Lec 5: algebraic groups & Lie algebras V/ Rep's of $SL_2, \mathfrak{sl}_2$, I

1) Universal enveloping algebras, cont'd
2) Representations of algebraic groups vs Lie algebras
3) Representations of $\mathfrak{sl}_2$ in characteristic 0.

1) Universal enveloping algebras, cont'd

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. In Sec 2 of Lec 4 we defined the universal enveloping algebra $\mathcal{U}(\mathfrak{g}) := \dfrac{T(\mathfrak{g})}{(x\otimes y - y\otimes x - [x,y] \mid x,y\in\mathfrak{g})}$

There's a natural Lie algebra homomorphism $\mathfrak{g}\to\mathcal{U}(\mathfrak{g})$ & we have the following universal property (also Sec 2 of Lec 2)

Lemma: Let A be an associative algebra & $\varphi:\mathfrak{g}\to A$ be a Lie algebra homomorphism. Then there is a unique associative algebra homomorphism $\tilde\varphi:\mathcal{U}(\mathfrak{g})\to A$ making the following diagram commutative:

$$\begin{tikzcd} \mathfrak{g} \arrow[d] \arrow[dr, "\varphi"] & \\ \mathcal{U}(\mathfrak{g}) \arrow[r, dashed, "\tilde\varphi"'] & A \end{tikzcd}$$

Note that one class of φ is representations $\mathfrak{g}\to\mathfrak{gl}(V)=\mathrm{End}(V)$. So a $\mathfrak{g}$-representation is the same thing as a $\mathcal{U}(\mathfrak{g})$-module (cf. reps of finite group vs group algebra modules) & a homomorphism of representations is the same thing as a $\mathcal{U}(\mathfrak{g})$-linear map.

Remark: Here's another important case of the lemma.

Let $\mathbb{F}$ be algebraically closed, G be an affine algebraic group$/\mathbb{F}$ & $\mathfrak{g}=\mathrm{Lie}(G)$. Recall, Claim (2) in Sec 0 of Lec 4, that we have a Lie algebra inclusion $\mathfrak{g}\subset D(G)$. By Lemma above it extends to an algebra homomorphism $\mathcal{U}(\mathfrak{g})\to D(G)$. This homomorphism is important & we'll return to it later.

1.1) PBW theorem

A crucial result that we'll need about $\mathcal{U}(\mathfrak{g})$ is a description of its basis. Assume for simplicity that $\dim\mathfrak{g}<\infty$. Let $x_1,\dots x_n$ be a basis in $\mathfrak{g}$. We can view any non-commutative polynomial in $x_1,\dots x_n$ as an element of $\mathcal{U}(\mathfrak{g})$. The following result is known as the Poincare-Birkhoff-Witt (PBW) theorem.

Fact: The <u>ordered</u> monomials $x_1^{\alpha_1}\dots x_n^{\alpha_n}$ form a basis in $\mathcal{U}(\mathfrak{g})$.

We'll prove this later for Lie algebras $\mathfrak{g}$ we actually care about.

Example: if $\mathfrak{g}$ is abelian, then $\mathcal{U}(\mathfrak{g})=\dfrac{T(\mathfrak{g})}{(x\otimes y-y\otimes x)}=S(\mathfrak{g})$, the symmetric algebra of $\mathfrak{g}$. The PBW theorem amounts to the classical claim that $S(\mathfrak{g})$ is the polynomial algebra $\mathbb{F}[x_1,\dots x_n]$.

2) Representations of algebraic groups vs Lie algebras

To a rational representation $\Phi: G\to GL(V)$ of an algebraic group G we assigned a representation $\varphi=T_e\Phi:\mathfrak{g}\to\mathfrak{gl}(V)$ of $\mathfrak{g}=\mathrm{Lie}(G)$.

One can ask how they are connected, e.g. if one can recover Φ from φ or tell if Φ is irreducible. Here's a basic result in this direction.

Lemma: 1) $\theta \in \mathrm{Hom}_{\mathbb{F}}(U,V)$ is G-linear $\Rightarrow \theta$ is $\mathfrak{g}$-linear.

2) $V' \subset V$ is G-stable $\Rightarrow V'$ is $\mathfrak{g}$-stable.

Proof:

1) Let $\Phi: G \to GL(V)$, $\Psi: G \to GL(U)$ be the corresponding homomorphisms. We can consider maps $g \mapsto \theta \circ \Phi(g), \Psi(g) \circ \theta : G \to \mathrm{Hom}(U,V)$. The condition that θ is G-linear means that they coincide. Their tangent maps are $\theta \circ \varphi$ & $\psi \circ \theta : \mathfrak{g} \to \mathrm{Hom}_{\mathbb{F}}(U,V)$ (b/c θ is a linear map independent of g). So θ is $\mathfrak{g}$-linear.

