Lec 7: Reps of SL_2 & $\mathfrak{sl}_2$, III

0) Overview

$\mathbb{F}$ continues to be algebraically closed field.

In Lec 6 we've completely described the finite dimensional representations of $\mathfrak{sl}_2(\mathbb{F})$ when $\operatorname{char}\mathbb{F}=0$, the structure is relatively easy: the irreducibles are in bijection w. $\mathbb{Z}_{\geqslant 0}$ via taking the highest weight and a general finite dimensional representation is completely reducible.

When $\operatorname{char}\mathbb{F}=p\,(>2)$, the situation is much more involved: the representations may fail to be completely reducible & there are uncountably many of them. One can ask why we should care about mod p representations. Here are somewhat subjective & personal answers:

a) The representations of simple algebraic groups & their Lie algebras in characteristic p are related in various ways to representations of "affine Kac-Moody algebras" (in char 0), a class of infinite dimensional Lie algebras important for Math Physics & Langlands program.

b) In the case of Lie algebras, the representation theory has rich and fruitful connections to certain geometric objects ("Springer resolutions", "Springer fibers" etc.) The same geometric objects appear in many other areas of Representation theory (in char 0) the study of representations of "finite groups of Lie type" (example of such: $GL_n(\mathbb{F}_q)$), "p-adic groups" ($GL_n(\mathbb{F}_q((t)))$) & "category $\mathcal{O}$." We'll study the 1st & the 3rd to some extent - although w/o connections to Geometry

Besides these two connections, there's also:

c) Applications to study congruences of Fourier coefficients of modular forms.

Our main goal in this & the next lecture is to classify the finite dimensional irreducible representations of $\mathfrak{sl}_2(\mathbb{F})$.

1) More central elements

1.0) Center & central character.

Definition: Let A be an associative $\mathbb{F}$-algebra. The center, $Z(A)$, is $\{z \in A \mid az = za \ \forall a \in A\}$, it's an $\mathbb{F}$-subalgebra of A.

The reason we care about the centers in Representation theory is the following consequence of the Schur lemma.

Claim/definition: Let V be a finite dimensional irreducible A-module. Then $\forall\, z \in Z(A)$ acts on V by a scalar to be denoted by $\chi_V(z)$.

The map $\chi_V: Z(A) \to \mathbb{F}$ is an algebra homomorphism called the central character of V.

It's convenient & useful to partition finite dimensional irred. A-modules according to their central characters.

Example: In Section 2 of Lec 6, we have constructed the Casimir element $C = ef + fe + \frac{1}{2}h^2 \in U(\mathfrak{sl}_2)$ satisfying $aC = Ca$ for $a = e, h, f$. Since e, h, f generate $U(\mathfrak{g})$, we get $C \in Z(U(\mathfrak{g}))$. In fact, as we'll see much later, $Z(U(\mathfrak{g})) = \mathbb{F}[C]$ if char $\mathbb{F} = 0$. We've seen that C acts on the irrep $L(\lambda)$ ($\lambda \in \mathbb{Z}_{\geqslant 0}$) by $\frac{1}{2}\lambda^2 + \lambda$, which completely determines $\chi_{L(\lambda)}$.

One feature (and not a bug) of char p setting is that $Z(U(\mathfrak{g}))$ becomes much larger.

1.1) Construction

Let G be an algebraic group over $\mathbb{F}$ (that admits an embedding to some $GL(V)$ but, as discussed in Sec 2 of Lec 2, this is automatic). Let $\mathfrak{g} = \mathrm{Lie}(G)$. Theorem in Sec 2 of Lec 2 shows that $x \in \mathfrak{g}$ ($\subset \mathfrak{gl}(V)$) $\Rightarrow x^p \in \mathfrak{g}$. The resulting map $\mathfrak{g} \to \mathfrak{g}$ to be denoted by $x \mapsto x^{[p]}$ is independent of the choice of an embedding $G \hookrightarrow GL(V)$.

