Lec 8: Reps of SL_2 & $\mathfrak{sl}_2$, IV.

1) Representations of $\mathfrak{sl}_2(\mathbb{F})$, finished
2) Representations of $SL_2(\mathbb{F})$

1.0) Recap & roadmap

Let $\mathbb{F}$ be an algebraically closed field of char $=p>2$.

We saw in Sec 1.1 of Lec 7 that the elements $e^p, h^p-h, f^p \in U(\mathfrak{sl}_2)$ are central. We also classified (Sec 2.2 of Lec 7) the $\mathfrak{g}$-irreps where these elements act by p-character $\chi=(\chi_e,\chi_h,\chi_f)$ as follows

(1) $$(0,0,0);\ (0,a,0),\ a\neq 0;\ (0,0,1)$$

Let's explain why the case of general χ reduces to one of these.

Let $\mathfrak{g}$ be a Lie algebra, $\varphi:\mathfrak{g}\to\mathfrak{gl}(V)$ a representation & $\tau:\mathfrak{g}\xrightarrow{\sim}\mathfrak{g}$ a Lie algebra automorphism. Then $\varphi^\tau:=\varphi\circ\tau$ is also a representation, irreducible iff φ is.

An important example of φ is as follows. Let G be an algebraic group w. $\operatorname{Lie}(G)=\mathfrak{g}$ & $g\in G$. We have the automorphism $a_g: G\to G$, $g'\mapsto gg'g^{-1}$ & its tangent map $\operatorname{Ad}(g):\mathfrak{g}\to\mathfrak{g}$, now a Lie algebra automorphism. If $G\subset GL(V)$, then $\operatorname{Ad}(g)$ is the matrix conjugation by g. See Example 2 in Sec 1.2 of Lec 4. We write φ^g for $\varphi^{\operatorname{Ad}(g)}$ & V^g for the corresponding $U(\mathfrak{g})$-module. We'll see that

(*) $\forall$ $\mathfrak{sl}_2$-irrep V we can find $g\in SL_2$ (by means of Linear algebra) s.t. the p-character of V^g is from (1).

This will complete our classification of irreps.

1.1) More on elements $x^p - x^{[p]}$

In this section $\mathfrak{g} = Lie(G)$, where G is an algebraic group.

Recall that we have the map $\mathfrak{g} \xrightarrow{\iota} Z(U(\mathfrak{g}))$, $x \mapsto x^p - x^{[p]}$, Sec 1.1 of Lec 7. Let's investigate its compatibilities with the G-representation structure on $\mathfrak{g}$. Note that since G acts on $\mathfrak{g}$ by automorphisms (via Ad), we get a G-representation on $U(\mathfrak{g})$ by algebra automorphisms (also called adjoint). Every automorphism preserves $Z(U(\mathfrak{g}))$.

Thm: 0) $\iota(ax) = a^p \iota(x)$, $\forall a \in \mathbb{F}, x \in \mathfrak{g}$

1) $\iota(Ad(g)x) = Ad(g)\iota(x)$ $\forall g \in G, x \in \mathfrak{g}$

2) $\iota(x+y) = \iota(x) + \iota(y)$ $\forall x, y \in \mathfrak{g}$.

Proof:

0) $\Leftarrow$ both $x^p, x^{[p]}$ are pth powers in suitable associative algebras

1): Since $Ad(g)$ is an automorphism of $U(\mathfrak{g})$, $(Ad(g)x)^p = Ad(g)(x^p)$. Now assume $G \subset GL(V)$. Then $Ad(g)$ = conjugation w. g so $(Ad(g)x)^{[p]} = (gxg^{-1})^p = gx^pg^{-1} = Ad(g)(x^{[p]})$ (equalities in $End(V)$). 1) follows.

2): will be proved later □

1.2) p-characters, conceptually.

Let V be a (fin. dim.) $\mathfrak{g}$-irrep $\rightsquigarrow$ central character $\chi_V: Z(U(\mathfrak{g})) \to \mathbb{F} \rightsquigarrow$ p-character $\chi_V \circ \iota: \mathfrak{g} \to \mathbb{F}$, so that $\chi_x = \chi_V \circ \iota(x)$, $x \in \{e, h, f\}$.

Exercise: $\mathcal{X}_{Vg} = \mathcal{X}_V \circ Ad(g)$ (hint: from constructions).

Recall the automorphism $Fr: \mathbb{F} \to \mathbb{F}$, $x \mapsto x^p$

Definition: We say that a function $\mathcal{X}: \mathfrak{g} \to \mathbb{F}$ is Fr-linear if $\mathcal{X}(x+y) = \mathcal{X}(x) + \mathcal{X}(y)$ & $\mathcal{X}(ax) = Fr(a)\mathcal{X}(x)$ $\forall x, y \in \mathfrak{g}, a \in \mathbb{F}$.
The set of such functions is denoted by $\mathfrak{g}^{*(1)}$.