2) Let $P = \{ g \in GL(V) \mid gV' = V' \}$. It's an algebraic subgroup (for a basis $v_1, \dots v_n \in V$ s.t. $V' = \mathrm{Span}(v_1, \dots v_m) \Rightarrow P = \left\{ \begin{pmatrix} * & * \\ \underbrace{0}_{m} & * \end{pmatrix} \right\}$) w. Lie algebra $\mathfrak{p} = \{ x \in \mathfrak{gl}(V) \mid xV' \subset V' \}$. The claim that V' is G-stable is equivalent to Φ factoring as $G \to P \hookrightarrow GL(V)$. Hence φ factors as $\mathfrak{g} \to \mathfrak{p} \hookrightarrow \mathfrak{gl}(V)$ & V' is $\mathfrak{g}$-stable. $\square$

In particular, if U, V are not isomorphic as $\mathfrak{g}$-representations, then they are not isomorphic as G-representations. And if V is irreducible over $\mathfrak{g}$, then it's irreducible over G.

Question: Are $\Leftarrow$ in 1) & 2) true?

Answer 1: not in general: if G is finite (a stupid exemple of an algebraic group), then $\mathfrak{g} = \{0\}$ & $\Leftarrow$ fails.

To remedy this assume G is irreducible as a variety ($\Leftrightarrow \mathbb{F}[G]$ is a domain). Easy examples include GL_n ($\mathbb{F}[GL_n] = \mathbb{F}[x_{ij}][\det^{-1}]$) & SL_2 ($\mathbb{F}[SL_2] = \mathbb{F}[a,b,c,d]/(ad-bc-1)$: $ad-bc-1$ is an irreducible polynomial – exercise). More advanced examples include SL_n, SO_n, Sp_n – but O_n is reducible, it has two components w. $\det = 1$ & $\det = -1$.

Answer 2: $\Leftarrow$ may fail if $\operatorname{char} \mathbb{F} = p$ even for irreducible G. Namely consider $Fr: GL_n \to GL_n, (a_{ij}) \mapsto (a_{ij}^p)$ (that we view as a representation). We claim that $T_e Fr = 0$. Indeed, $[T_e Fr(\xi)](x_{ij}) = \xi(Fr^* x_{ij} = x_{ij}^p)$ $= p x_{ij}(e)^{p-1} \xi(x_{ij}) = 0$. Since $T_e Fr = 0$, $\Leftarrow$ (miserably) fails.

More generally for a representation $\varphi: G \to GL(V)$ we consider its Frobenius twist $\varphi^{(1)} = Fr \circ \varphi$. Then $T_e \varphi^{(1)} = 0$ (by chain rule)

Answer 3: If $\operatorname{char} \mathbb{F} = 0$ & G is irreducible as a variety, then $\Leftarrow$ in 1) & 2) holds.

We'll elaborate on why this is the case later in the course.

To summarize, if $\operatorname{char} \mathbb{F} = 0$, the representations of G & $\mathfrak{g}$ are very closely related. If $\operatorname{char} \mathbb{F} = p$, there's still a connection, but it's less immediate & holds for less basic reasons.

3) Representations of $\mathfrak{sl}_2$ in characteristic 0.

We proceed to the 2nd major part of the course: the study of rational representations of the algebraic group $SL_2(\mathbb{F})$ & finite dimensional representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$. Here $\mathbb{F}$ is an algebraically closed field. Before getting to details, we'll put this study in some context.

3.1) Simple algebraic groups & Lie algebras

As a part of the general ideology, we care about the structure and representation theory of "simple" algebraic groups & their relatives ("semisimple" & "reductive") groups.

Definition 1: An algebraic group G is simple if

1) G is irreducible as a variety
2) $\forall$ normal algebraic subgroup of G is finite (most important condition).
3) G is not abelian (modulo 1) & 2), this excludes exactly $\mathbb{G}_a$ & $\mathbb{G}_m$).

For example, SL_n, $n\geq 2$, SO_n, $n=3$ or $n\geq 5$, Sp_{2n}, $n\geq 1$, are simple (SO_4 is "semisimple" but not simple). Basically, there are just five more examples, the exceptional groups: G_2, F_4, E_6, E_7, E_8. We may discuss more on that much later.

We now proceed to the simplicity of Lie algebras

Definition 2: Let $\mathfrak{g}$ be a Lie algebra.

