Lazy approach to categories $\mathcal{O}$, IV

1) Quantum categories $\mathcal{O}$
2) Highest weight structure.
3) Deformation & subgeneric behaviour.
4) Whittaker coinvariants.

1) Quantum categories $\mathcal{O}$

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. We can consider the Drinfeld-Jimbo quantum group $\mathcal{U}_q(\mathfrak{g})/\mathbb{C}(q)$ and the mixed (a.k.a. hybrid) $\mathbb{C}[q^{\pm1}]$-lattice $\mathcal{U}_q^{mix}(\mathfrak{g})$ generated by:

$$F_i, K_i^{\pm1}, E_i^{(\ell)} = E_i^{(\ell)}/[\ell]_q! \quad (i \in I, \ell \in \mathbb{Z}_{\geqslant 0})$$

indexing set for simple roots.

We can also consider De Concini-Kac lattice $\mathcal{U}_q^{DK}(\mathfrak{g})$ (generated by $F_i, K_i^{\pm1}, E_i$) & Lusztig lattice $\mathcal{U}_q^L(\mathfrak{g})$ (generated by $F_i^{(\ell)}, K_i^{\pm1}, E_i^{(\ell)}$).

For $\varepsilon \in \mathbb{C}^\times \rightsquigarrow \mathcal{U}_\varepsilon^{mix}(\mathfrak{g}) = \mathcal{U}_q^{mix}(\mathfrak{g})/(q-\varepsilon)$, all these algebras are graded by the root lattice, Λ. Fix the G-invariant form $(\cdot,\cdot)$ on $\mathfrak{h}$ w. $(\alpha^\vee, \alpha^\vee) = 2$ for all short coroots α. Set $n_\alpha := (\alpha^\vee, \alpha^\vee)/2$. Identify $\mathfrak{h}$ w $\mathfrak{h}^*$ using $(\cdot,\cdot)$, so that $\alpha^\vee = n_\alpha \alpha$.

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Def: Pick $p \in \mathbb{C}^\times$ w. $\varepsilon = \exp(\pi\sqrt{-1}/p)$ & $\nu \in \mathfrak{h}^*$. Category $\mathcal{O}_{p,\nu}$ = full subcategory in Λ-graded fin. generated $\mathcal{U}_\varepsilon^{mix}$-modules consisting of all $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ s.t.

- $\dim M_\lambda < \infty \ \forall \lambda$.
- $\{\lambda \mid M_\lambda \neq 0\}$ is bounded from above.
- K_i acts on M_λ by $\exp(\pi\sqrt{-1}(\lambda+\nu, \alpha_i)/p)$.

The most interesting case is when $p \in \mathbb{Q}$ ($\Leftrightarrow \varepsilon = \sqrt[\cdot]{1}$) & ν "has small denominator".

Rem: One can introduce Verma modules $\Delta_{p,\nu}(\lambda)$, $\lambda \in \Lambda$, in the same way as for the usual category $\mathcal{O}$. Their simple quotients, $L_{p,\nu}(\lambda)$ give a complete list of irreps in $\mathcal{O}_{p,\nu}$. If $p \in \mathbb{Z} + \frac{1}{2}$ & $\nu = 0$ (we should be able to require just that $p \in \mathbb{Q}$) all $L_{p,\nu}(\lambda)$ are finite dimensional. In particular the objects $\Delta_{p,\nu}(\lambda)$ have infinite length.

2) Highest weight structure.

In Sec 1 of Lec 1 we have introduced the notion of a highest weight category with <u>finite poset</u>. The usual category $\mathcal{O}$ doesn't quite fit this definition, but it splits as the direct sum of infinitesimal blocks that do. While $\mathcal{O}_{p,\nu}$ splits as the direct sum of infinitesimal blocks, for interesting parameters (p,ν), they are still infinite. So we need to generalize the definition of a highest weight category.

Def: Let $\mathcal{T}$ be a poset. We say that $\mathcal{T}$ is interval finite (resp. coideal finite) if $\forall\, \tau_1, \tau_2 \in \mathcal{T} \Rightarrow [\tau_1, \tau_2] := \{\tau \mid \tau_1 \leq \tau \leq \tau_2\}$ is finite (resp., $\{\tau \geq \tau_1\}$ is finite)

Example: • Λ w. the usual order is interval finite.
• If $\mathcal{T}$ is interval finite, then $\mathcal{T}(\leq \tau_2) = \{\tau \in \mathcal{T} \mid \tau \leq \tau_2\}$ is coideal finite $\forall\, \tau_2 \in \mathcal{T}$.

$\mathcal{O}_{p,\nu}$ should be a highest weight category with poset Λ & standards $\Delta_{p,\nu}(\lambda)$, $\lambda \in \Lambda$ – we just need to say what this means formally.

Let $\mathbb{F}$ be a field & $\mathcal{C}$ be an abelian category w. objects $\Delta(\tau), \tau\in\mathcal{T}$, where $\mathcal{T}$ is interval finite. To a poset ideal $\mathcal{T}_0\subset\mathcal{T}$ we assign the Serre span $\mathcal{C}_{\mathcal{T}_0}$ of $\Delta(\tau), \tau\in\mathcal{T}_0$.

Def: We say that $\mathcal{C}$ is highest weight w. poset $\mathcal{T}$ & standard objects $\Delta(\tau)$ if the following hold:

(I) - properties of $\mathcal{C}$ itself:

(I.1) $\mathcal{C}$ is Noetherian,

(I.2) Hom's are finite dimensional

(II) - highest wt. structure for $\mathcal{C}_{\mathcal{T}_0}$ - familiar axioms:

(II.1) $\mathrm{Hom}_{\mathcal{C}}(\Delta(\tau_1), \Delta(\tau_2))\neq 0 \Rightarrow \tau_1\leqslant\tau_2$

(II.2) $\mathrm{End}_{\mathcal{C}}(\Delta(\tau))\xleftarrow{\sim}\mathbb{F}$

(II.3) $\forall M\in\mathcal{C}, \neq 0\ \exists\ \tau \mid \mathrm{Hom}_{\mathcal{C}}(\Delta(\tau), M)\neq 0$.

(II.4) $\forall$ coideal finite poset ideal $\mathcal{T}_0\subset\mathcal{T}$ $\forall\ \tau\in\mathcal{T}_0$ $\exists$ projective object P_τ in $\mathcal{C}_{\mathcal{T}_0}$ w. $P_\tau\twoheadrightarrow\Delta(\tau)$ & ker filtered by $\Delta(\tau')$ w $\tau'\in\mathcal{T}_0, \tau'>\tau$ (a finite filtration)

Premium exercise: $\mathcal{O}_{p,\nu}$ is highest weight w. poset Λ &

standards $\Delta_{p,\nu}(\lambda), \lambda\in\Lambda$.

3) Deformation & subgeneric behaviour.

We want to analyze $\mathcal{O}_{p,\nu}$ using the same techniques as we used for the BGG cat. $\mathcal{O}$, namely

- construct a deformation over a formal power series algebra.
- understand the subgeneric behavior.
- construct a "nice" functor to a "combinatorial" category

3.1) Deformation.

The formal power series ring will be in $r+1$ (where $r=\operatorname{rk}\mathfrak{g}$) variables, r corresponding to deforming ν and one corresponds to deforming p. Namely, let $\hat{\mathfrak{h}}$ be the affine Cartan, $\hat{\mathfrak{h}}=\mathfrak{h}\oplus\mathbb{C}\hbar$ (where we use $\hbar$ to denote a central element in the Kac-Moody algebra corresp. to $(\cdot,\cdot)$). Set $R=\mathbb{C}[[\hat{\mathfrak{h}}^*]]$.

We can consider the deformation $\mathcal{O}_{p,\nu,R}$ similarly to $\mathcal{O}_{\nu,R}$ in Sec 1.3 in Lec 1.

Set $\hat{\varepsilon} = \exp(\pi\sqrt{-1}/(p+\hbar)) \in \mathbb{C}[[\hbar]] \subset R$ and form the algebra $\mathcal{U}_{\hat{\varepsilon}}^{mix}/\mathbb{C}[[\hbar]]$. We still have the natural inclusion $\iota: \mathfrak{h} \longrightarrow R$.

By def'n, $\mathcal{O}_{p,\nu,R}$ consists of certain Λ-graded fin. generated $\mathcal{U}_{\hat{\varepsilon}}^{mix}$-modules (cf. $\mathcal{O}_{\nu,R}$ in Sec 1.3). The condition for the action of K_i's on M_λ is that it acts by the following element of R:

$$\exp\left(\pi\sqrt{-1}\left[(\lambda+\nu, \alpha_i) + \iota(\alpha_i)\right]/(p+\hbar)\right)$$

The category $\mathcal{O}_{p,\nu,R}$ is highest weight over R, the details of the definition are left as an exercise.

3.2) Subgeneric behavior.

We consider the affine root system $\{\alpha + n\delta \mid n \in \mathbb{Z}\}$.

Def: The integral roots for (ν, p) are the affine roots $\alpha + n\delta$ with $\alpha \neq 0$ & $n_\alpha((\alpha,\nu) + np) \in \mathbb{Z}$, where $n_\alpha = \frac{(\alpha^\vee, \alpha^\vee)}{2}$.

Expectation: 1) $\mathcal{O}_{\nu,p}$ is semisimple $\Leftrightarrow$ there are no integral roots.
2) When the integral root system consists of exactly two (mutually opposite) roots, $\mathcal{O}_{\nu,p}$ is equivalent to $\oplus$ of blocks of $\mathcal{O}(\mathfrak{sl}_2)$. This is the subgeneric behavior.

Remark: In the full generality this is not in the literature but in interesting (and sufficiently broad) special cases it is. The case when ε is not a root of 1 should be done using a suitable version of twisting functors (constituting an action of Br_W) and their t-exactness morally similar to I.L.'s work with Dhillon. With this, the proof reduces to the case when $\varepsilon=\sqrt[p]{1}$: we expect that for generic ν, $\mathcal{O}_{p,\nu}$ is s/simple. This is known when the order of ε is odd (& coprime to 3 if $\mathfrak{g}$ is G_2), and is based on understanding the Azumaya locus in $Z(\mathcal{U}_\varepsilon^{DK})$. An informal reason why $\mathcal{O}_{\varepsilon,\nu}$ w. ν generic should be s/simple is as follows.

Let $\mathfrak{l}\subset\mathfrak{g}$ be a Levi. Suppose that $(\nu,\alpha)+np$ is suff. generic for all α that are not roots of $\mathfrak{l}$ & all $n\in\mathbb{Z}$.

Then we expect that the parabolic induction functor $\mathcal{O}_{p,\nu}(\mathfrak{l}) \longrightarrow \mathcal{O}_{p,\nu}(\mathfrak{g})$ is an equivalence. But $\mathcal{O}_{p,\nu}(\mathfrak{h})$ is semisimple.

4) Whittaker coinvariants.

We now assume:

- $\nu = 0$
- ε = primitive $\sqrt[d]{1}$ w. d odd (& coprime to 3 for $\mathfrak{g} = G_2$).

To handle the general case, some modification may be needed.

It will be convenient to modify the Cartan part: let $\Lambda_w^\vee$ be the coweight lattice: we replace $\mathrm{Span}(K_\mu \mid \mu \in \Lambda)$ w. $\mathrm{Span}(K_\mu \mid \mu \in \Lambda_w^\vee)$. It still naturally acts on modules from $\mathcal{O}_{p,\nu,R}$.

Our functor will still be Whittaker coinvariants.

Note that the quantum Serre relations imply that there's no homomorphism $\mathcal{U}^- \xrightarrow[\varphi]{} \mathbb{C}$ w. $\varphi(F_i) \neq 0$ $\forall i$ (exercise).