Example: Thx to 0) & 2) of Theorem, $\mathcal{X}_V \circ \iota \in \mathfrak{g}^{*(1)}$.

Some basic remarks on $\mathfrak{g}^{*(1)}$ are in order.
i) Just as with linear functions, an element of $\mathfrak{g}^{*(1)}$ is determined by its values on a basis, say e, h, f.

ii) G acts on $\mathfrak{g}^{*(1)}$: $g.\mathcal{X} := \mathcal{X} \circ Ad(g^{-1})$. We care about this b/c
$g^{-1}.(\mathcal{X}_V \circ \iota) = \mathcal{X}_V \circ \iota \circ Ad(g) = [1) \text{ of Thm}] = \mathcal{X}_V \circ Ad(g) \circ \iota = [\text{Exercise}] = \mathcal{X}_{Vg} \circ \iota$

iii) we have a G-equivariant bijection $\mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}^{*,(1)}: \alpha \mapsto Fr \circ \alpha$.
On the other hand, we claim that $\exists$ G-representation isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$. Indeed, we have a symmetric bilinear form $(\cdot,\cdot): \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$: $(x,y) = tr(xy)$. It's non-degenerate - its matrix in the basis e, h, f is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ & G-invariant: $(Ad(g)x, Ad(g)y) = tr(gxg^{-1}gyg^{-1}) = (x,y)$
This yields an isomorphism $\mathfrak{g} \to \mathfrak{g}^*$: $x \mapsto [y \mapsto tr(xy)]$ & hence $\mathfrak{g} \to \mathfrak{g}^{*(1)}$: $x \mapsto [y \mapsto Fr(tr(xy))]$, both are G-equivariant.

Example: Consider the element $\frac{1}{2}Fr^{-1}(a)h \in \mathfrak{g}$ for $a \in \mathbb{F}$. Its image in $\mathfrak{g}^{*(1)}$ vanishes on e, f and sends h to $Fr(\frac{1}{2}Fr^{-1}(a)\,tr(h^2)) = a$. We get two of p-characters in (1) of Sec 1.0. The third, $(0,0,1)$, is the image of e (exercise).

Now we can establish (*). By the JNF thm, every G-orbit in $\mathfrak{g}$ contains either bh ($b \in \mathbb{F}$) or e. Since $a \mapsto \frac{1}{2}Fr^{-1}(a): \mathbb{F} \to \mathbb{F}$ is bijective, and $\mathfrak{g} \to \mathfrak{g}^{*(1)}$ is G-equivariant, we see that the G-orbit of any element of $\mathfrak{g}^{*(1)}$ contains one of elements in (1). ii) implies (*).

2) Representations of $SL_2(\mathbb{F})$

We now proceed to studying the finite dimensional rational representations of $SL_2(\mathbb{F})$ (for simplicity, char $\mathbb{F} \neq 2$). We'll see that

- if char$(\mathbb{F}) = 0$, then the rep. theories of SL_2 & $\mathfrak{sl}_2$ are the same.
- while for char $\mathbb{F} > 0$, they are quite different: in fact, oversimplifying one can say that the representations of $SL_2(\mathbb{F})$ are closer to the characteristic 0 story (but are <u>not</u> completely reducible).

2.1) Some (ir)reducible representations

Consider the representation of $G = SL_2$ in homogeneous degree n polynomials, $M(n) := \mathrm{Span}_{\mathbb{F}}(x^n, x^{n-1}y, \ldots, y^n)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . f(x,y) = f(ax+cy,\ bx+dy).$$

Lemma: $M(n)$ is rational & its tangent representation is given

by $e(x^i y^{n-i}) = (n-i)x^{i+1}y^{n-i-1}$, $f(x^i y^{n-i}) = i x^{i-1} y^{n-i+1}$, $h(x^i y^{n-i}) = (2i-n)x^i y^{n-i}$.
where $e, h, f \in \mathfrak{sl}_2$ is a basis as in Example in Sec 3.1.

Proof:

For $n=1$ we get the tautological representation of $SL_2 / \mathfrak{sl}_2$.

In general, $M(n) = \mathrm{Sym}^n M(1)$ hence a quotient of $M(1)^{\otimes n}$, hence rational. Under the epimorphism $M(1)^{\otimes n} \twoheadrightarrow M(n)$ we have $x^{\otimes i} \otimes y^{n-i} \mapsto x^i y^{n-i}$, hence $e(x^{\otimes i} \otimes y^{n-i}) = \sum_{j=1}^{n-i} x^{\otimes i} \otimes y^{j-1} \otimes x \otimes y^{n-i-j} \mapsto (n-i)x^{i+1}y^{n-i-1}$.
The formulas for h & f are obtained similarly $\square$

Proposition: $M(n)$ is irreducible and isomorphic to $L(n)$ over $\mathfrak{g}$ if
1) char $\mathbb{F} = 0$
2) or char $\mathbb{F} > 2$ & $n < p$.

