Lec 6: Reps of SL_2 & $\mathfrak{sl}_2$, II

0) Introduction.
1) Classification of irreducibles
2) Complete reducibility

0) Introduction.

Let $\mathbb{F}$ be an algebraically closed field, char $\mathbb{F}=0$. Our goal is to describe the finite dimensional representations of $\mathfrak{g}=\mathfrak{sl}_2$. We'll see that:

- the irreducibles are in bijection w. $\mathbb{Z}_{\geqslant 0}$: $\forall n\in\mathbb{Z}_{\geqslant 0}$ $\exists!$ (up to iso) $\mathfrak{g}$-irrep $L(n)$ w. $\dim L(n)=n+1$.
- all finite dimensional representations are completely reducible $\Leftrightarrow$ isomorphic to direct sums of irreducible.

To aid us with these tasks we obtained (Sec 3.2 of Lec 5) the following formulas in $\mathcal{U}(\mathfrak{g})$ (where $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}\in\mathfrak{g}$):

$$P(h)e=eP(h+2),\ \forall\ P\in\mathbb{F}[x] \qquad (i)$$

$$P(h)f=fP(h-2),\ \forall\ P\in\mathbb{F}[x] \qquad (ii)$$

$$ef^n=nf^{n-1}(h-(n-1))+f^ne \qquad (iii')$$

$$e^nf^n=ae+n!\prod_{i=0}^{n-1}(h-i) \text{ for some } a\in\mathcal{U}(\mathfrak{sl}_2) \qquad (iii'')$$

1) Classification of irreducibles

1.1) Weight decomposition

Let $\mathbb{F}$ be an algebraically closed field, V a fin. dim. $\mathbb{F}$-vector space & $A\in\operatorname{End}_{\mathbb{F}}(V)$. Then to $\lambda\in\mathbb{F}$ we assign the generalized λ-eigenspace $V_\lambda(A)=\{v\in V \mid \exists\, n>0\ (A-\lambda\,\mathrm{Id})^n v=0\}$. It's known (e.g.

from the JNF theorem) that $V=\bigoplus_{\lambda\in\mathbb{F}} V_\lambda(A)$ ($V_\lambda(A)\neq\{0\} \Leftrightarrow \lambda$ is an e-value).

Now let V be a finite dimensional representation of $\mathfrak{g}$.

Definition: The λ-weight space in V is $V_\lambda := V_\lambda(h)$.
We say: λ is a weight of V if $V_\lambda \neq \{0\}$.

Note that $V=\bigoplus_{\lambda\in\mathbb{F}} V_\lambda$.

Lemma: $eV_\lambda \subset V_{\lambda+2}$, $fV_\lambda\subset V_{\lambda-2}$.

Proof: Let $v\in V_\lambda$. Need to prove $\exists\, m>0$ s.t. $(h-(\lambda+2))^m ev =$ [(i) in Sec 0] $= e(h-\lambda)^m v = 0$ for $m\gg 0$ $\Rightarrow eV\subset V_{\lambda+2}$; $fV\subset V_{\lambda-2}$ follows from (ii) there. $\square$

1.2) Highest weight.

Until the end of the lecture, assume char $\mathbb{F}=0$

Definition: A weight λ of V is called a highest weight if $\lambda+2k$ is not a weight of V $\forall\, k>0$.

Note that this notion is meaningless if char $\mathbb{F}=p$ ($\lambda+2p=\lambda$)
Since $\dim V<\infty$, the set of weights of V is finite, so $\exists$ highest weight (if $V\neq\{0\}$).

Proposition 1: Let λ be a highest weight of V & $v \in V_\lambda$. Then
(1) $ev=0$.
(2) $\lambda \in \mathbb{Z}_{\geq 0}$ & $hv=\lambda v$ (i.e. V_λ is an eigenspace for h).