1) By an ideal in $\mathfrak{g}$ we mean a subspace $\mathfrak{h}\subset\mathfrak{g}$ s.t. $x\in\mathfrak{g}, y\in\mathfrak{h} \Rightarrow [x,y]\in\mathfrak{h}$. As usual, these are exactly the possible kernels of Lie algebra homomorphisms.

2) $\mathfrak{g}$ is simple if its only ideals are $\{0\}$ & $\mathfrak{g}$ & $\mathfrak{g}$ is not 1-dim. abelian.

Example: Consider $\mathfrak{g} = \mathfrak{sl}_2$. It has basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ w. commutation relations

(1) $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$

These relations imply that if char $\mathbb{F} \neq 2$, then any nonzero ideal $\mathfrak{h} \subset \mathfrak{g}$ contains both e & f, hence h (e.g $\forall\, x \neq 0$ one of x, $[x,e]$, $[x,[x,e]]$ is $\neq 0$-multiple of e). So $\mathfrak{sl}_2$ is simple. On the other hand if char $\mathbb{F} = 2$, then $[h,x] = 0$ $\forall\, x \in \mathfrak{sl}_2$, hence $\mathbb{F}h \subset \mathfrak{sl}_2$ is an ideal.

The following exercise connects the notions of simplicity for groups & algebras.

Exercise: If $H \subset G$ is a normal algebraic subgroup, then $\mathrm{Lie}(H) \subset \mathrm{Lie}(G)$ is an ideal (hint: $\mathrm{Lie}(H)$ is $\mathrm{Ad}(G)$-stable hence $\mathrm{ad}(\mathfrak{g})$-stable).

From here one can deduce that for irreducible G, if $\mathfrak{g}$ is simple, then G is simple. If char $\mathbb{F} = 0$, then the converse is true as well (but this may fail in positive characteristic).

The algebraic group SL_2 and its Lie algebra $\mathfrak{sl}_2$ are the simplest simple algebraic group and Lie algebra (e.g of smallest possible dimension). We will study

1) The representation theory of $\mathfrak{sl}_2$ when char $\mathbb{F} = 0$.

2) The representation theory of $\mathfrak{sl}_2$ for char $\mathbb{F} > 2$.

3) The representation theory of SL_2 for char $\mathbb{F}\neq 2$ (the case of char $\mathbb{F}=2$ is essentially the same as other positive characteristics).

These cases already illustrate the essential features of the representation theory of (semi)simple algebraic groups and their Lie algebras (but have none of the complexity of the general case), 1 & 3 are also used to understand the general case.

3.2) Some relations in $U(\mathfrak{sl}_2)$

For now we place no restrictions on $\mathbb{F}$. We have

$$U(\mathfrak{sl}_2)=\underbrace{\mathbb{F}\langle e,h,f\rangle}_{\text{free algebra w. generators } e,h,f}/([h,e]=2e,\ [h,f]=-2f,\ [e,f]=h)$$

Lemma: In $U(\mathfrak{sl}_2)$ we have the following identities:

$$P(h)e=eP(h+2),\ \forall\ P\in\mathbb{F}[x] \qquad (i)$$

$$P(h)f=fP(h-2),\ \forall\ P\in\mathbb{F}[x] \qquad (ii)$$

$$e^mf^n=\sum_{j=0}^{\min(m,n)}\binom{m}{j}\binom{n}{j}j!\ f^{n-j}\Big(\prod_{i=0}^{j-1}(h-(m+n)+2j-i)\Big)e^{m-j} \qquad (iii)$$

Sketch of proof: (i): enough to assume $P(h)=h^n$. Then $h^ne=[he=e(h+2)]$ $=h^{n-1}e(h+2)=\ldots=e(h+2)^n$. (ii) is similar.

To prove (iii) we move e past f: $ef^n=[e,f]f^{n-1}+fef^{n-1}=hf^{n-1}+fhf^{n-2}+$ $+\ldots f^{n-1}h+f^ne=[(ii)]\ f^{n-1}(h-2(n-1)+h-2(n-2)+\ldots+h)+f^ne=nf^{n-1}(h-(n-1))$ $+f^ne$, which is (iii) for $m=1$. The general case is by induction on m using (i) & some combinatorics, the details are left as exercise. $\square$

We'll need two special cases of (iii): $m=1$ (in the proof) & $m=n$:

$$ef^n = nf^{n-1}(h-(n-1)) + f^n e \qquad \text{(iii')}$$

$$e^n f^n = \underbrace{ae}_{\text{terms w. } j<n} + \underbrace{n!\prod_{i=0}^{n-1}(h-i)}_{j=n} \quad \text{for some } a \in \mathcal{U}(\mathfrak{sl}_2) \qquad \text{(iii'')}$$