Proof:

The map $L(n) \to M(n)$, $f^i v_\lambda \mapsto \frac{1}{i!} x^{n-i} y^i$ $(i = 0, \ldots n)$ is an isomorphism of $\mathfrak{sl}_2$-reps (exercise: use Lemma) so $M(n)$ is irreducible as a $\mathfrak{g}$-representation. By Section 2 of Lec 5, $M(n)$ is irreducible over G $\square$

Example: $M(p)$ is not irreducible if char $\mathbb{F} = p$. Indeed, $\mathrm{Span}_{\mathbb{F}}(x^p, y^p) \subset M(p)$ is a subrepresentation, e.g. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^p = (ax+cy)^p = a^p x^p + c^p y^p$. More generally, if $n \geqslant p$, then $M(n)$ is irreducible $\Leftrightarrow$ $n = p^k - 1$ for some $k \geqslant 1$ (shouldn't be clear at this point).

2.2) Classification in characteristic 0

Theorem: Assume char $\mathbb{F} = 0$. Then every finite dimensional rational

G-representation, V, is completely reducible and the irreducibles are classified by $\mathbb{Z}$ via $n \mapsto M(n)$.

Proof: As a $\mathfrak{g}$-representation, $V \simeq \bigoplus_{i=1}^{k} M(n_i)$ for some n_i thx to the results of Lecture 6 & Proposition in Sec 2.1. Now we envoke Answer 3 in Sec 2 of Lec 5 to deduce an isomorphism of G-reps $\square$

2.3) Irreducibles in characteristic p.

Assume now $\operatorname{char} \mathbb{F} = p > 2$. Recall that to a representation $\varphi: G \to GL(V)$ we can assign its Frobenius twist $\varphi^{(1)} := Fr \circ \varphi$. Note that $Fr: GL(V) \to GL(V)$ is defined using an isomorphism $GL(V) \to GL_n$ that we get identifying V w. $\mathbb{F}^n$ but different identifications (i.e. choices of basis) result in conjugating Fr by an element of $GL(V)$ hence in isomorphic representations (exercise).

Exercise: Define $Fr: \mathbb{F}^n \to \mathbb{F}^n$, $(a_1, \ldots a_n) \mapsto (a_1^p, \ldots, a_n^p)$. Prove that $U \subset \mathbb{F}^n$ is a subrep'n for $\varphi \Leftrightarrow Fr(U)$ is a subrep'n for $\varphi^{(1)}$. In particular, φ is irreducible $\Leftrightarrow \varphi^{(1)}$ is.

Example: $\operatorname{Span}_{\mathbb{F}}(x^p, y^p) \simeq M(1)^{(1)}$ as G-representations.

Proposition: If V is irreducible, then $M(i) \otimes V^{(1)}$ is irreducible $\forall i \in \{0, \ldots p-1\}$.

Proof: Note that $\mathfrak{g}$ acts on $V^{(1)}$ by 0 (by Answer 2 in Sec 2 of Lec 5) So $M(i) \otimes V^{(1)} (\simeq M(i)^{\oplus \dim V})$ is a completely reducible $U(\mathfrak{g})$-module.

Prop 2.17 in Chapter 0 of notes on the website allows to describe its $\mathfrak{g}$-submodules: they are of the form $M(i)\otimes V'$ for a subspace $V'\subset V^{(1)}$. Note that $g(M(i)\otimes V') = M(i)\otimes gV'$. So $M(i)\otimes V'$ is G-stable $\Leftrightarrow$ $V'\subset V^{(1)}$ is G-stable. Since V (hence $V^{(1)}$) is irreducible, we are done. $\square$

This gives rise to the following inductive construction. For $k>0$, we write $\bullet^{(k)}$ for $\bullet^{(1)}$ repeated k times.

Exercise: For rational representations V_1, V_2 of G, we have $(V_1\otimes V_2)^{(1)} \simeq V_1^{(1)}\otimes V_2^{(1)}$ (hint: look at matrix coefficients)

Corollary (Steinberg tensor product theorem) For $0\leq \lambda_0,\dots,\lambda_k\leq p-1$, the G-representation $M(\lambda_0)\otimes M(\lambda_1)^{(1)}\otimes\dots\otimes M(\lambda_k)^{(k)}$ is irreducible.

In the next lecture we'll that every irreducible rational representation of G is isomorphic to exactly one of these tensor products.