On the other hand, we consider x as an element of $U(\mathfrak{g}) \rightsquigarrow x^p \in U(\mathfrak{g})$. The following is the main result of this section

Theorem: $x\in\mathfrak{g} \Rightarrow x^p - x^{[p]} \in Z(U(\mathfrak{g}))$

Example: Let $G = SL_2$. Consider its tautological embedding into GL_2. Then $e^{[p]} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^p = 0$. Similarly, $f^{[p]} = 0$ & $h^{[p]} = h$. One can check directly (exercise) that $e^p, h^p - h, f^p$ commute w. $\forall x\in\mathfrak{g}$, hence are central in $U(\mathfrak{sl}_2)$.

In the proof of Thm we'll need a lemma. Let A be an associative algebra over $\mathbb{F}$. For $x\in A$ we write ad_x for the element of $End_{\mathbb{F}}(A)$ given by $y \mapsto xy - yx$.

Lemma: $(ad_x)^p = ad_{x^p}$.

Proof: Let $\ell_x, r_x \in End_{\mathbb{F}}(A)$ be given by $\ell_x(y) = xy$, $r_x(y) = yx$. In particular, $\ell_x r_x = r_x \ell_x$ (from associativity) & $ad_x = \ell_x - r_x$. Then $(ad_x)^p = (\ell_x - r_x)^p = [\ell_x r_x = r_x \ell_x \Rightarrow$ binomial formula applies$] = (\ell_x)^p + \sum_{i=1}^{p-1} \underbrace{\binom{p}{i}}_{=0 \text{ in } \mathbb{F}} (-1)^i \ell_x^{p-i} r_x^i - (r_x)^p = [(\ell_x)^p = \ell_{x^p}$, same for r?$] = \ell_{x^p} - r_{x^p} = ad_{x^p}$ □

Proof of Thm: The elements $y\in\mathfrak{g}$ generate $U(\mathfrak{g})$ so it's enough to show $x^p - x^{[p]}$ commutes w. y $\forall x,y\in\mathfrak{g} \Leftrightarrow x^p y - y x^p = [x^{[p]}, y]$. If $\mathfrak{g}\subset A$ as a Lie subalgebra in an associative algebra A, then for $x,y\in\mathfrak{g}$, $ad_{x^p}(y) = ad_x^p(y)$ in A. But the r.h.s. is the commutator in $\mathfrak{g}$, so doesn't depend on A. For $A = U(\mathfrak{g})$, we get $x^p y - y x^p = ad_x^p(y)$ & for $A = End(V)$, we get $ad_{x^{[p]}}(y) = ad_x^p(y) \Rightarrow x^p y - y x^p = [x^{[p]}, y]$ □

2) Baby Verma modules

Let V be a finite dimensional irreducible $\mathfrak{g}$-module so that we can assign its central character $\chi_V: Z(U(\mathfrak{g})) \to \mathbb{F}$. By the p-character of V we mean the triple $(\chi_e, \chi_h, \chi_f) \in \mathbb{F}^3$, where for $x=e,h,f$, we have $\chi_x := \chi_V(x^p - x^{[p]})$ (for the time being - in the next lecture we'll describe the p-character more conceptually). In this section we will classify V with the following p-characters:

- $(0,0,0)$
- $(0,0,1)$
- $(0,a,0)$, $a \neq 0$.

The reason for these is that the general case reduces to one of these cases, we'll explain how in the next lecture.

And to do the classification for these p-characters, we need a family of representations known as baby Verma modules.

2.1) Construction & basic properties

For $\lambda \in \mathbb{F}$ one defines the Verma module $\Delta(\lambda) = U(\mathfrak{g})/\underbrace{U(\mathfrak{g})(h-\lambda, e)}_{I_\lambda}$

The following properties hold just as in characteristic 0 (Sec 1.3 in Lec 6):

I) Universal property: $\forall$ $U(\mathfrak{g})$-module V:

$$\mathrm{Hom}_{U(\mathfrak{g})}(\Delta(\lambda), V) \xrightarrow{\sim} \{v \in V \mid ev=0, hv=\lambda v\}, \; \varphi \mapsto \varphi(v_\lambda) \; (v_\lambda = 1 + I_\lambda)$$

II) Basis: $f^i v_\lambda$ $(i \in \mathbb{Z})$ w. $hf^i v_\lambda = (\lambda - 2i)v_\lambda$, $ef^i v_\lambda = i(\lambda - i + 1) f^{i-1} v_\lambda$.

