The deadline is Oct 1.

Problem 1

The setup is the following. Let F be an algebraic group, Y a smooth variety, and $\pi : \widetilde{Y} \to Y$ be a principal F-bundle (for simplicity, locally trivial in Zariski topology). Our main example is $F = B$ and $\pi : G \to G/B$. In particular, we can talk about the (automatically, weakly) F-equivariant quasi-coherent sheaves on $\widetilde{Y}$, denote their category by $\mathrm{QCoh}^F(\widetilde{Y})$. You are allowed to use the following fact:

(*) The categories $\mathrm{QCoh}^F(\widetilde{Y})$ and $\mathrm{QCoh}(Y)$ are equivalent via the following functors: π^* that naturally upgrades to $\mathrm{QCoh}(Y) \to \mathrm{QCoh}^F(\widetilde{Y})$ (note that $\pi^*(\mathcal{F})$ carries a natural F-equivariant structure) and $\pi_*(?)^F : \mathrm{QCoh}^F(\widetilde{Y}) \to \mathrm{QCoh}(Y)$ (note that $\pi_*(\mathcal{G})$ carries an action of F by automorphisms).

Let $D_Y\text{-mod}$ denote the category of D-modules on Y (that are quasi-coherent over $\mathcal{O}_Y$) and $D_{\widetilde{Y}}\text{-mod}^F$ denote the category of strongly F-equivariant D-modules on $\widetilde{Y}$. The goal of this problem is to establish an equivalence of these categories.

a). Prove that every object $\mathcal{G} \in D(F)\text{-mod}^F$ (for the action of F on itself by left translations) decomposes as $\mathbb{K}[F] \otimes \mathcal{G}^F$ (where the 2nd factor is the F-invariants). *Hint: $D(F)$ is generated by $\mathbb{K}[F]$ and $\mathfrak{f}$.* Deduce the desired equivalence for $\widetilde{Y} = F$.

b). Let Y be affine and $\widetilde{Y} = F \times Y$. Prove that $D(\widetilde{Y}) = D(F) \otimes D(Y)$ and use part a) to establish an equivalence $D(\widetilde{Y})\text{-mod}^F \xrightarrow{\sim} D(Y)\text{-mod}$.

c). Now suppose we are in the general case. For $\mathcal{F} \in D_Y\text{-mod}$, equip the F-equivariant coherent sheaf $\pi^*(\mathcal{F})$ on $\widetilde{Y}$ with a natural structure of an object in $D_{\widetilde{Y}}\text{-mod}^F$. This gives a functor $D_Y\text{-mod} \to D_{\widetilde{Y}}\text{-mod}^F$.

d). Assume the same generality. Conversely, for $\mathcal{G} \in D_{\widetilde{Y}}\text{-mod}^F$, equip the quasi-coherent sheaf $\pi_*(\mathcal{G})^F$ on Y with a natural D_Y-module structure. This gives a functor $D_{\widetilde{Y}}\text{-mod}^F \to D_Y\text{-mod}$.

e). Finally, prove that the functors in c) and d) are mutually inverse equivalences between $D_Y\text{-mod}$ and $D_{\widetilde{Y}}\text{-mod}^F$.