Proof: By Lemma in Sec 1.1, $ev \in V_{\lambda+2} = \{0\}$ b/c λ is highest. This proves (1). To prove (2) observe that there's $n>0$ s.t. $\lambda-2n$ is not a weight of V - b/c the set of weights is finite. So $f^n v \in V_{\lambda-2n} = \{0\}$. Consider the vector $e^n f^n v = 0$. By (iii'') in Sec 0, the l.h.s. equals $aev + n!\left(\prod_{i=0}^{n-1}(h-i)\right)v = 0$. Since $ev=0$, we get $h(h-1)\dots(h-n+1)v=0$. Since $v \in V_\lambda$, $(h-i)$ is an invertible operator on V_λ unless $i=\lambda$. So $\lambda = i \in \{0,1,\dots n-1\}$ & $(h-i)$ acts by 0 on $V_\lambda \rightsquigarrow 2)$ $\square$

1.3) Verma modules.

It turns out that if V is, in addition, irreducible, then it has unique highest wt, and sending an irrep to its highest weight gives a bijection between fin. dim. $\mathfrak{g}$-irreps & $\mathbb{Z}_{\geq 0}$ mentioned in Sec 0

In order to prove these claims we need to study the $\mathcal{U}(\mathfrak{g})$-module w. one generator, v_λ, & relations $ev_\lambda=0$, $hv_\lambda = \lambda v_\lambda$ ($\lambda \in \mathbb{F}$). Here's a more formal

Definition: Let $\lambda \in \mathbb{F}$. By the Verma module $\Delta(\lambda)$ we mean the quotient $\Delta(\lambda) = \mathcal{U}(\mathfrak{g})/I_\lambda$, where $I_\lambda := \mathrm{Span}_{\mathcal{U}(\mathfrak{g})}(e, h-\lambda)$.

Proposition 2: 1) Universal property: $\forall$ $\mathcal{U}(\mathfrak{g})$-module V

$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\Delta(\lambda), V) \xrightarrow{\sim} \{v \in V \mid ev=0,\ hv=\lambda v \Leftrightarrow I_\lambda v = 0\},\ \varphi \mapsto \varphi(1+I_\lambda)$.

2) Basis: if $v_\lambda := 1+I_\lambda \in \Delta(\lambda)$, then the vectors $f^i v_\lambda$, $i \geqslant 0$, form a basis in $\Delta(\lambda)$.

3) Submodules: $\Delta(\lambda)$ is irred. if $\lambda \notin \mathbb{Z}_{\geqslant 0}$ and has a unique proper submodule, $K(\lambda) := \mathrm{Span}(f^i v_\lambda \mid i \geqslant \lambda+1)$ else.

Proof: 1) – the isomorphism is given by $\varphi \mapsto \varphi(v_\lambda)$, details are left as exercise

2) By the PBW theorem, the elements $f^k h^\ell e^m$, $k,\ell,m \geqslant 0$, form a basis in $\mathcal{U}(\mathfrak{g})$. We claim that the elements $f^k(h-\lambda)^\ell e^m$ also form a basis. This is because the binomial formula for $h^\ell = (h-\lambda+\lambda)^\ell$ gives a unique expansion for $f^k h^\ell e^m$ in terms of $f^{k'}(h-\lambda)^{\ell'}e^{m'}$, details are an exercise.

Note that $I_\lambda = \mathrm{Span}_{\mathbb{F}}(f^k(h-\lambda)^\ell e^{m+1}, f^k(h-\lambda)^\ell e^m(h-\lambda)) = \mathrm{Span}_{\mathbb{F}}(f^{k'}(h-\lambda)^{\ell'}e^{m'} \mid m'>0 \text{ or } [m'=0 \ \& \ \ell'>0]) \Rightarrow f^k + I_\lambda = f^k v_\lambda$ form a basis in $\Delta(\lambda)$.

