}$, exercise)
- Jacobi identity: $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$.

Once one knows the skew-symmetry, the Jacobi id'y is equivalent to:

$$[[x,y],z]=[x,[y,z]]-[y,[x,z]] \qquad (1)$$

- A ==Lie algebra homomorphism== is an $\mathbb{F}$-linear map $\varphi:\mathfrak{g}\to\mathfrak{h}$ s.t. $[\varphi(x),\varphi(y)]=\varphi([x,y])$

Example 0: Abelian Lie algebra: $[\cdot,\cdot]=0$.

Example 1: Let A be an associative algebra. Then $[a,b]:=ab-ba$ is a Lie bracket (exercise). An important special case: $A=Mat_n(\mathbb{F})$ (or $End(V)$ for a vector space V). The resulting Lie algebra is denoted by $\mathfrak{gl}_n$ (or $\mathfrak{gl}(V)$).

Example 2: Let $G\subset GL_n$ be an algebraic subgroup. Then $\mathfrak{g}:=T_eG$ is a Lie subalgebra in $\mathfrak{gl}_n$ ((1) in Thm from Sec 0) so is a Lie algebra. Moreover, by (2) of that Thm, for an algebraic group homomorphism $\Phi: G\to H$, its tangent map $\varphi:=T_e\Phi$ is a Lie algebra homomorphism $\mathfrak{g}\to\mathfrak{h}$.

We say that $\mathfrak{g}$ is the Lie algebra of G and write $\mathfrak{g}=Lie(G)$.

Example 2 together w. examples of algebraic groups (Sec 2.1 in Lec 1) and computation of their tangent spaces (Sec 1.3 of Lec 2) allow us to give many examples of Lie subalgebras of $\mathfrak{gl}_n$ (or $\mathfrak{gl}(V)$): $\mathfrak{sl}_n$ (or $\mathfrak{sl}(V)$), $\mathfrak{so}_n$ (or $\mathfrak{so}(V,B)$), $\mathfrak{sp}_n$ (n is even; or $\mathfrak{sp}(V,\omega)$), the subalgebras of upper triangular, strictly upper triangular $=Lie\{\begin{pmatrix}1 & & *\\ & \ddots & \\ 0 & & 1\end{pmatrix}\}$ diagonal matrices.

1.2) Representations of Lie algebras.

As usual, a representation of a Lie algebra in a vector space V is a Lie algebra homomorphism $\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

One way to get such for $\mathfrak{g} = \mathrm{Lie}(G)$ is to consider $\varphi = T_e \Phi$: $\mathfrak{g} \to \mathfrak{gl}(V)$ for a rational representation $\Phi: G \to GL(V)$.

Example 0: $V = \mathbb{F}$ & zero $\mathfrak{g} \to \mathfrak{gl}(\mathbb{F}) = \mathbb{F}$. Note that if $\mathfrak{g} = \mathrm{Lie}(G)$, then this is the tangent map of the trivial representation of G.

