1. Quick reminder on reductive groups and Lie algebras

1.1. **Definitions and examples.** Let G be a connected linear algebraic group (affine group scheme of finite type) over an algebraically closed field $\mathbb{F}$. Main examples for us are $GL_n, PGL_n, SL_n, Sp_{2n}, SO_n, B_n, U_n, T_n, \mathbb{G}_m, \mathbb{G}_a$, where $B_n \subset GL_n$ (resp. $U_n \subset GL_n$) is the group of upper triangular (resp. strictly upper triangular) matrixes and $T_n \subset GL_n$ is a group of diagonal matrixes.

We will denote by $\mathbb{F}[G]$ the ring of functions of G. The multiplication $G \times G \to G$ defines a comultiplication $\Delta\colon \mathbb{F}[G] \to \mathbb{F}[G] \otimes \mathbb{F}[G]$ on $A = \mathbb{F}[G]$. A is a $G \times G$ - bimodule via the action $(g_1, g_2) \cdot f(x) = f(g_1^{-1} x g_2)$. Using comultiplication Δ one can check that this action is locally finite. By a finite dimensional representation of a group G we will always mean a homomorphism of algebraic groups $G \to GL(V)$. By a subgroup H of G we will always mean a closed algebraic subgroup.

Theorem 1.1. *Let G be an algebraic group acting on a variety X. Then the orbits of G are locally closed subvarieties of X. Moreover any orbit of minimal dimension is closed, in particular, the set of closed orbits is always nonempty.*

Proof. The first claim follows from the Chevalley's theorem about the image of a constructible set and the second claim is an exersise. □

Theorem 1.2. *The quotient G/H has a structure of a quasi projective algebraic variety such that the natural morphism $G \to G/H$ is a geometric quotient in the category of varieties over $\mathbb{F}$. Moreover if H is a normal subgroup of G then G/H with the natural group structure and the variety structure as above is an linear algebraic group.*

Proof. To define a variety structure on G/H one should construct (using the algebra $\mathbb{F}[G]$) a finite dimensional representation W of G with vector $w \in W$ such that the the stabilizer of the line $\mathbb{F}w \subset W$ is H then $G/H := G \cdot [w] \subset \mathbb{P}(W)$ and we apply Theorem 1.1. □

Definition 1.3. *We say that an element $g \in G$ is semisimple (resp. unipotent) if there exists a closed embedding $G \hookrightarrow GL_N$ such that the image of x is semisimple (resp. unipotent).*

Theorem 1.4 (Jordan decomposition)**.** *(1) If $g \in G$ is semi-simple (resp. unipotent) then it acts semi-simply (resp. unipotently) on any finite dimensional representation of G.*

(2) Any element $g \in G$ has a unique (Jordan) decomposition $g = g_s g_u$ such that g_s is semi-simple, g_u is unipotent and $g_s g_u = g_u g_s$. Moreover an element $x \in G$ commutes with g iff it commutes with both g_s and g_u.

(3) Let $\varphi\colon G_1 \to G_2$ be a homomorphism of algebraic groups. Then if $g = g_s g_u$ is the Jordan decomposition of g then $\varphi(g) = \varphi(g_s)\varphi(g_u)$ is the Jordan decomposition of $\varphi(g)$.

Proof. Consider the Jordan decomposition of g acting on $\mathbb{F}[G]$ via $f \mapsto (x \mapsto f(g^{-1}x))$. □

Definition 1.5. *A group is called unipotent if it consists of unipotent elements.*

Theorem 1.6. *Any unipotent group G can be closely embedded in some U_N.*

Corollary 1.7. *Let G be a unipotent group and V a finite dimensional representation of G. Then the space V^G is nonzero.*

Lemma 1.8. *Consider an exact sequence of groups $1 \to U \to G \to U' \to 1$. Then the group G is unipotent iff U and U' are.*

Proof. Use Theorem 1.4. □

It follows from Lemma 1.8 that any algebraic group G contains the maximal normal unipotent subgroup $G_u \subset G$.

Definition 1.9. *Group G is called reductive if $G_u = \{e\}$ i.e. it has no nontrivial unipotent subgroups.*

Remark 1.10. *If $\operatorname{char}\mathbb{F} = 0$ then any simply connected group G is isomorphic to the semi-direct product of its maximal unipotent subgroup G_u and a reductive group $L = G/G_u$: $G \simeq L \ltimes G_u$. This is a theorem of Mostow. It is not true in positive characteristic.*

Basic examples of reductive groups are $\mathbb{G}_m^r(\mathbb{F})$, $GL_n(\mathbb{F})$, $SL_n(\mathbb{F})$, $SO_n(\mathbb{F})$, $Sp_{2n}(\mathbb{F})$. The simplest example of a nonreductive group is $\mathbb{G}_a$ or, more generally, groups U_n, $B_n = T_n \ltimes U_n$.

Remark 1.11. *In the case when $\operatorname{char}(\mathbb{F}) = 0$ group G is reductive iff the category $\operatorname{Rep}(G)$ of finite dimensional representations of G is semi-simple.*

Remark 1.12. *We will prove in Proposition 1.27 that any representation of $T = \mathbb{G}_r^m$ is completely reducible (even if $\operatorname{char}\mathbb{F} \neq 0$).*

1.2. **Borel subgroups, maximal tori, Weyl group.** We now fix an arbitrary linear algebraic group G (we do not assume it to be reductive in this section).

1.2.1. *Borel and parabolic subgroups and subalgebras.*

Definition 1.13. *Subgroup $B \subset G$ is called Borel subgroup if it is connected, solvable and maximal with these properties.*

Example 1.14. *For $G = GL_n$ any Borel subgroup is conjugate to the group $B_n \subset GL_n$.*

Definition 1.15. *Subgroup $P \subset G$ is called parabolic subgroup if G/P is a proper variety i.e. for any variety Y the projection morphism $(G/P) \times Y \to Y$ maps closed subsets to closed subsets.*

Example 1.16. *For $G = GL(V)$ and a subspace $W \subset V$ the group $P = \{f \in GL(V),\ f(W) \subset W\}$ is parabolic. Indeed G/P is isomorphic to the Grassmannian $\operatorname{Gr}(w, v)$ via the map $[g] \mapsto g(W)$, here $w = \dim W$, $v = \dim V$.*

Proposition 1.17. *Group G does not contain proper parabolic subgroups iff G is solvable.*

Proof. Exersise. □

Corollary 1.18 (Borel fixed point theorem)**.** *Let G be a solvable algebraic group acting on a proper variety X. Then the set X^G is nonempty.*

Proof. Let $O \subset X$ be a closed G-orbit. It follows that O is proper. Fix $x \in X$ and consider the stabilizer $G_x \subset G$. Note that $G/G_x \simeq O$ so $G_x \subset G$ is parabolic. By Proposition 1.17, $G_x = G$, hence, $x \in X^G$. □

Corollary 1.19. *If G is solvable then any finite dimensional representation V of G has a filtration $F^\bullet V$ by G-submodules such that F^i/F^{i-1} are one dimensional. As a corollary, any solvable group G can be closely embedded in B_N for some N.*

Proof. Follows by induction on $\dim V$ from the Corollary 1.18 applied to $G \curvearrowright \mathbb{P}[V]$. □

Theorem 1.20 (Lie theorem)**.** *Let $\mathfrak{g}$ be a solvable Lie algebra. Then any finite dimensional representation V of $\mathfrak{g}$ has a filtration $F^\bullet V$ by $\mathfrak{g}$-submodules such that F^i/F^{i-1} are one dimensional.*

Proposition 1.21. *(1) Subgroup $P \subset G$ is parabolic iff it contains a Borel subgroup. In other words Borel subgroups are minimal parabolic subgroups of G.*

(2) If $B_1, B_2 \subset G$ are Borel subgroups of G then there exists an element $g \in G$ such that $B_2 = gB_1g^{-1}$.

Proof. Part (1) follows from Proposition 1.17. Part (2) follows from Corollary 1.18 applied to $G = B_1, B_2$ and $X = G/B_2, G/B_1$. □

We can now define Borel and parabolic subalgebras.

Definition 1.22. *Subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is called Borel if it is a maximal solvable subalgebra.*

Definition 1.23. *A Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is called a parabolic Lie algebra if it contains a Borel subalgebra.*

Proposition 1.24. *The map $P \mapsto \operatorname{Lie} P$ defines a bijection between the set of parabolic (resp. Borel) subgroups and parabolic (resp. Borel) subalgebras.*

1.2.2. *Tori and maximal tori.*

Definition 1.25. *Connected subgroup $S \subset G$ is called a torus if it is commutative an consists of semisimple elements.*

Example 1.26. *For $G = GL_n$ all maximal tori are conjugate to the subgroup of diagonal matrixes $T_n \subset GL_n$.*

Proposition 1.27. *Any representation of S is completely reducible. Irreducible representations of S are one dimensional and in bijection with the character lattice $\Lambda = \operatorname{Hom}(S, \mathbb{G}_m)$.*

Proof. Let V be a representation of T. We prove Proposition by induction on $v = \dim(V)$. Fix $t \in V$ such that t acts on V not via multiplication by an element from $\mathbb{F}^\times$. It follows from Theorem 1.4 that we can decompose $V = \bigoplus_{\lambda_i} V_{\lambda_i}$, where $V_{\lambda_i} = \{v \in V \,|\, t(v) = \lambda_i v\}$. Note that each V_{λ_i} is a T-module of dimension less then v. Proposition follows. □

Corollary 1.28. *Let S be a torus. Then there exists a closed embedding $S \hookrightarrow T_N$ for some N.*

Proposition 1.29. *Let S be a torus then $S \simeq \mathbb{G}_m^r$, where $r = \dim S$.*

Proof. Consider the action $S \curvearrowright \mathbb{F}[S]$ via left multiplication. Recall that this action is locally finite so we can apply Proposition 1.27. It follows that $\mathbb{F}[S] = \bigoplus_{\lambda \in \Lambda} \mathbb{F}[S]_\lambda$, where λ runs through some torsion free abelian subgroup $\Lambda \subset \operatorname{Hom}(S, \mathbb{G}_m)$. It can be deduced from Corollary 1.28 that Λ is finitely generated i.e. $\Lambda \simeq \mathbb{Z}^r$ for some r. Note that any $f \in \dim \mathbb{F}[S]_\lambda$ is uniquelly determined by $f(1) \in \mathbb{F}$ so $\dim \mathbb{F}[S]_\lambda = 1$. Let $\lambda_i \in \Lambda$ be a generators of Λ. Fix $f_i \in \mathbb{F}[S]_{\lambda_i}$. The morphism $T \to \mathbb{G}_m^r$, $t \mapsto (f_1(t), \ldots, f_r(t))$ is an isomorphism of algebraic groups. □

Theorem 1.30. *All maximal tori are cojugate.*

Proposition 1.31. *Let $S \subset G$ be a subtorus. Then $Z_G(S)$ is connected. Moreover if G is reductive then $Z_G(S)$ is reductive.*

Corollary 1.32. *Let $T \subset G$ be a maximal subtorus of a reductive group G. Then $Z_G(T) = T$.*

Definition 1.33. *We define a Weyl group of G: $W = W(G) := N_G(T)/T$, where $N_G(T) \subset G$ is the normalizer of T.*

In our examples we have $W = S_n$ for $G = GL_n, SL_n$, for $G = Sp_{2n}, SO_{2n+1}$ we have $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ and for $G = SO_{2n}$ we have $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$.

1.2.3. *Semisimple groups.* Let $Z = Z(G) \subset G$ be the center of G and $\mathfrak{z} := \operatorname{Lie} Z$. Note that Z is a commutative semisimple algebraic group, in particular by Proposition 1.29 the connected component $Z^0 \subset Z$ of $1 \in Z$ is isomorphic to $\mathbb{G}_m^{\dim Z}$.

Definition 1.34. *Reductive group G is called semisimple if Z is finite i.e. $\dim Z = 0$ or equivalently if $\mathfrak{z} = \{0\}$.*

Example 1.35. *Group $G = GL_n$ is reductive but not semisimple. Center of GL_n is isomorphic to $\mathbb{G}_m$ via the map $t \mapsto \operatorname{diag}(t, \dots, t)$. Groups PGL_n and SL_n are semisimple, $Z(PGL_n) = \{\operatorname{Id}\}$, $Z(SL_n) = \{\xi \cdot \operatorname{Id} \mid \xi \in \mathbb{F}^\times, \xi^n = 1\}$. Other examples of semisimple groups are Sp_{2n}, SO_n.*

Remark 1.36. *We will see later that the Weyl group $W(G)$ depends only on the Lie algebra $\mathfrak{g}/\mathfrak{z}$ of G/Z. In particular, $W(GL_n) = W(PGL_n) = W(SL_n) = S_n$.*

1.3. **Roots and root data.**

1.3.1. *Root decomposition.* Let G be a reductive group with Lie algebra $\mathfrak{g}$. We fix a maximal torus $T \subset G$ of dimension r and denote by $\Lambda := \operatorname{Hom}(T, \mathbb{G}_m)$ the character lattice of T. It follows from Proposition 1.29 that Λ is noncanonically isomorphic to $\mathbb{Z}^r$. Consider the adjoint action $T \curvearrowright \mathfrak{g}$. We denote by $\Delta \subset \Lambda$ the set of nonzero characters of T which appear as weights of this action. It can be deduced from Proposition 1.27 and Corollary 1.32 that we have a weight decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{h} = \operatorname{Lie}(T)$. For any root $\alpha \in \Delta$ the root subspace $\mathfrak{g}_\alpha$ is one dimensional.

Example 1.37. *For $G = GL_n$ we have $\mathfrak{gl}_n = \mathfrak{h}_n \oplus \bigoplus_{1 \leqslant i,j \leqslant n} \mathbb{F} e_{ij}$. The weight of e_{ij} is $\varepsilon_i - \varepsilon_j$, where $\varepsilon_p \colon T_n \to \mathbb{C}^\times$ sends $\operatorname{diag}(t_1, \dots, t_n)$ to t_p.*

1.3.2. *Positive, simple roots and dominance order.* Choice of a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ corresponds to the choice of positive roots $\Delta_+ \subset \Delta$ such that $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$.

Example 1.38. *For $G = GL_n$ and $B = B_n$ we have $\Delta_+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant n\}$.*

Proposition 1.39. *Let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra containing $\mathfrak{h}$. Then there exists a unique subset $J \subset \Delta_+$ of positive roots such that $|J| = \dim(\mathfrak{h}) - \dim(\mathfrak{z})$ and any root $\alpha \in \Delta_+$ can be uniquely expressed as a linear combination of elements from S with non negative coefficients. The set J is called the set of simple roots.*

Example 1.40. *For $G = GL_n$ and $B = B_n$ we have $J = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n-1\}$, $\varepsilon_i - \varepsilon_j = \alpha_i + \ldots + \alpha_{j-1}$ for $i < j$.*

It follows from Proposition 1.39 that if G is semi-simple, then J is a basis of the vector space $\Lambda_\mathbb{Q} = \Lambda \otimes_\mathbb{Z} \mathbb{Q}$. Therefore $\{\alpha^\vee \mid \alpha \in J\}$ is a basis of $\Lambda^\vee_\mathbb{Q} = \Lambda^\vee \otimes_\mathbb{Z} \mathbb{Q}$.

Definition 1.41. *Let $(\omega_\alpha)_{\alpha \in J} \in \Lambda_\mathbb{Q}$ be the weights such that $\langle \omega_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$ for all $\alpha, \beta \in J$. These ω_α are called the fundamental weights.*

Example 1.42. *For $G = SL_n$ there are $n-1$ fundamental weights. We have $\omega_i = \varepsilon_1 + \ldots + \varepsilon_i$, $1 \leqslant i < n$.*

Definition 1.43 (Dominance order)**.** *We can define an order relation $\leqslant$ on Λ by $\mu \leqslant \lambda \Leftrightarrow \lambda - \mu \in \sum_{\alpha \in J} \mathbb{N}\alpha$.*

Example 1.44. *For $G = SL_n$ we have $\mu_1\varepsilon_1 + \ldots + \varepsilon_n\mu_n \leqslant \lambda_1\varepsilon_1 + \ldots + \varepsilon_n\lambda_n$ iff $\mu_1 + \ldots + \mu_i \leqslant \lambda_1 + \ldots + \lambda_i$ for any $i = 1, \dots, n-1$. More general for G semi-simple we have $\mu \leqslant \lambda$ iff $\langle \mu, \omega_\alpha^\vee \rangle \leqslant \langle \lambda, \omega_\alpha^\vee \rangle$ for any $\alpha \in J$, here $\omega_\alpha^\vee \in \Lambda_\mathbb{Q}^\vee$ are fundamental coweights defined by $\langle \alpha, \omega_\beta^\vee \rangle = \delta_{\alpha,\beta}$ for $\alpha, \beta \in J$.*

1.3.3. *Root subgroups, Weyl group and root datum.*

Proposition 1.45. *For each $\alpha \in \Delta$ there is a root homomorphism $x_\alpha\colon \mathbb{G}_a \to G$ with $tx_\alpha(a)t^{-1} = x_\alpha(\alpha(t)a)$ for any $t \in T$, $a \in \mathbb{G}_a$ and such that $dx_\alpha\colon \operatorname{Lie}(\mathbb{G}_a) \xrightarrow{\sim} \mathfrak{g}_\alpha$.*

Example 1.46. *For $G = GL_n$ we have $x_{\varepsilon_i - \varepsilon_j}(a) = \operatorname{Id} + ae_{ij}$.*

Proposition 1.47. *We set $U_\alpha := x_\alpha(\mathbb{G}_a)$. Group G is generated by subgroups $U_\alpha, \alpha \in J$ and T.*

For any $\alpha \in \Delta$ there is a homomorphism $\varphi_\alpha\colon SL_2 \to G$ such that

$$\varphi_\alpha \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = x_\alpha(a),\ \varphi_\alpha \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = x_{-\alpha}(a).$$

Definition 1.48. *Set $\Lambda^\vee := \operatorname{Hom}(\mathbb{G}_m, T)$. To any $\alpha \in \Delta$ we can associate a coroot $\alpha^\vee \in \Lambda^\vee$: $\alpha^\vee(t) = \varphi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. We denote by $\Delta^\vee \subset \Lambda^\vee$ the set of coroots of G.*

Remark 1.49. *Kernel of φ_α is a proper normal subgroup of SL_2. There are only two of them: $\{\operatorname{Id}\}$, $\{\pm \operatorname{Id}\}$. If $-\operatorname{Id} \in \ker \varphi_\alpha$ then $\langle \lambda, \alpha^\vee \rangle \in 2\mathbb{Z}$. We have $\ker \varphi_\alpha = \{\pm \operatorname{Id}\}$ iff $\alpha^\vee \in 2\Lambda^\vee$ and φ_α is injective otherwise.*

The image of φ_α is normalized by T and $\varphi_\alpha \cap T = \alpha^\vee(\mathbb{G}_m)$ as any element in $\varphi_\alpha^{-1}(T)$ centralizes all diagonal matrixes. So we have inside G the product

$$G_\alpha = T\varphi_\alpha(SL_2) \simeq (T \ltimes \varphi_\alpha(SL_2))/\alpha^\vee(\mathbb{G}_m).$$

Each G_α is a reductive group with maximal torus T and root system $\{\alpha, -\alpha\}$. One has $G_\alpha = Z_G(\ker(\alpha))$. At the level of Lie algebras we have $\operatorname{Lie} G_\alpha = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. The Weyl group of G_α is isomorphic to $S_2 = \{1, s\}$ via the map

$$s \mapsto s_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in N(T).$$

Proposition 1.50. *The group W is generated by the equivalence classes of s_α, $\alpha \in J$ to be denoted by the same symbols $s_\alpha \in W$.*

Example 1.51. *It is well-known that the group S_n is genrated by transpositions $(i, i+1)$.*

Definition 1.52. *A root datum is a quadruple $(M, R, M^\vee, R^\vee)$ satisfying the following conditions:*

(1) The sets M and $M^\vee$ are free $\mathbb{Z}$-modules of finite rank with $M^\vee = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. We denote by $\langle m, f \rangle = f(m)$ for $m \in M$, $f \in M^\vee$ the pairing.

(2) The sets R and $R^\vee$ are finite subsets of M and $M^\vee$ respectively such that the following conditions hold: $\langle \alpha, \alpha^\vee \rangle = 2$ and $\sigma_\alpha(R) = R$, $\sigma_{\alpha^\vee}(R^\vee) = R^\vee$ where $\sigma_\alpha(m) = m - \langle m, \alpha^\vee \rangle \alpha$, $\sigma_{\alpha^\vee}(f) = f - \langle \alpha, f \rangle \alpha^\vee$.

Root datum is called reduced if R does not contain 2α for any $\alpha \in R$.

Proposition 1.53. *The quadruple $(\Lambda, \Delta, \Lambda^\vee, \Delta^\vee)$ is a reduced root datum.*

Proof. Note that the group W acts on Λ (resp. $\Lambda^\vee$) via its conjugation action on T. We claim that $s_\alpha \in W$ acts on Λ via σ_α. Indeed $s_\alpha \in Z_G(\ker \alpha)$ and $s_\alpha U_\alpha s_\alpha^{-1} = U_{-\alpha}$. It follows that $\sigma_\alpha(\Delta) = \Delta$ (because the action of s_α on $\mathfrak{g}$ must send a root to a root). To see that $s_{\alpha^\vee}(\Delta^\vee) = \Delta^\vee$ one should note that $(w\alpha)^\vee = w(\alpha^\vee)$ for any $w \in W$. □

Proposition 1.54. *The map $s_\alpha \mapsto \sigma_\alpha$ (resp. $s_{\alpha^\vee} \mapsto \sigma_{\alpha^\vee}$) extends to the isomorphism $W \xrightarrow{\sim} \langle \sigma_\alpha \mid \alpha \in J \rangle$ (resp. $W \xrightarrow{\sim} \langle \sigma_{\alpha^\vee} \mid \alpha \in J \rangle$).*

Remark 1.55. *As a corollary of Proposition 1.54 we have* $\langle \sigma_\alpha \,|\, \alpha \in J\rangle \xrightarrow{\sim} \langle \sigma_{\alpha^\vee} \,|\, \alpha \in J\rangle$ *via* $s_\alpha \mapsto s_{\alpha^\vee}$.

1.3.4. *Length in W and the longest element.*

Definition 1.56. *Hyperplanes* $H_\alpha := \ker \subset \Lambda_\mathbb{R} := \Lambda \otimes_\mathbb{Z} \mathbb{R}$ *are called walls and* $\{H_\alpha \,|\, \alpha \in \Delta\}$ *is called a hyperplane arrangement of* $\Lambda_\mathbb{R}$. *Walls* H_α *divide* $\Lambda_\mathbb{R}$ *into chambers. We denote by* $\Lambda_\mathbb{R}^+$ *the dominant chamber consisting of* λ *such that* $\langle \lambda, \alpha^\vee\rangle \geqslant 0$ *for any* $\alpha \in J$. $\Lambda_\mathbb{R}^+$ *is a fundamental domain for* $W \curvearrowright \Lambda_\mathbb{R}$ *i.e. chambers are in bijection with* W *via the map* $w \mapsto w(\Lambda_\mathbb{R}^+)$.

Example 1.57. *For* $G = GL_n$ *walls* $H_{i,j}$ *are hyperplanes consisting of* $k_1\varepsilon_1 + \ldots + k_n\varepsilon_n$ *such that* $k_i = k_j$. *The dominant chamber* $\Lambda_\mathbb{R}^+$ *consists of* $k_1\varepsilon_1 + \ldots + k_n\varepsilon_n$ *such that* $k_1 \geqslant \ldots \geqslant k_n$.

Definition 1.58. *Fix* $w \in W$. *Let* A, B *be elements in the interiors of* $\Lambda_\mathbb{R}^+$ *and* $w(\Lambda_\mathbb{R}^+)$ *respectively. The length* $l(w)$ *of* $w \in W$ *is the number of walls which the segment* $[A, B]$ *intersects.*

Remark 1.59. *The length* $l(w)$ *of* $w \in W$ *equals to the smallest* m *such that there exist* $\alpha_1, \ldots, \alpha_m \in J$ *with* $w = s_{\alpha_1} \ldots s_{\alpha_m}$.

Lemma 1.60. *We have* $l(w) = l(w^{-1})$.

Proof. Exersise. □

Lemma 1.61. *One has for all* $w \in W$, $\alpha \in J$

$$l(ws_\alpha) = \begin{cases} l(w)+1 & \text{if } w(\alpha) > 0 \\ l(w)-1 & \text{if } w(\alpha) < 0 \end{cases} \qquad l(s_\alpha w) = \begin{cases} l(w)+1 & \text{if } w^{-1}(\alpha) > 0 \\ l(w)-1 & \text{if } w^{-1}(\alpha) < 0. \end{cases}$$

Proof. It is enough to deal with ws_α, second case will follow from Lemma 1.60. Fix a point A in the interior of the face $\Lambda_\mathbb{R}^+ \cap s_\alpha(\Lambda_\mathbb{R}^+)$. Fix also a point B in the interior of $w(\Lambda_\mathbb{R})$. Note that the segment $[A, B]$ must intersect the interior of $\Lambda_\mathbb{R}^+$ or the interior of $s_\alpha(\Lambda_\mathbb{R}^+)$ and it can not intersect both of them. It follows that $l(ws_\alpha) = l(w) \pm 1$. It is an exersise to check that $[A, B]$ intersects the interior of Λ^+ iff $w(\alpha) < 0$ and $[A, B]$ intersects the interior of $s_\alpha(\Lambda_\mathbb{R}^+)$ iff $w(\alpha) > 0$. □

Lemma 1.62. *For* $\alpha \in J$ *we have* $s_\alpha(\Delta_+) = (\Delta_+ \cap \{-\alpha\}) \setminus \{\alpha\}$

Proof. Consider a vector space $\mathfrak{p}_\alpha := \mathfrak{g}_{-\alpha} \oplus \mathfrak{b}$. This is a subalgebra of $\mathfrak{g}$. Note that $\mathfrak{b} \subset \mathfrak{p}_\alpha$ so $\mathfrak{p}_\alpha$ is parabolic. Let $P_\alpha \subset G$ be the corresponding parabolic subgroup. Note that $G_\alpha \subset P_\alpha$, hence, s_α normalizes P_α and so acts on $\mathfrak{p}_\alpha$ via adjoint action. The claim follows. □

Remark 1.63. $\mathfrak{p}_\alpha, P_\alpha, \alpha \in J$ *are called subminimal parabolic subalgebras, subgroups.*

Proposition 1.64. *We have* $l(w) = |w(\Delta_+) \cap \Delta_-|$.

Proof. It is enough to prove the first equality of Lemma 1.61 for the function $l'(w) = |w(\Delta_+) \cap \Delta_-|$. By Lemma 1.62 we have $|ws_\alpha(\Delta_+) \cap \Delta_-| = |w((\Delta_+ \cup \{-\alpha\}) \setminus \{\alpha\}) \cap \Delta_-|$ and the Proposition follows. □

Example 1.65. *Proposition 1.64 implies that for* $W = S_n$ *the length of a permutation* σ *coincides with the number of pairs* $1 \leqslant i < j \leqslant n$ *such that* $\sigma(i) > \sigma(j)$.

Note now that $\Lambda_- = -\Lambda_+$ is a chamber of our hyperplane arrangement. It follows that there exists the unique element $w_0 \in W$ such that $l(w_0) = |\Delta_+|$ and $w_0(\Lambda_+) = -\Lambda_+$ i.e. $w_0(\Delta_+) = -\Delta_+$. It follows from the definition that $l(ww_0) = l(w_0w) = |\Delta_+| - l(w)$ for any $w \in W$. So w_0 can be uniquely determined as the *longest* element in W. We also have $w_0^2(\Delta_+) = \Delta_+$, hence, $w_0^2 = 1$.

Example 1.66. *For* $G = GL_n$ *we have* $w_0(i) = n + 1 - i$.

1.3.5. *Element* ρ.

Definition 1.67. *We set* $\rho := \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$.

Lemma 1.68. *Element* ρ *has the following property:* $\langle\rho,\alpha^\vee\rangle = 1$ *for any* $\alpha \in J$. *If* G *is semi-simple then* ρ *is uniquelly determined by this property.*

Proof. Follows from Lemma 1.62. □

Proposition 1.69. ρ *is dominant and lies in the interior of* $\Lambda^+ \subset \Lambda$.

Proof. Immediately follows from Lemma 1.68. □

Remark 1.70. *It follows that for* G *semi-simple we have* $\rho = \sum_{\alpha\in J}\omega_\alpha$, *where* ω_α *are fundamental weights (see Example 1.44).*

Example 1.71. *For* $G = GL_n$ *we have* $\rho = \frac{n-1}{2}\varepsilon_1 + \ldots + \frac{n-2k+1}{2}\varepsilon_k + \ldots + \frac{-n+1}{2}\varepsilon_n$.

Remark 1.72. *The element* $n\varepsilon_1 + (n-1)\varepsilon_2 + \ldots + \varepsilon_1$ *is integer and meets the property of Lemma 1.68, hence, is dominant and lies in the interior of* Λ^+. *Sometimes this element is denoted by* ρ.

1.3.6. *Homomorphisms of Root Data.* Let us now return to the notion of root datum. Let G, G' be two reductive groups with maxima tori T, T' and root datums $(\Lambda,\Delta,\Lambda^\vee,\Delta^\vee), (\Lambda',\Delta',\Lambda'^\vee,\Delta'^\vee)$ respectively.

Definition 1.73. *A homomorphism of root data (from that of* G *to that of* G'*) is a group homomorphism* $\psi\colon \Lambda' \to \Lambda$ *that maps* R' *bijectively to* R *and such that the dual homomorphism* $\psi^\vee\colon \Lambda^\vee \to \Lambda'^\vee$ *maps* $f(\beta)^\vee$ *to* $\beta^\vee$.

Note that the map ψ defines a homomorphism $f\colon T \to T'$.

Proposition 1.74. *Any map* ψ *between root data of* G *and* G' *gives rise to the unique homomorphism* $\varphi\colon G \to G'$ *with* $\varphi|_T = f$, $\ker(\varphi) = \ker(f) \subset Z(G)$. *In particular if* f *is an isomorphism then so is* φ.

Proof. Idea is to check that for any $\alpha \in \Delta$, the map f extends uniquely to the map $G_\alpha \to G_{\psi(\alpha)}$ and these maps are compatible (note that by Proposition 1.47, groups G_α generate G). □

Example 1.75. *Consider the authomorphism of the root datum of given by* $\Lambda \ni \lambda \mapsto -\lambda \in \Lambda$. *The corresponding authomorphism* $\tau\colon G \xrightarrow{\sim} G$ *is called Cartan involution. For* $G = GL_n$ *it is given by* $A \mapsto (A^T)^{-1}$ *and* $A \mapsto -A^T$ *at the level of Lie algebra.*

Corollary 1.76. *Two reductive groups* G, H *are isomorphic iff the corresponding root data are isomorphic.*

Remark 1.77. *The map* $G \mapsto (\Lambda,\Delta,\Lambda^\vee,\Delta^\vee)$ *is a bijection between reductive algebraic groups and reduced root data.*

Definition 1.78. *Let* $G^\vee$ *be the algebraic group with a root data* $(\Lambda,\Lambda^\vee,\Delta,\Delta^\vee)$. *Group* $G^\vee$ *is called Langlands dual to* G.

Remark 1.79. *By Remark 1.55 Langlands dual groups have isomorphic Weyl groups.*

1.3.7. *Parabolic and Levi subgroups in reductive groups.* We can now describe parabolic subgroups of G in terms of root systems. Let $I \subset J$ be a subset of the set of simple roots. We denote by $\Delta_I \subset \Delta$ the set of roots generated by I. Note that the quadruple $(\Lambda, \Lambda^\vee, \Delta_I, \Delta_I^\vee)$ is a reduced root datum. It follows from Remark 1.77 that there exists a reductive group L_I with this root datum. Set

$$\mathfrak{l}_I := \operatorname{Lie} L_I = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha,\ \mathfrak{u}_I := \bigoplus_{\alpha \in \Delta \setminus \Delta_I} \mathfrak{g}_\alpha,\ \mathfrak{p}_I := \mathfrak{l}_I \oplus \mathfrak{u}_I.$$

Note that $\mathfrak{b} \subset \mathfrak{p}_I$, hence, $\mathfrak{p}_I$ is a parabolic subalgebra. Note also that $\mathfrak{u}_I$ is a nilpotent radical of $\mathfrak{p}_I$ and $\mathfrak{l}_I \simeq \mathfrak{p}_I/\mathfrak{u}_I$. Let P_I, U_I be the corresponding algebraic groups. We have a Levi decomposition $P_I = L_I \ltimes U_I$. Subgroups $L_I \subset P_I \subset G$ are called standard Levi and parabolic subgroups.

Proposition 1.80. *Any parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is conjugate to a standard parabolic $\mathfrak{p}_I$ for some $I \subset S$.*

Proof. Let $\mathfrak{b}' \subset \mathfrak{p}$ be a Borel subalgebra. Let $g \in G$ be an element such that $\operatorname{Ad}(g)(\mathfrak{b}') = \mathfrak{b}$ then $\operatorname{Ad}(g)(\mathfrak{p})$ is standard. □

1.4. **Flag variety, Bruhat decomposition.** We fix a Borel subgroup $B \subset G$ and recall the flag variety $\mathfrak{B} := G/B$. This is a smooth projective variety of dimension $|\Delta_+|$.

Remark 1.81. *Variety $\mathfrak{B}$ can be identified with the variety of Borel subgroups of G via the map $gB \mapsto gBg^{-1}$. This gives a definition of $\mathfrak{B}$ that does not depend on the choice of B.*

We fix a subset $I \subset J$ and consider the corresponding subgroups $L = L_I$, $P = P_I$. We consider the projective variety G/P. Note that for $I = \varnothing$ we have $P = B$ and $G/P = \mathfrak{B}$.

Example 1.82. *In the case $G = GL_n$ the variety $\mathfrak{B}$ is isomorphic to the variety of full flags. Variety G/P is isomorphic to the variety of partial flags.*

We have the action $G \curvearrowright \mathfrak{B}, G/P$ via the left multiplication. We have the induced actions $T, B \curvearrowright \mathfrak{B}, G/P$.

Proposition 1.83. *(1) The fixed points $\mathfrak{B}^T$ (resp. $(G/P)^T$) coincide with W (resp. W/W_L) via the map $w \mapsto wB$ (resp. $[w] \mapsto wP$). For $w \in W$ (resp. $[w] \in W/W_I$) we denote by X_w (resp. $X_{[w]}$) the corresponding B-orbit.*

(2) We have the decomposition $\mathfrak{B} = \bigsqcup_{w \in W} X_w$ (resp. $G/P = \bigsqcup_{[w] \in W/W_L} X_{[w]}$). Variety X_w (resp. $X_{[w]}$) is an affine space of dimension $l(w) = |w(\Delta_+) \cap \Delta_-|$ (resp. $|w(\Delta_+) \cap (\Delta_- \setminus \Delta_{I,-})|$).

(3) There is a unique open dence B-orbit in $\mathfrak{B}$ (resp. in G/P). This orbit is X_{w_0} (resp. $X_{[w_0]}$).

Corollary 1.84. *We have the decompositions*

$$G = \bigsqcup_{w \in W} BwB,\ G = \bigsqcup_{w \in W/W_I} BwP \tag{1.85}$$

Subvarieties $Bw_0B \subset G$, $Bw_0P \subset G$ are open and dence.

Example 1.86. *Note that for $G = GL_n$ the first decomposition of 1.85 is the Gauss decomposition which claims that any invertible matrix A has the form $B_1 \sigma B_2$, where B_1, B_2 are upper triangular and σ is a permutation matrix.*

1.4.1. *Orders and longest/shortest elements in cosets.*

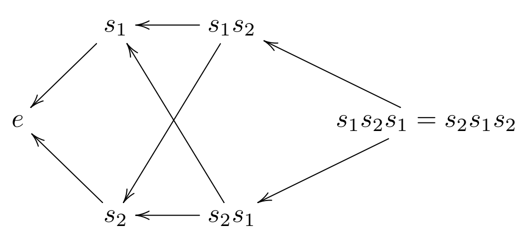
Definition 1.87 (Bruhat order on W)**.** *For $w, w' \in W$, $w' \preceq w$ iff $X_{w'} \subset \overline{X}_w$. The partial order $\preceq$ is called a Bruhat order.*

Lemma 1.88. *We have $w' \preceq w$ iff $w_0 w' \succeq w_0 w$.*

Proposition 1.89. *The order $\preceq$ can be described combinatorially as follows: this is a transitive closure of the relation $w' \leftarrow w$, where we write $w' \leftarrow w$ if $l(w) = l(w') + 1$ and $w = s_\alpha w'$ for some $\alpha \in \Delta$.*

Remark 1.90. *One can show that $w' \preceq w$ iff some substring of some (any) reduced word for w is a reduced word for w'.*

Example 1.91. *For $G = SL_3, W = S_3$, the Bruhat order is the order on the vertices of the following graph:*



Lemma 1.92. *For any $w \in W$ and $\alpha \in J$ we have:*
(1) $ws_\alpha \succ w$ (resp. $s_\alpha w \succ w$) if $l(ws_\alpha) = l(w) + 1$ (resp. $l(s_\alpha w) = l(w) + 1$),
(2) $ws_\alpha \prec w$ (resp. $s_\alpha w \prec w$) if $l(ws_\alpha) = l(w) - 1$ (resp. $l(s_\alpha w) = l(w) - 1$).

Proof. Proof is very similar to the one of Lemma 1.61. □

Definition 1.93 (Bruhat order on Λ)**.** *For $\lambda, \mu \in \mathfrak{h}^*$ we write $\mu \preceq \lambda$ if $\mu = \lambda$ or there exist $\alpha_1, \dots, \alpha_n \in \Delta_+$ such that*

$$\mu = s_{\alpha_1} \dots s_{\alpha_n} \lambda < s_{\alpha_2} \dots s_{\alpha_n} \lambda < \dots < s_{\alpha_n} \lambda < \lambda.$$

Note that $w' \preceq w$ iff $w'\lambda \preceq w\lambda$ for λ in the interior of Λ^+ (for example for $\lambda = \rho$).

Proposition 1.94. *Fix $w \in W$, the coset wW_I contains an element $u \in wW_I$ such that $u \succ v$ (resp. $u \prec v$) for any other $v \in wW_I$.*

Proof. Consider the locally trivial fibration $\pi \colon G/B \twoheadrightarrow G/P$. Note that $\pi^{-1}(X_{[w]})$ is B-invariant and irreducible. We have a decomposition $\pi^{-1}(X_{[w]}) = \bigsqcup_{v \in wW_I} X_v$. It follows that there exists the unique open orbit X_u for some $u \in wW_I$. We have $\overline{X}_u = \pi^{-1}(X_{[w]}) \supset X_v$ for any $v \in wW_I$ i.e. $u \succeq v$. Let $u' \in w_0 w W_I$ be the maximal element. It now follow from 1.88 that $w_0 u' \preceq v$ for any $v \in wW_I$. □

Remark 1.95. *One can show that if $u \in uW_I$ is the minimal (resp. the maximal) element then for any $w \in W_I$ we have $l(uw) = l(u) + l(w)$ (resp. $l(uw) = l(u) - l(w)$).*

1.5. **Finite dimensional representation theory over $\mathbb{C}$.** Let us now assume that $\operatorname{char} \mathbb{F} = 0$. In this case the category $\operatorname{Rep}(G)$ is semi-simple. So to describe the category $\operatorname{Rep}(G)$ it is enough to understand simple modules. Fix $T \subset B \subset G$. Let V be a G-module. We say that a vector $v \in V$ is highest of weight $\lambda \colon T \to \mathbb{C}^\times$ if B acts on v_λ via multiplication by the character $B \twoheadrightarrow B/U \xrightarrow{\lambda} \mathbb{C}^\times$.

Proposition 1.96. *For each $\lambda \in \Lambda^+$ there exists the unique irreducible module $L(\lambda)$ with highest vector $v_\lambda \in L(\lambda)$ of weight λ. Module $L(\lambda)$ is generated by v_λ via the action of the unipotent radical $U_- \subset B_-$.*

Let us mention a geometric construction of representation $L(\lambda)$. For $\lambda \in \Lambda_+$ we denote by $\mathcal{L}(\lambda)$ the induced line bundle $G \times_B \mathbb{F}_{-\lambda} = (G/U) \times_T \mathbb{F}_{-\lambda}$ on the variety $\mathfrak{B}$. The following holds.

Proposition 1.97. *a) For $\lambda \in \Lambda^+$, $L(\lambda) \simeq H^0(\mathfrak{B}, \mathcal{L}(\lambda))$.*
b) We have $\mathbb{C}[G/U] = H^0(G/U, \mathcal{O}_{G/U}) = \bigoplus_{\lambda\in\Lambda^+} L(\lambda)$ so G/U is a model for G.
(c) $\mu \in \Lambda$ appears as a weight of $L(\lambda)$ iff $\overline{\mu} \leqslant \lambda$, where $\overline{\mu}$ is the dominant representative of μ in $W\mu$.
(d) For any $w \in W$, $\mu \in \Lambda$ we have $\dim(L_\lambda(\mu)) = \dim(L_\lambda(w\mu))$.

Example 1.98. *For $G = SL_n$ the fundamental weights are $\omega_i = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n$ and we have $L(\omega_i) = \Lambda^i(\mathbb{C}^n)$, $v_{\omega_i} = e_1 \wedge \ldots \wedge e_i$, here $\{e_k\}$ is the standard basis in $\mathbb{C}^n$. For any $\lambda \in \Lambda^+$ decompose $\lambda = \sum k_i\omega_i$ with $k_i \in \mathbb{Z}_{\geqslant 0}$. Representation $L(\lambda) \subset V$ is generated by*

$$v_{\omega_1}^{\otimes k_1} \otimes v_{\omega_2}^{\otimes k_2} \otimes \ldots \otimes v_{\omega_r}^{\otimes k_{n-1}} \in (\mathbb{C}^n)^{\otimes k_1} \otimes (\Lambda^2(\mathbb{C}^n))^{k_2} \otimes \ldots \otimes (\Lambda^{n-1}(\mathbb{C}^n))^{k_{n-1}}.$$

Example 1.99. *For $G = SL_2$ we take $T = T_n, B = B_n$ and we have the identification $\Lambda_+ \simeq \mathbb{Z}_{\geqslant 0}$. With respect to this identifications $L(n) = S^n(\mathbb{F}^2)$. So the direct sum $\bigoplus_{n\geqslant 0} L(n)$ identifies with $\mathbb{F}[\mathbb{A}^2]$. $\mathbb{A}^2$ is nothing else but the affinization of $SL_2/U = \mathbb{A}^2 \setminus \{0\}$.*

Definition 1.100. *Let V be a representation of G such that for each $\lambda \in \Lambda$ the weight space V_λ is finite dimensional. Then we can define $\operatorname{ch} V := \sum_{\lambda\in\Lambda} \dim(V_\lambda)e(\lambda)$, where $e(\lambda)$ are variables with $e(\lambda+\mu) = e(\lambda)e(\mu)$.*

Proposition 1.101 (Weyl character formula)**.** *For $\lambda \in \Lambda^+$ we have*

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w\in W}(-1)^{l(w)}e(w(\lambda+\rho))}{\prod_{\alpha\in\Delta_+} e(\alpha/2) - e(-\alpha/2)} = \frac{\sum_{w\in W}(-1)^{l(w)}e(w(\lambda+\rho))}{\sum_{w\in W}(-1)^{l(w)}e(w(\rho))}.$$

Example 1.102. *For $G = SL_2$ we have $\rho = 1$, $\alpha = 2$ and we have $\operatorname{ch}(S^n(\mathbb{C}^2)) = e(n) + e(n-2) + \ldots + e(2-n) + e(-n) = \frac{e(n+1)-e(-n-1)}{e(1)-e(-1)}$.*

2. BGG category $\mathcal{O}$

2.1. **Definition, Verma modules, simple modules.** In this section we assume that $\operatorname{char} \mathbb{F} = 0$. Recall that $\mathfrak{g}$ is a semi-simple Lie algebra with a triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, here $\mathfrak{n} := \operatorname{Lie} U$, $\mathfrak{n}_- := \operatorname{Lie} U_-$.

Definition 2.1. *We define the universal enveloping algebra of $\mathfrak{g}$ as follows: $\mathcal{U}(\mathfrak{g}) := T^\bullet(\mathfrak{g})/I$, where I is a two-sided ideal in the tensor algebra $T^\bullet(\mathfrak{g})$ generated by $\{x \otimes y - y \otimes x - [x,y] \,|\, x, y \in \mathfrak{g}\}$.*

The natural $\mathbb{Z}_{\geqslant 0}$-grading on $T^\bullet(\mathfrak{g})$ induces an increasing $\mathbb{Z}_{\geqslant 0}$-filtration $F^\bullet\mathcal{U}(\mathfrak{g})$.

Theorem 2.2 (PBW decomposition)**.** *There exists an isomorphism of algebras $\operatorname{gr} F^\bullet\mathcal{U}(\mathfrak{g}) \simeq S^\bullet(\mathfrak{g})$ which can be described as follows. Let $x_1, \ldots, x_N$ be a basis in $\mathfrak{g}$. Then we send $[x_1^{k_1} \ldots x_N^{k_N}] \mapsto x_1^{k_1} \ldots x_N^{k_N}$.*

Proof. We have to prove that the elements $[x_1^{k_1} \ldots x_N^{k_N}]$ form a basis of $\operatorname{gr} F^\bullet\mathcal{U}(\mathfrak{g})$. It is easy to see that they span the whole $\operatorname{gr} F^\bullet\mathcal{U}(\mathfrak{g})$. To see that they are linearly independent one should embed $\mathcal{U}(\mathfrak{g})$ in the the algebra of differential operators on G and to note that symbols of the corresponding differential operators being restricted to a small neighbourhood of $1 \in G$ are linearly independent. $\square$

Corollary 2.3. *Let $\mathfrak{g}$, $\mathfrak{l}$ be Lie algebras then $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{l}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{g} \oplus \mathfrak{l})$ via the map $g \otimes l \mapsto gl$. This is an isomorphism of $\mathcal{U}(\mathfrak{g})$-$\mathcal{U}(\mathfrak{l})$ bimodules.*

For $\lambda \in \mathfrak{h}^*$ we define Verma modules as follows $M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}^\lambda$, where $\mathbb{C}^\lambda$ is a one dimensional module on which $\mathfrak{b}$ acts via $\mathfrak{b} \twoheadrightarrow \mathfrak{h} \xrightarrow{\lambda} \mathbb{C}$.

Proposition 2.4. *Module $M(\lambda)$ has the following universal property: for any object $M \in \mathcal{O}$ we have $\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M) \simeq \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}^\lambda, M)$.*

By construction, $M(\lambda)$ is generated over $\mathfrak{g}$ by a vector $v_\lambda \in M_\lambda$ which is annihilated by $\mathfrak{n}$ and which has $\mathfrak{h}$-weight λ.

Lemma 2.5. *The vector v_λ freely generates $M(\lambda)$ over $\mathfrak{n}_-$ i.e. the map $\mathcal{U}(\mathfrak{n}_-) \to M_\lambda$, $x \mapsto xv_\lambda$ is an isomorphism (of left $\mathcal{U}(\mathfrak{n}_-)$-modules).*

Proof. Use the isomorphism $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{n}_-) \otimes \mathcal{U}(\mathfrak{b})$ as $\mathcal{U}(\mathfrak{n}_-)$-$\mathcal{U}(\mathfrak{b})$ bimodules. □

Proposition 2.6. *The action of $\mathfrak{h}$ on $M(\lambda)$ is locally finite and semisimple. The eigenvalues are of the form $\lambda - \sum_{\alpha \in \Delta_+} n_\alpha \alpha$, $n_\alpha \in \mathbb{Z}_{\geqslant 0}$.*

Proof. Follows from Lemma 2.5 and the PBW-decomposition for $\mathcal{U}(\mathfrak{n}_-)$. □

Theorem 2.7. *The Verma module $M(\lambda)$ admits a unique irreducible quotient module $L(\lambda)$.*

Proof. Let N be the union of all proper submodules of $M(\lambda)$. Note that $N_\lambda = 0$, hence, N is the maximal proper submodule. It follows that $L(\lambda) = M(\lambda)/N$ is the unique irreducible quotient. □

Lemma 2.8. *For $\lambda \neq \lambda'$ the modules $L(\lambda), L(\lambda')$ are non-isomorphic.*

Proof. If $L(\lambda) \simeq L(\lambda')$ then by Proposition 2.6 we have $\lambda \in \lambda' - Q^+$, $\lambda' \in \lambda - Q^+$, hence, $\lambda = \lambda'$. □

We will now define the main object of our study – the category $\mathcal{O}$. It will contain all Verma modules. Simples in the category $\mathcal{O}$ will be precisely $L(\lambda)$, $\lambda \in \mathfrak{h}^*$.

Definition 2.9. *We denote by $\mathcal{O}$ the category of finitely generated $\mathfrak{g}$-modules M such that the action of $\mathfrak{h}$ is diagonalizable, each $\mathfrak{h}$-weight has finite multiplicity and the $\mathfrak{h}$-weights are bounded from the above i.e. there exist $\lambda_1, \dots, \lambda_N \in \mathfrak{h}^*$ such that any weight λ of M lies in $\lambda_i - \sum_{\alpha \in \Delta_+} \mathbb{N}\alpha$ for some $= 1, \dots, N$.*

Lemma 2.10. *Verma modules $M(\lambda)$ lie in the category $\mathcal{O}$.*

Proof. Follows from Proposition 2.6. □

Proposition 2.11. *Every object in the category $\mathcal{O}$ is a quotient of a finite successive extension of Verma modules.*

Proof. Fix a module $M \in \mathcal{O}$ and denote by $W \subset M$ a finite dimensional subspace which generates M over $\mathfrak{g}$. Set $W' := \mathcal{U}(\mathfrak{b}) \cdot W$. Note that W' is finite dimensional. We have a surjection $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} W' \twoheadrightarrow M$. It remains to note that by 1.20 the module W' is a successive extension of 1-dimensional $\mathfrak{b}$-modules. □

Proposition 2.12. *For $\lambda, \mu \in \mathfrak{h}^*$, $\mathrm{Ext}^1(M(\mu), M(\lambda)) \neq 0$ implies $\mu < \lambda$.*

Proof. Consider an exact sequence

$$0 \to M(\lambda) \to M \to M(\mu) \to 0 \tag{2.13}$$

and assume that it does not split. Let $v \in M_\mu$ be a preimage of the highest weight $v_\mu \in M(\mu)$. Sequence 2.13 does not split so v is not annihilated by $\mathfrak{n}$. It follows that $\mu \leqslant \lambda$. □

2.2. **Harish-Chandra isomorphism and block decomposition. Finite length. Structure of K_0.** Let us now describe the center of the algebra $\mathcal{U}(\mathfrak{g})$. Let $Z(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$ be the center. Note that $Z(\mathfrak{g}) = \{u \in \mathcal{U}(\mathfrak{g}),\ [x,u] = 0\ \forall x \in \mathfrak{g}\}$. Let us now construct a homomorphism $Z(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h}) \simeq S^\bullet(\mathfrak{h})$. Fix $z \in Z(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$. Note that the element z acts on $M(\lambda)$ via some scalar $z(\lambda)$. We obtain a polynomial function $\mathfrak{h}^* \to \mathbb{C}$ i.e. an element of $S^\bullet(\mathfrak{h})$. It follows from the construction that have a homomorphism of algebras $\phi\colon Z(\mathfrak{g}) \to S^\bullet(\mathfrak{h})$. We introduce a new (dotted) action of W on $\mathfrak{h}$ as follows: $w \cdot \lambda = w(\lambda+\rho) - \rho$.

Theorem 2.14 (Harish-Chandra)**.** *The map ϕ induces the isomorphism $Z(\mathfrak{g}) \xrightarrow{\sim} S^\bullet(\mathfrak{h})^W$, where invariants are taken with respect to the dotted action.*

Example 2.15. *Let us consider the case $\mathfrak{g} = \mathfrak{sl}_2$. We have $W = S_2 = \{e, s\}$, $\rho = 1$ and the center is generated by the Casimir element*

$$c = 2ef + 2fe + h^2 = 4fe + 2h + h^2.$$

We see that c acts on $M(\lambda)$ via $\lambda(\lambda+2)$. The dotted action is given by $s\cdot\lambda = s(\lambda+1)-1 = -\lambda-2$. We see that $\mathbb{F}[\lambda]^W = \mathbb{F}[\lambda(\lambda+2)]$ so everything works.

Corollary 2.16. *We can identify $\operatorname{Spm}(Z(\mathfrak{g})) = \operatorname{Spm}(S^\bullet(\mathfrak{h})^W) = \mathfrak{h}^*/W$, where quotient and invariants are taken with respect to the dotted action.*

Definition 2.17. *For $\lambda \in \mathfrak{h}^*$ we denote by $[\lambda] \in \mathfrak{h}^*/W$ the image of λ under the projection morphism $\mathfrak{h}^* \twoheadrightarrow \mathfrak{h}^*/W$, where quotient is taken with respect to the dotted action. Note that $M_\lambda \in \mathcal{O}_{[\lambda]}$.*

Proposition 2.18. *The action of $Z(\mathfrak{g})$ on every object $M \in \mathcal{O}$ factors through an ideal of finite codimension.*

Proof. By Proposition 2.11 the assertion reduces to the case when $M = M(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. In the latter case the action factors through the kernel of the map $Z(\mathfrak{g}) \hookrightarrow S^\bullet(\mathfrak{h}) \xrightarrow{\lambda} \mathbb{C}$, which is a maximal ideal. □

Corollary 2.19. *Every object M of $\mathcal{O}$ splits as a direct sum $M \simeq \bigoplus_{\chi \in \operatorname{Spm}(Z(\mathfrak{g}))} M_\chi$, where the action of $Z(\mathfrak{g})$ on M_χ factors through some power of the maximal ideal which corresponds to χ. We also have $\operatorname{Hom}(M_\chi, M_{\chi'}) = 0$ for $\chi \neq \chi'$.*

So we have a decomposition $\mathcal{O} \simeq \bigoplus_{\chi \in \operatorname{Spm}(Z(\mathfrak{g}))} \mathcal{O}_\chi$, where $\mathcal{O}_\chi$ consists of modules annihilated by some power of the ideal $\mathfrak{m}_\chi$.

Corollary 2.20. *(1) Verma modules $M(\lambda)$, $M(\mu)$ lie in the same block $\mathcal{O}_\chi$ iff $\mu = w \cdot \mu$ for some $w \in W$.*

(2) Simple modules $L(\lambda)$, $L(\mu)$ lie in the same block $\mathcal{O}_\chi$ iff $\mu = w \cdot \mu$ for some $w \in W$.

(3) If $L(\mu)$ is isomorphic to a subquotient of $M(\lambda)$ then $\mu = w \cdot \lambda$ for some $w \in W$.

Corollary 2.21. *Every object of $\mathcal{O}$ has a finite length.*

Proof. By Proposition 2.11 it is enough to show that $M(\lambda)$ has a finite length for any $\lambda \in \mathfrak{h}^*$. This follows from Corollary 2.20. □

Definition 2.22. *Character $\alpha \in \mathfrak{h}^*$ is called regular if $\operatorname{Stab}_W(\alpha+\rho) = \{1\}$.*

Lemma 2.23. *Character $\alpha \in \mathfrak{h}^*$ is regular iff $\langle \lambda+\rho, \alpha\rangle \neq 0$ for any $\alpha \in \Delta$ i.e. $\lambda+\rho$ does not lie on any wall H_α of our hyperplane arrangement.*

Proposition 2.24. *Fix $\lambda \in \mathfrak{h}^*$. We have the idenification $W/\operatorname{Stab}_W(\lambda+\rho) \xrightarrow{\sim} \operatorname{Irr}(\mathcal{O}_{[\lambda]})$ via $w \mapsto w \cdot \lambda$.*

Proof. Follows from Corollary 2.20. ☐

Corollary 2.25. *For a regular* $\lambda \in \mathfrak{h}^*$*. We have* $W \xrightarrow{\sim} \mathrm{Irr}(\mathcal{O}_{[\lambda]})$, $\dim(K_0(\mathcal{O}_\lambda)) = |W|$.

Corollary 2.26. *We have* $M(-\rho) = L(-\rho) = M^\vee(-\rho)$*. Moreover* $\mathcal{O}_{-\rho} \simeq \mathrm{Vect}$ *via* $M(-\rho) \mapsto \mathbb{F}$*.*

Proof. Exersise. ☐

2.3. **Duality, dual Verma modules.** Let $\tau\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ be a Cartan involution (see Example 1.75). It is uniquelly determined by the property: τ is an authomorphism of $\mathfrak{g}$ such that $\tau(\mathfrak{n}) = \mathfrak{n}_-$, $\tau|_{\mathfrak{h}} = -\mathrm{Id}$. For $\mathfrak{g} = \mathfrak{gl}_n$ it is given by $A \mapsto -A^T$.

For $M \in \mathcal{O}$ we define another object $M^\vee$ as follows: $M^\vee := \bigoplus_\mu M(\mu)^*$ the action $x \cdot f(m) = f(-\tau(x)m)$. Note that $\tau^2 = \mathrm{Id}$ so we have

$$(M^\vee)^\vee = M. \tag{2.27}$$

Proposition 2.28. *For any* $\lambda \in \mathfrak{h}^*$ *we have* $L(\lambda)^\vee \simeq L(\lambda)$*.*

Proof. It follows from 2.27 that the module $L(\lambda)^\vee$ is irreducible. The highest weight of $L(\lambda)^\vee$ is λ because $\mathrm{ch}\, L(\lambda)^\vee = \mathrm{ch}\, L(\lambda)$, hence, $L(\lambda)^\vee \simeq L(\lambda)$. ☐

Corollary 2.29. *We have* $[M(\lambda)] = [M(\lambda)^\vee]$ *in* $K_0(\mathcal{O})$*.*

Corollary 2.30. *If* M *belongs to* $\mathcal{O}$ *then so does* $M^\vee$*. Moreover if* $M \in \mathcal{O}_\chi$ *for some* χ *then so does* $M^\vee$*.*

Proposition 2.31. *The functor* $\bullet^\vee\colon \mathcal{O} \xrightarrow{\sim} \mathcal{O}$ *is a contravariant (involutive) self-equivalence.*

Proof. Follows from 2.27. ☐

We shall now study dual Verma modules $M^\vee_\lambda$. We begin with their universal property.

Proposition 2.32. *For any* $M \in \mathcal{O}$ *we have* $\mathrm{Hom}(M, M^\vee(\lambda)) \simeq (M/M\mathfrak{n}_-)^*_\lambda$*.*

Proof. By Proposition 2.4 and Proposition 2.31 we have $\mathrm{Hom}(M, M^\vee(\lambda)) \simeq \mathrm{Hom}(M(\lambda), M^\vee) \simeq \mathrm{Hom}_{\mathfrak{b}}(\mathbb{C}^\lambda, M^\vee)$. The later space is the set of functionals $f \in M^\vee$ which are annihilated by $\tau(\mathfrak{n}) = \mathfrak{n}_-$ and have weight λ. This space is canonically isomorphic to $(M/M\mathfrak{n}_-)^*_\lambda$. ☐

Theorem 2.33. *(1) The module* $M^\vee(\lambda)$ *has* $L(\lambda)$ *as its unique irreducible subquotient.*

(2) $\mathrm{Hom}(M(\lambda), M^\vee(\lambda)) = \mathbb{C}$*, such that* $1 \in \mathbb{C}$ *corresponds to the composition*

$$M(\lambda) \twoheadrightarrow L(\lambda) \hookrightarrow M^\vee(\lambda).$$

(3) For $\lambda \neq \mu$ $\mathrm{Hom}(M(\lambda), M^\vee(\mu)) = 0$*.*

(4) $\mathrm{Ext}^1(M(\lambda), M^\vee(\mu)) = 0$ *for all* λ, μ*.*

Proof. Point (1) follows from Theorem 2.7. By Proposition 2.32 we have $\mathrm{Hom}(M(\lambda), M^\vee(\mu)) \simeq (M(\lambda)/M(\lambda) \cdot \mathfrak{n}_-)^*_\mu$. Note that $M(\lambda)/M(\lambda) \cdot \mathfrak{n}_- = \mathbb{C}_\lambda$. Points (2), (3) follow. To prove (4) consider a short exact sequence

$$0 \to M^\vee(\mu) \to M \to M(\lambda) \to 0. \tag{2.34}$$

We need to show that it splits. Consider a canonical $\mathfrak{b}_-$-equivariant functional $M^\vee(\mu) \to \mathbb{C}^\mu$ and denote by M' its kernel. We have a short exact sequence

$$0 \to \mathbb{C}^\mu \to M/M' \to M(\lambda) \to 0 \tag{2.35}$$

of $\mathfrak{b}_-$-modules. By Proposition 2.32 sequence 2.34 of $\mathfrak{g}$-modules splits iff sequence 2.35 of $\mathfrak{b}_-$-modules split. It remains to note that M_λ is a free $\mathfrak{n}_-$-module (see Lemma 2.5) so its enough to split 2.35 as a sequence of $\mathfrak{h}$-modules. This is possible since the action of $\mathfrak{h}$ is semi-simple. ☐

2.4. **Projective functors and projective objects,** BGG **reciprocity, order.**

2.4.1. *Projective functors.* Let V be a finite dimensional module. We can consider the functor $T_V\colon \mathfrak{g}\text{-mod} \to \mathfrak{g}\text{-mod}$ given by $M \mapsto M \otimes V$. This functor sends $\mathcal{O}$ to itself. This functor is exact and its both left an right adjoint is T_{V^*}. In particular, T_V sends projectives to projectives and injectives to injectives.

Definition 2.36. *We say that an object $M \in \mathcal{O}$ is standardly (resp. costandardly) filtered if there exists a finite filtration of M by $\mathfrak{g}$-submodules such that the associated graded quotients are $M(\lambda)$ (resp. $M^\vee(\lambda)$). We denote by $\mathcal{O}^\Delta$ (resp. $\mathcal{O}^\nabla$) the full subcategory of standardly (resp. costandardly) filtered modules.*

We will need the following Lemma in the next subsection.

Lemma 2.37. *(1) A direct summand of an object admitting a standard filtration itself admits a standard filtration.*

(2) Kernel of an epimorphism between standardly filtered modules is standardly filtered.

Lemma 2.38. *The module $M_\lambda \otimes V$ admits a filtration whose subquotients are isomorphic to $M_{\lambda+\mu}$. Moreover* $\mathrm{mult}(M_{\lambda+\mu}, M_\lambda \otimes V) = \dim(V(\mu))$.

Proof. As a $\mathfrak{b}$-module we have $M_\lambda \otimes V \simeq \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} (V \otimes \mathbb{C}^\lambda)$. The claim follows. □

Definition 2.39. *For $\mu \in \mathfrak{h}^*$ we denote by $\mathrm{pr}_\mu\colon \mathcal{O} \to \mathcal{O}_{[\mu]}$ the projection functor. We also denote by $\iota_\mu\colon \mathcal{O}_{[\mu]} \hookrightarrow \mathcal{O}$ the natural embedding.*

Definition 2.40. *Fix $\lambda, \mu \in \Lambda$ such that $\lambda + \rho, \mu + \rho \in \Lambda^+$ (i.e. λ, μ lie in the dominant chamber for dotted action). Set $\nu = \mu - \lambda$ and define the translation functor $T_{\lambda\to\mu}\colon \mathcal{O}_{[\lambda]} \to \mathcal{O}_{[\mu]}$ by $T_{\lambda\to\mu}(M) = \mathrm{pr}_\mu(\iota_\lambda(M) \otimes V(\overline{\nu}))$, where $\overline{\nu}$ is the dominant representative in $W\nu$.*

Remark 2.41. *Note that for any $\chi \in \mathfrak{h}^*/W$ there exists a unique $\lambda \in \mathfrak{h}^*$ such that $\lambda + \rho \in \Lambda^+$ and $[\lambda] = \chi$.*

Example 2.42. *Consider the example: $\mathfrak{g} = \mathfrak{sl}_2, \lambda = 0, \mu = -1$. So $V(\overline{\nu}) = \mathbb{C}^2 = L(1)$. Let us compute the images of Verma and simple modules $M(0)$, $M(-2) = L(-2), L(0) = \mathbb{C} \in \mathcal{O}_0$. By Lemma 2.38 $M(0) \otimes \mathbb{C}^2$ admits a filtration whose subquotients are $M(1), M(-1)$. Note that they lie in different blocks and $M_{-1} \in \mathcal{O}_{-1}$, hence, $T_{0\to-1}(M(0)) = M(-1)$. By Lemma 2.38 $M(-2) \otimes \mathbb{C}^2$ admits a filtration whose subquotients are $M(-1), M(-3)$. So $T_{0\to-1}(M_{-2}) = M(-1)$. Note now that $\mathbb{C}\otimes\mathbb{C}^2 \simeq \mathbb{C}^2 = L(1)$ and does not lie in the category $\mathcal{O}_{-1}$ so $T_{0\to-1}(L_0) = 0$. Let us now compute $T_{-1\to0}(M(-1))$. Again by Lemma 2.38 $M(-1) \otimes \mathbb{C}^2$ admits a filtration whose subquotients are $M(0), M(-2)$. Note that both $M(0)$, $M(-2)$ lie in $\mathcal{O}_0$. It follows that $T_{-1\to0}(M(-1)) = M(-1) \otimes \mathbb{C}^2 =: P(-2)$. Recall that it includes into the following short exact sequence*

$$0 \to M_0 \to P_{-2} \to M_{-2} \to 0.$$

Lemma 2.43. *Fix $\lambda, \mu \in \Lambda^+$ such that the face F containing λ contains μ in its closure and set $\nu = \lambda - \mu$. Set $\overline{\nu} = \Lambda^+ \cap W\nu$. