Representations of SL_2 & $\mathfrak{sl}_2$, V

0) Recap
1) Weight decomposition
2) $M(n)$ as an induced representation.

0) Recap

Below $\mathbb{F}$ is an algebraically closed field of characteristic >2 & $G = SL_2(\mathbb{F})$. In Sec 2 of Lec 0 we started to study rational G-reps

a) We introduced the G-rep'n $M(n) = Span(x^n, x^{n-1}y, \dots, y^n)$ $(n \in \mathbb{Z}_{\geq 0})$ and saw that it's irreducible for $n<p$.

b) For a representation $\varphi: G \to GL(V)$ we considered its Frobenius twist $\varphi^{(1)} := Fr \circ \varphi: G \to GL(V)$, where $Fr: GL(V) \to GL(V)$ depends on a choice of basis, $v_1, \dots v_n \in V$. Note that $\varphi^{(1)}$ can be described in terms of its matrix coefficients $c^{(1)}_{\alpha,v}(g) = \langle \alpha, \varphi^{(1)}(g) v \rangle$: if $\alpha_1, \dots \alpha_n \in V^*$ is the dual basis, then

(1) $c^{(1)}_{\alpha_i, v_j} = c^{p}_{\alpha_i, v_j}$, where $c_{\alpha,v}$ is matrix coeff of φ.

c) $V^{(1)}$ is irrep. iff V is. More generally, if V is irrep, then $M(i) \otimes V^{(1)}$ is irrep. $\forall i \in \{0, \dots p-1\}$ (proved in the notes)

d) Note that $(U \otimes V)^{(1)} \simeq U^{(1)} \otimes V^{(1)}$ $\forall$ rational reps U, V of G. This follows e.g. from looking at the matrix coefficients: we use (1) & $c_{\beta \otimes \alpha, u \otimes v} = c_{\beta,u} c_{\alpha,v}$ (important exercise in Sec 2.2 of Lec 1)

e) Now we can use c) & d) to define, for $n \geq 0$, the irreducible representation $L(n)$ as follows: we write $n = n_0 + pn_1 + \ldots + p^k n_k$ ($n_i \in \{0, 1, \ldots, p-1\}$) & set $L(n) = M(n_0) \otimes M(n_1)^{(1)} \otimes \ldots \otimes M(n_k)^{(k)}$ (w. $\bullet^{(k)} = (\bullet^{(k-1)})^{(1)}$).

The main result for this lecture is:

Thm: $n \mapsto L(n)$ is a bijection between $\mathbb{Z}_{\geq 0}$ & {rational G-irreps}.

1) Weight decomposition

An approach one uses to prove the theorem is related to what we used to classify fin. dim. irreps of $\mathfrak{sl}_2$ in char 0 and can also be called "highest weight theory".

1.1) Construction

Consider the subgroup $T = \{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \} \subset G$ that is identified w. $\mathbb{G}_m$ via $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

Lemma: 1) $\forall$ rational representation $\varphi: \mathbb{G}_m \to GL(V)$ is completely reducible, the irreducibles are 1-dimensional,

2) and are given by $t \mapsto t^i$ ($i \in \mathbb{Z}$).

Proof: Set $S := \{ z \in \mathbb{G}_m \mid \exists \ell$ coprime to p s.t $z^\ell = 1\}$. Note that $\varphi(S) \subset GL(V)$ consists of pairwise commuting linear operators. They are simultaneously diagonalizable by the following

Exercise: 1) $\forall s \in GL(V)$ of order ℓ w. $\gcd(\ell, p) = 1$ is diagonalizable.

2) Any family of pairwise commuting diagonalizable operators on a finite dimensional $\mathbb{F}$-vector space is diagonalizable simultaneously (hint: use that $AB=BA \Rightarrow BV_\lambda(A) \subset V_\lambda(A)$ & induct on $\dim V$).

Now let $v_1, \ldots v_n \in V$ be a basis diagonalizing $\varphi(S)$ & $\alpha_1, \ldots \alpha_n \in V^*$ be the dual basis. Note that $i \neq j \Rightarrow \underbrace{c_{\alpha_i, v_j}}_{\in \mathbb{F}[t^{\pm 1}]}|_S = 0 \Rightarrow [|S| = \infty] \Rightarrow c_{\alpha_i, v_j} = 0$. This proves 1).

To prove 2) note that a rational G_m-rep'n of $\dim = 1$ is exactly a Laurent polynomial, say f, s.t. $f(st) = f(s)f(t)$. Any such is $t \mapsto t^i$ $(i \in \mathbb{Z})$, exercise. □

Definition: Let V be a rational representation of G. For $i \in \mathbb{Z}$, define the i-weight space $V_i = \{v \in V : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} v = t^i v\}$.

By Lemma, $V = \bigoplus_{i \in \mathbb{Z}} V_i$. Moreover $\forall\ \varphi \in \mathrm{Hom}_G(V, U)$ we have

(2) $\varphi(V_i) \subset U_i$.

Example: $M(n)_{n-2i} = \mathbb{F} x^{n-i} y^i$ $(i = 0, \ldots n)$. The other weight spaces are 0.

Exercise: 1) if V, W are rational representations of G, then

$$(V \otimes W)_i = \bigoplus_{j \in \mathbb{Z}} V_j \otimes W_{i-j}$$

2) If $Fr: GL(V) \to GL(V)$ is defined using eigenbasis for $\Phi(T)$ then $V_i = [V^{(1)}]_{pi}$ $\forall\ i \in \mathbb{Z}$ (hint: $Fr(A) = A^p$ $\forall$ diagonal $A \in Mat_n$)

3) The maximal (w.r.t. usual order on $\mathbb{Z}$) weight of $L(n)$ is n.

The following theorem implies Thm in Sec 0.

Thm': Two irreducible rational representations of G with the same max. weight are isomorphic.

2) $M(n)$ as an induced representation

2.1) Motivation

Our first step in proving Thm' is:

Lemma: If V is an irrep. w. max. weight n that admits a nonzero homomorphism to $M(n)$, then $V \simeq Span_{\mathbb{F}}(G.x^n) \subset M(n)$, hence V is uniquely determined up to iso.

Proof:

Let $\varphi: V \to M(n)$ be a nonzero homomorphism. Since V is irreducible, φ is injective. It remains to prove $\operatorname{im}\varphi = Span_{\mathbb{F}}(G.x^n)$, which, thx to irreducibility of V amounts to $x^n \in \operatorname{im}\varphi$. But $V_n \neq \{0\}$ & $M(n)_n = \mathbb{F}x^n$. By (2), $\varphi(V_n) \subset M(n)_n$ & $\varphi|_{V_n}$ is injective $\Rightarrow x^n \in \varphi(V_n)$ □

So it remains to show that $\forall$ irrep V w. highest weight $n \Rightarrow Hom_G(V, M(n)) \neq 0$. In order to see this we'll realize $M(n)$ as an induced module.

2.2) Induced modules in algebraic context

Recall that if $H \subset G$ are finite groups & U is a representation of H, then the induced representation $Ind_H^G U$ is defined by

$$\mathrm{Ind}_H^G U := \{\text{maps } f: G \to U \mid f(hg) = hf(g),\ \forall h \in H, g \in G\}$$
w. G-action given by $[gf](g') = f(g'g)$. We have Frobenius reciprocity:

(3) $$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G U) \xrightarrow{\sim} \mathrm{Hom}_H(V, U)$$

Now let $H \subset G$ be algebraic groups, and U be a rational H-rep.

Definition: The (algebraic) *induced representation* is
$$\mathrm{Ind}_H^G U := \{\text{morphisms } f: G \to U \mid f(hg) = hf(g)\}$$
$G \curvearrowright \mathrm{Ind}_H^G U$ as above (f is morphism $\Rightarrow$ so is gf, exercise).

Lemma: $\forall$ rational V, (3) holds.

Sketch of proof: we'll sketch maps in both directions. An extended exercise to prove they are well-defined & mutually inverse.

- $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G U) \longrightarrow \mathrm{Hom}_H(V, U)$: consider $ev: \mathrm{Ind}_H^G U \to U$, $f \mapsto f(e)$. The map $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G U) \longrightarrow \mathrm{Hom}_H(V, U)$ we need is $\varphi \mapsto ev \circ \varphi$
- $\mathrm{Hom}_H(V, U) \longrightarrow \mathrm{Hom}_G(V, \mathrm{Ind}_H^G U)$: $\psi \in \mathrm{Hom}_H(V, U) \rightsquigarrow G \times V \longrightarrow U$, $(g, v) \mapsto \psi(gv) \rightsquigarrow \forall v \in V$ we have $\psi_v: G \to U$, $\psi_v(g) = \psi(gv)$. It's a morphism b/c $(g, v) \mapsto gv: G \times V \to V$ is & it's H-equivariant b/c ψ is. And $v \mapsto \psi_v: V \to \mathrm{Ind}_H^G U$ is G-linear. □

2.3) Realization of $M(n)$.

Now we explain how to construct $M(n)$ as an induced representation. Consider the subgroup $B = \left\{\begin{pmatrix} t & 0 \\ u & t^{-1} \end{pmatrix}\right\} \subset G$. Let $\mathbb{F}_n$ be its 1-dimensional representation where $\begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix}$ acts by t^n.

Proposition: $M(n) \simeq \mathrm{Ind}_B^G \mathbb{F}_n$ as G-reps.

Proof: R.h.s. is $\{f \in \mathbb{F}[G] \mid f\left(\begin{pmatrix} t & 0 \\ u & t^{-1} \end{pmatrix} g\right) = t^n f(g)\}$

Step 1: We first examine f satisfying

$$f\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} g\right) = f(g), \ \forall g \in G, u \in \mathbb{F} \qquad (4)$$

The algebra $\mathbb{F}[a,b]$ admits a natural homomorphism to $\mathbb{F}[G]$ (evaluate an element of $\mathbb{F}[a,b]$ on the 1st row). It's an embedding b/c $\forall$ nonzero first row, there's an element of SL_2, so we view $\mathbb{F}[a,b]$ as a subalgebra of $\mathbb{F}[G]$

Since $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c+ua & d+ub \end{pmatrix}$, $\forall f \in \mathbb{F}[a,b]$ satisfies (4). We claim that the converse is true.

Consider the open subset $G_a = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \neq 0\}$. Since a is not 0-divisor, the restriction map $\mathbb{F}[G] \to \mathbb{F}[G_a]$ is injective. The product map $\{(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix})\} \longrightarrow G_a$, $(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}) \longmapsto \begin{pmatrix} a & b \\ ua & a^{-1}+ub \end{pmatrix}$ is iso of varieties $\rightsquigarrow \mathbb{F}[G_a] \xrightarrow{\sim} \mathbb{F}[u, a^{\pm 1}, b]$. And under isomorphism w. $\{(a,u,b) \mid a \neq 0\}$, the action of $\{\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}\}$ on G_a becomes $(a,u,b) \mapsto (a, u+v, b)$. So (4) implies that $f|_{G_a}$ is independent of u hence lies in $\mathbb{F}[a^{\pm 1}, b]$.

Notice that the diagram of inclusions

$$\begin{array}{ccc} \mathbb{F}[a,b] & \hookrightarrow & \mathbb{F}[G] \\ \downarrow & & \downarrow \\ \mathbb{F}[a^{\pm 1}, b] & \hookrightarrow & \mathbb{F}[G_a] \end{array}$$

is commutative. So we need to show that $\mathbb{F}[a,b]$ coincides w. $\mathbb{F}[a^{\pm 1}, b] \cap \mathbb{F}[G]$ in $\mathbb{F}[G_a]$

Let $f' \in \mathbb{F}[a^{\pm 1}, b]$ correspond to $f|_{G_a}$. Assume $f' \notin \mathbb{F}[a,b]$ & take minimal $k>0$ w. $\tilde{f}' = a^k f' \in \mathbb{F}[a,b] \Rightarrow \tilde{f}'(0,b) \neq 0$. Take $\tilde{f} = a^k f \in \mathbb{F}[G]$. Note that a vanishes on $\begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix}$ but $\exists\, \beta$ s.t. $\tilde{f}\begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix} = \tilde{f}'(0,\beta) \neq 0$, contradiction (w. $k>0$).

Step 2: We have $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ta & tb \\ t^{-1}c & t^{-1}d \end{pmatrix}$. So a polynomial in a, b lies in $\mathrm{Ind}_B^G \mathbb{F}_n \iff$ it's in $\mathrm{Span}_{\mathbb{F}}(a^n, a^{n-1}b, \ldots b^n) \xrightarrow{\sim} M(n)$: $a \mapsto x$, $b \mapsto y$, G-linear (exercise). □