And here are new features of characteristic p

i) Since $e^p v_\lambda = 0$ & $(h^p - h)v_\lambda = (\lambda^p - \lambda)v_\lambda$ & the elements $e^p, h^p - h$ are central, e^p acts by 0 and $h^p - h$ acts by $\lambda^p - \lambda$ on the entire module $\Delta(\lambda)$ ($= \mathrm{Span}_{\mathbb{F}}(f^i v_\lambda)$).

ii) $\Delta(\lambda)$ is a free module over the subalgebra $\mathbb{F}[f^p] \subset Z(U(\mathfrak{g}))$ w. basis $f^i v_\lambda$ $(0 \le i \le p-1)$

Note that $\forall\ z \in \mathbb{F} \Rightarrow (f^p - z)\Delta(\lambda) \subset \Delta(\lambda)$ is a $\mathfrak{g}$-subrep & the quotient $\underline{\Delta}^z(\lambda) := \Delta(\lambda)/(f^p - z)\Delta(\lambda)$ has dimension p. This quotient is called the baby Verma module. The following result is an analog of Prop 2 from Lec 6 for baby Vermas modules

Proposition: 0) Let v^i be the image of $f^i v_\lambda$ in $\underline{\Delta}^z(\lambda)$ for $i = 0, \dots, p-1$. These elements form a basis in $\underline{\Delta}^z(\lambda)$ &

$$e v^i = i(\lambda - i + 1)v^{i-1}$$

$$h v^i = (\lambda - 2i)v^i$$

$$f v^i = \begin{cases} v^{i+1} & \text{if } i \neq p-1 \\ z v^0 & \text{if } i = p-1 \end{cases}$$

1) Universal property: let V be a module over $U(\mathfrak{g})$ s.t. $f^p - z$ acts by 0. Then

$$\mathrm{Hom}_{U(\mathfrak{g})}(\underline{\Delta}^z(\lambda), V) \xrightarrow{\sim} \{v \in V \mid ev = 0,\ hv = \lambda v\},\quad \varphi \mapsto \varphi(\underline{v}^0).$$

2) $\ker e$: If $\lambda \in \{0, \dots, p-2\}$, then $\ker_{\underline{\Delta}^z(\lambda)}(e) = \mathrm{Span}_{\mathbb{F}}(v^0, v^{\lambda+1})$, otherwise $\ker_{\underline{\Delta}^z(\lambda)}(e) = \mathbb{F}v^0$.

3) Submodules: $\underline{\Delta}^z(\lambda)$ is irreducible if $z\neq 0$ or $\lambda\notin\{0,\dots,p-2\}$; for $z=0$ and $\lambda\in\{0,\dots,p-2\}$, $\underline{\Delta}^z(\lambda)$ has unique proper submodule: $\underline{K}(\lambda):=\mathrm{Span}_{\mathbb{F}}(v^i \mid i>\lambda)$

Sketch of proof: 0), 1), 2) are exercises (hints: 0) follows from I), 1) from II & 2) from 0)). To prove 3) note that:

- The weight spaces of h are 1-dim'l so as in the proof of 3) of Proposition 2 in Lec 6, any h-stable subspace of $\underline{\Delta}^z(\lambda)$ is spanned by some $\underline{v}^i$.
- If $z\neq 0$, then $\mathrm{Span}(f^j\underline{v}^i \mid j=0,\dots,p-1)=\underline{\Delta}^z(\lambda)$, which settles this case. If $z=0$, then $\mathrm{Span}(f^j\underline{v}^i)=\mathrm{Span}(\underline{v}^i,\dots,\underline{v}^{p-1})$
- If $\lambda\notin\{0,\dots,p-2\}$, $\mathrm{Span}(e^j\underline{v}^i)=\mathrm{Span}(\underline{v}^i,\dots,\underline{v}^0)$. This & previous part $\Rightarrow\underline{\Delta}(\lambda)$ is irreducible. The case of $\lambda\in\{0,\dots,p-2\}$ is handled just like for Verma modules in char 0, Sec. 1.3 of Lec 6. □

2.2) Classification of irreps.