Example 1 (adjoint representation): for $x \in \mathfrak{g}$, let $\mathrm{ad}(x): \mathfrak{g} \to \mathfrak{g}$ denote the operator $z \mapsto [x,z]$; $\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a representation — $\mathrm{ad}([x,y]) = [\mathrm{ad}(x), \mathrm{ad}(y)]$ is (1) in Sec. 1.1 — called the adjoint representation. Moreover, if G is an algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$, then this representation arises as the tangent map of a rational representation of G in $\mathfrak{g}$ (also called the adjoint representation), $\mathrm{Ad}: G \to GL(\mathfrak{g})$ constructed as follows. G acts on itself by the adjoint action: $a_g(g') = gg'g^{-1}$. Since $a_g(e) = e$, $g \mapsto \mathrm{Ad}(g) = T_e a_g$ gives a representation $G \to GL(\mathfrak{g})$. It's rational & $T_e \mathrm{Ad} = \mathrm{ad}$. Indeed, first consider $G = GL_n$. Then $\mathrm{Ad}(g)z = gzg^{-1}$ $(z \in \mathfrak{g} = \mathfrak{gl}_n) \Rightarrow T_e \mathrm{Ad}(x) z = [x,z]$ (exercise). For the general G, embed $G \hookrightarrow GL_n$, and consider the representations $\widetilde{\mathrm{Ad}}$ of G in $\mathfrak{gl}_n$: $\widetilde{\mathrm{Ad}}(g)z = gzg^{-1}$ & $\widetilde{\mathrm{ad}}$ of $\mathfrak{g}$ in $\mathfrak{gl}_n$ so that $T_e \widetilde{\mathrm{Ad}} = \widetilde{\mathrm{ad}}$. Then $\widetilde{\mathrm{Ad}}$ is rational, as it's restricted from a rational representation of GL_n. And since $G \subset GL_n$ is stable under a_g for $g \in G$, we set that $\mathfrak{g} = T_e G$ is stable under $\widetilde{\mathrm{Ad}}(g) = \mathrm{Ad}(g)$ $\forall g \in$

G. Moreover, since $T_e\mathrm{Ad} = \mathrm{ad}$ for GL_n, we see that the same is true for G.

For future applications we record the following observation. Recall, Sec 2.2 of Lec 1, that to a rational representation $\Phi: G \to GL(V)$ we assign its matrix coefficients $c_{\alpha,v} \in \mathbb{F}[G]$ ($\alpha \in V^*, v \in V$): $c_{\alpha,v}(g) = \alpha(\Phi(g)v)$

Exercise 1: $\varphi = T_e\Phi: \mathfrak{g} \to \mathfrak{gl}(V)$ is uniquely determined by

$$x(c_{\alpha,v}) = \alpha(\varphi(x)v), \ \forall x \in \mathfrak{g}$$

where in l.h.s. we view $\mathfrak{g}$ as the subspace of e-derivations in $\mathbb{F}[G]^*$.

Example 2: Let $\varphi_i: \mathfrak{g} \to \mathfrak{gl}(V_i)$ be representations. Then

$$\varphi: \mathfrak{g} \to \mathfrak{gl}(V_1 \otimes V_2),\ x \mapsto \varphi_1(x) \otimes id + id \otimes \varphi_2(x)$$

defines a representation of $\mathfrak{g}$ (the tensor product $\varphi_1 \otimes \varphi_2$ of φ_1, φ_2): $[\varphi(x), \varphi(y)] = [\varphi_1(x) \otimes id + id \otimes \varphi_2(x), \varphi_1(y) \otimes id + id \otimes \varphi_2(y)] =$ $= [(\varphi_i(?) \otimes id)(id \otimes \varphi_j(\cdot)) = \varphi_i(?) \otimes \varphi_j(\cdot) = (id \otimes \varphi_j(\cdot))(\varphi_i(?) \otimes id) \rightsquigarrow$ cancellations$] = [\varphi_1(x), \varphi_1(y)] \otimes id + id \otimes [\varphi_2(x), \varphi_2(y)] = \varphi([x,y])$

The motivation for this is as follows: let $\mathfrak{g} = \mathrm{Lie}(G)$, $\Phi_i: G \to GL(V_i)$ be rational representations & $\varphi_i = T_e\Phi_i$. Then we claim that $\varphi_1 \otimes \varphi_2 = T_e(\Phi_1 \otimes \Phi_2)$. Indeed, by Example 1(iii) in Sec 2.2 of Lec 2, the matrix coefficients for $\Phi = \Phi_1 \otimes \Phi_2$ are

$$c_{\alpha,v} = c_{\alpha_1,v_1} c_{\alpha_2,v_2} \text{ for } \alpha = \alpha_1 \otimes \alpha_2,\ v = v_1 \otimes v_2$$