However, Sevostyanov proved that there are elements

$\nu_i \in P^\vee$ s.t. $\tilde{U}^-$ generated by the elements $\tilde{F}_i = K_{\nu_i} F_i$ admits $\psi: \tilde{U}^- \to \mathbb{C}$ w. $\psi(\tilde{F}_i) = 1$. Then we can consider the functor $Wh: \mathcal{O}_{p,\nu,R} \to R$, $M \mapsto M / \sum_{i \in I} \mathrm{im}(\tilde{F}_i - 1)$.

The properties of this functor are similar to its non-quantum counterpart, the main difference is that the finite Weyl group is replaced w. $W^{a,\vee} = W \ltimes \Lambda$. Namely,

I) $Wh: \mathcal{O}^\Delta_{p,\nu} \to Vect$ is faithful.

II) The center $Z(U^{DK}_{\hat{\varepsilon}})$ ← still generic acts on modules from $\mathcal{O}_{p,\nu,R}$ commuting w. $U^{DK}_{\hat{\varepsilon}}$ & making Wh a functor

$$\mathcal{O}_{p,\nu,R} \longrightarrow Z(U^{DK}_{\hat{\varepsilon}}) \otimes R\text{-mod}$$

It is fully faithful on $\mathcal{O}^\Delta_{p,\nu,R}$.

III) We have $Z(U^{DK}_{\hat{\varepsilon}}) \simeq \mathrm{Span}_{\mathbb{C}[[\hbar]]}(K_{2\mu} \mid \mu \in P)^{(W,\cdot)}$ via HC isomorphism. This implies infinitesimal block decomposition for $\mathcal{O}_{p,0}$ (& $\mathcal{O}_{p,0,R}$). Namely consider the action of $W \ltimes \Lambda$ on Λ, where W acts by the dot-action & Λ acts by $t_\lambda . \mu = \mu + d\lambda$ (where d is the order of ε). Then $\mathcal{O}_{p,0} = \bigoplus_{\Xi} \mathcal{O}_{p,0,\Xi}$, where Ξ runs over the set of $W \ltimes \Lambda$-orbits in Λ. This can be deduced, for example from Sec 3.2. Every orbit has unique point in the anti-dominant $\mathcal{d}$-alcove: minimal coroot

$$A_-=\{\lambda\in\Lambda \mid \langle\lambda+\rho,\alpha_i^\vee\rangle\le 0,\ \langle\lambda+\rho,\alpha_0^\vee\rangle\ge -d\}$$

Let λ_- be the unique point of $A_-\cap\Xi$ & $W^\circ\subset W\ltimes\Lambda$ be its stabilizer, a finite reflection group.

IV) $W\ltimes\Lambda$ acts on $\hat{\mathfrak{h}}=\mathfrak{h}\oplus\mathbb{C}\hbar$ s.t. W acts by the default action & $t_\mu(\xi+z\hbar)=\mu+(z+\langle\mu,\xi\rangle)\hbar$ ($\mu\in\Lambda,\xi\in\mathfrak{h}$). For $\lambda\in\Xi$ let γ_λ denote the homomorphism $Z(\mathcal{U}^{\mathcal{D}k}_{\hat{\varepsilon}})\longrightarrow R$ given by action on $\Delta_{p,0,R}(\lambda)$. Then γ_{λ_-} gives an identification of the completion $Z(\mathcal{U}^{\mathcal{D}k}_{\hat{\varepsilon}})^{\wedge\Xi}$ at the maximal ideal corresponding to Ξ & R^{W°. Moreover, for $x\in W^{a,\vee}$, $\gamma_{x\cdot\lambda_-}=x\gamma_{\lambda_-}$ giving an identification of $\Delta_{p,0,R}(\lambda)$ w. R^{W°-R-bimodule R_x.

Rem: One can ask to describe the order on $(W\ltimes\Lambda)/W^\circ$ coming from the highest wt stucture on $\mathcal{O}_{p,0,\Xi}$ in terms of the Bruhat order, $\le$. Note that it cannot coincide w $\le$ b/c the highest wt. order is preserved under left multiplication by the t_λ's. It turns out that we have the following well-defined order: $x \leq^{st} y$ if $t_\mu x \leq t_\mu y$ for all μ sufficiently dominant. This is the highest wt. order for $\mathcal{O}_{p,0,\Xi}$.