We need the following Lemma.

Lemma: Let V be a finite dimensional $U(\mathfrak{g})$-module s.t. e^p acts by 0 & h^p-h by $a\in\mathbb{F}$. Then $\exists\, v\in V \mid ev=0$ & $hv=\lambda v$, where $\lambda^p-\lambda=a$.

Proof:

Since $e^p=0$, the only e-value of e in V is 0, hence $\ker_V(e)\neq 0$. Since $eh=he-2e$, $\ker_V(e)$ is h-stable, so $\exists$ e-vector $v\in\ker_V(e)$. The e-value λ must satisfy $\lambda^p-\lambda=a$. □

The following is our main classification result.

Thm: 1) $\mathfrak{g}$-irreducibles with p-character $(0,0,0)$ are in bijection w. $\{0,1,..,p-1\}$ via $\lambda \mapsto L(\lambda) := \underline{\Delta}^0(\lambda)/\underline{K}(\lambda)$ (where $\underline{K}(p-1)=\{0\}$).

2) Every $\mathfrak{g}$-irreducible w. p-character $(0,0,1)$ is of the form $\underline{\Delta}^1(\lambda)$ w. $\lambda\in\{0,1,..,p-1\}$, and $\underline{\Delta}^1(\lambda)\simeq\underline{\Delta}^1(\lambda') \Leftrightarrow \lambda+\lambda'=-2$ in $\mathbb{F}_p$ (so that there are $(p+1)/2$ pairwise non-isomorphic irreps).

3) $\mathfrak{g}$-irreducibles with p-character $(0,a,0)$ for $a\in\mathbb{F}\setminus\{0\}$ are in bijection w. $\{\lambda\in\mathbb{F} \mid \lambda^p-\lambda=a\}$ via $\lambda\mapsto\underline{\Delta}^0(\lambda)$

Proof:

1) Let V be a $\mathfrak{g}$-irrep killed by e^p, h^p-h, f^p. By Lemma, $\exists\, v\in V\setminus\{0\}$ w. $ev=0$, $hv=\lambda v$ for λ satisfying $\lambda^p-\lambda=0 \Leftrightarrow \lambda\in\mathbb{F}_p$. By 1) of Prop. we have a nonzero hence surjective homomorphism $\underline{\Delta}^0(\lambda)\twoheadrightarrow V$. The only possibility for the kernel is $\underline{K}(\lambda) \Rightarrow V\simeq L(\lambda)$. The modules $L(\lambda)$ are pairwise non-isomorphic b/c $\dim L(\lambda)=\lambda+1$ (here we view $\lambda\in\mathbb{Z}_{\geq 0}$).

2) Similarly to 1), if e^p, h^p-h, f^p-1 acts by 0 on $V \Rightarrow V\simeq\underline{\Delta}^1(\lambda)$ for $\lambda\in\mathbb{F}_p$. To figure out when two such modules are isomorphic, notice that by 2) of Proposition, for $\lambda\neq p-1$, the e-values of h in $\ker_{\underline{\Delta}^1(\lambda)}(e)$ are λ & $-2-\lambda \rightsquigarrow$ nonzero homomorphism $\underline{\Delta}^1(-2-\lambda)\to\underline{\Delta}^1(\lambda)$ that must be an isomorphism. And if $\underline{\Delta}^1(\lambda)\xrightarrow{\sim}\underline{\Delta}^1(\lambda') \Rightarrow \{\lambda,-2-\lambda\}=\{\lambda',-2-\lambda'\} \Leftrightarrow \lambda'=\lambda$ or $-2-\lambda$.

3) exercise. □

Exercise: $\underline{K}(\lambda)\simeq L(-2-\lambda)$ & $\underline{\Delta}^0(\lambda)$ is not completely reducible for $\lambda\in\mathbb{F}_p\setminus\{p-1\}$.