Now we use Exercise 1:

identity matrix

$\alpha(T_e\Phi(x)v) = x(c_{\alpha,v} = c_{\alpha_1,v_1} c_{\alpha_2,v_2}) = [\text{Leibniz}] = c_{\alpha_1,v_1}(e)\, x(c_{\alpha_2,v_2}) +$

$$x(c_{\alpha_1,v_1})c_{\alpha_2,v_2}(e) = \alpha_1(v_1)\alpha_2(\varphi_2(x)v_2) + \alpha_1(\varphi_1(x)v_1)\alpha_2(v_2) =$$
$$\alpha_1\otimes\alpha_2([\varphi_1(x)\otimes id + id\otimes\varphi_2(x)](v_1\otimes v_2)] \Rightarrow T_e\Phi(x) = \varphi_1(x)\otimes id + id\otimes\varphi_2(x).$$

Example 3: Let $\varphi: \mathfrak{g}\to\mathfrak{gl}(V)$ be a representation. Then we have the representation $\varphi^*: \mathfrak{g}\to\mathfrak{gl}(V^*)$ (dual represent'n) determined by
$$\langle\varphi^*(x)\alpha, v\rangle = -\langle\alpha, \varphi(x)v\rangle \quad (\alpha\in V^*, v\in V, \langle\alpha,v\rangle = \alpha(v)).$$

Exercise 2: 1) Show that φ^* is indeed a representation

2) Let $\varphi = T_e\Phi$ for a rational representation $G\to GL(V)$. Then $\varphi^* = T_e(\Phi^*)$. Hint: differentiate $\langle\Phi^*(g)\alpha, \Phi(g)v\rangle = \langle\alpha, v\rangle$.

2) Universal enveloping algebra

This is an associative algebra, whose role in the study of representations of $\mathfrak{g}$ is similar to the role of the group algebras in the study of representations of finite groups.

Definition: Define $\mathcal{U}(\mathfrak{g}) := \dfrac{T(\mathfrak{g})}{(x\otimes y - y\otimes x - [x,y] \mid x,y\in\mathfrak{g})}$, where $T(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$,
$$T(\mathfrak{g}) = \bigoplus_{i=0}^{\infty}\mathfrak{g}^{\otimes i}$$

The composition $\mathfrak{g}\hookrightarrow T(\mathfrak{g})\twoheadrightarrow\mathcal{U}(\mathfrak{g})$ is a Lie algebra homomorphism thx to the relations we imposed. Here is the universal property of $\mathcal{U}(\mathfrak{g})$ (and this homomorphism).

Lemma: Let A be an associative algebra (hence a Lie algebra, Ex 1 in Sec 1.1) and let $\varphi: \mathfrak{g} \to A$ be a Lie algebra homomorphism. Then there is a unique associative algebra homomorphism $\tilde{\varphi}: \mathcal{U}(\mathfrak{g}) \to A$ making the following diagram commutative:

$$\begin{tikzcd} \mathfrak{g} \arrow[d] \arrow[dr, "\varphi"] & \\ \mathcal{U}(\mathfrak{g}) \arrow[r, dashed, "\tilde{\varphi}"'] & A \end{tikzcd}$$

Proof: Since φ is an $\mathbb{F}$-linear map, $\exists!$ assoc. algebra homomorphism $\hat{\varphi}: T(\mathfrak{g}) \longrightarrow A$ s.t. $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \xrightarrow{\hat{\varphi}} A$ coincides w. φ. The condition that φ is a Lie algebra homomorphism means that $\hat{\varphi}(x \otimes y - y \otimes x - [x,y]) = [\varphi(x), \varphi(y)] - \varphi([x,y]) = 0$ so $\hat{\varphi}$ (uniquely) factors through the quotient $\mathcal{U}(\mathfrak{g})$ of $T(\mathfrak{g})$. This gives the required $\tilde{\varphi}$. $\square$