3) We have $(h-(\lambda-2i))f^i v_\lambda = [(ii) \text{ in Sec } 0] = f^i(h-\lambda)v_\lambda = 0$. We claim that any submodule $N \subsetneq \Delta(\lambda)$ is the span of some $f^i v_\lambda$ (if $v = \sum_{i=1}^{j} a_{\ell_i} f^{\ell_i} v_\lambda \in N$ w. $a_{\ell_i} \neq 0$ & pairwise distinct ℓ_i $\Rightarrow (h-\lambda+2\ell_j)v = \sum_{i=1}^{j-1} 2(\ell_j-\ell_i)a_{\ell_i} f^{\ell_i} v_\lambda \in N$, then we induct on j)

Also if $f^i v_\lambda \in N$, then $f^{i+1}v_\lambda \in N$. So $N = \mathrm{Span}(f^i v_\lambda \mid i \geqslant k)$ for some $k>0$. if $N \neq \{0\}$. If $N \neq \Delta(\lambda)$, then $k>0$ & $ef^k v_\lambda = 0$. By (iii') in Sec 0, $ef^k v_\lambda = f^k e v_\lambda + k f^{k-1}(h-(k-1))v_\lambda = 0 + k(\lambda-(k-1))f^{k-1}v_\lambda$

Note that $ef^k v_\lambda \in N$, while $k(\lambda-(k-1))f^{k-1}v_\lambda \in N \Leftrightarrow \lambda = k-1$. In partiular, for $\lambda \neq k-1$, there is no proper submodule, while for $\lambda = k-1$, $K(\lambda) = \mathrm{Span}(f^i v_\lambda \mid i \geq \lambda+1)$ is the unique proper submodule. 3) is proved. $\square$

1.4) Modules $L(\lambda)$.

For $\lambda \in \mathbb{Z}_{\geq 0}$ set $L(\lambda) = \Delta(\lambda)/K(\lambda)$. Note that $\dim K(\lambda) = \infty$, $\dim L(\lambda) = \lambda+1$. Set $\ell_\lambda := v_\lambda + K(\lambda) \in L(\lambda)$.

Thm: 1) $L(\lambda), K(\lambda)$ are irreducible $\forall \lambda \geq 0$.

2) Universal property: $\forall$ finite dimensional $\mathcal{U}(\mathfrak{g})$-module V

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(L(\lambda), V) \xrightarrow{\sim} \{v \in V \mid ev = 0, hv = \lambda v\}, \varphi \mapsto \varphi(\ell_\lambda)$$

3) $\lambda \mapsto L(\lambda)$ is a bijection between $\mathbb{Z}_{\geq 0}$ & the set of isom. classes of finite dimensional irreducible $\mathfrak{sl}_2$-reps.

Proof:

1): follows from 3) of Proposition 2 (exercise).

2): We'll use the following general property of irreps

(*) $\forall$ nonzero homomorphism from (resp. to) irrep is injective (resp. surjective).

Since $\dim K(\lambda) = \infty$, 1) $\Rightarrow$ $\forall$ $\mathfrak{g}$-linear $K(\lambda) \to V$ is 0 if $\dim V < \infty$. So $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(L(\lambda), V) \xrightarrow{\sim} \{\varphi: \Delta(\lambda) \to V \mid \varphi(K(\lambda)) = 0\} = \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(\Delta(\lambda), V)$ & we are done by 1) of Proposition 2.

3) Injectivity: $\lambda \neq \lambda' \Rightarrow \dim L(\lambda) \neq \dim L(\lambda') \Rightarrow L(\lambda) \not\cong L(\lambda')$

Surjectivity: let V be irrep, $\lambda \in \mathbb{Z}_{\geq 0}$ be a highest weight & $v \in V_\lambda$,

$v\neq 0$. By Proposition 1, $ev=0$ & $hv=\lambda v$. By 2) of Thm, $\exists !$ homom. $\varphi: L(\lambda)\to V$ w. $\varphi(\ell_\lambda)=v$. It's nonzero, so isomorphism by (*) $\square$

2) Complete reducibility

Thm: Let V be a finite dimensional $\mathfrak{sl}_2$-rep. Then $V\simeq\bigoplus$ of irreps.

Proof:

Step 1, Casimir element: Set $C=fe+ef+\frac{1}{2}h^2=2fe+\frac{1}{2}h^2+h\in U(\mathfrak{g})$. Then (important exercise)

(**) $\qquad C$ commutes w. e,h,f.

Step 2, C acts on $\Delta(\lambda)$, hence on $L(\lambda)$ by scalar $(\frac{1}{2}\lambda^2+\lambda)$: $\Delta(\lambda)=\mathrm{Span}(f^iv_\lambda)$ by 2) of Prop 2 & $Cf^iv_\lambda=[Cf=fC]=f^iCv_\lambda=[ev_\lambda=0, hv_\lambda=\lambda v_\lambda]=(\frac{1}{2}\lambda^2+\lambda)f^iv_\lambda$.

Step 3 (decomposition). (**) $\Rightarrow\forall\mu\in\mathbb{F}$ the generalized e-space $V_\lambda(C)$ is e,h,f-stable (i.e. subrep). Thx to $V=\bigoplus_\mu V_\mu(C)$, we can reduce to $V=V_\mu(C)$ for some μ.

Step 4: Let λ be a highest weight of $V=V_\mu(C)$ & $v\in V_\lambda\setminus\{0\}\rightsquigarrow$ nonzero hence injective $L(\lambda)\hookrightarrow V\Rightarrow C$ acts on $L(\lambda)$ w. e-value $\mu\Rightarrow$ [Step 2] $\mu=\frac{1}{2}\lambda^2+\lambda$. Note that equation in λ, $\mu=\frac{1}{2}\lambda^2+\lambda$, has unique solution $\geqslant 0\Rightarrow\lambda$ is uniquely determined by μ.

Step 5: Let $v_1,\dots v_k\in V_\lambda$ be basis; 2) of Thm $\rightsquigarrow\varphi_i: L(\lambda)\to V$, $\varphi_i(\ell_\lambda)=v_i\rightsquigarrow\varphi: U=L(\lambda)^{\oplus k}\to V$, $\varphi(u_1,\dots u_k)=\sum_{i=1}^k\varphi_i(u_i)$. We'll show φ is an isomorphism.

Step 6: Note that $L(\lambda)=\mathrm{Span}(f^i\ell_\lambda\,|\,i=0,\dots\lambda)\Rightarrow L(\lambda)_\lambda=\mathbb{F}\ell_\lambda\Rightarrow$ U_λ has basis $\ell_{\lambda,1},\dots\ell_{\lambda,k}$ & $\varphi(\ell_{\lambda,i})=v_i$ -basis in $V_\lambda\Rightarrow\varphi: U_\lambda\xrightarrow{\sim}V_\lambda$.

Step 7 (φ is injective) $K := \ker\varphi$; $\varphi : U_\lambda \xrightarrow{\sim} V_\lambda \Rightarrow K_\lambda = U_\lambda \cap K = \{0\}$.

C acts on K by $\frac{1}{2}\lambda^2+\lambda$. By Step 4, λ is the only possible highest weight & since $K_\lambda = \{0\} \Rightarrow K = \{0\}$

Step 8 (φ is surjective) Let $N = \operatorname{coker}\varphi\,(= V/\operatorname{im}\varphi)$. By Step 6, $V_\lambda \subset \operatorname{im}\varphi \Rightarrow \bigoplus_{\lambda_1 \neq \lambda} V_{\lambda_1} \twoheadrightarrow N \Rightarrow \lambda$ is not e-value of h in $N \Leftrightarrow N_\lambda = \{0\}$.
Then we argue as in Step 7 to show $N = \{0\}$ finishing the proof. $\square$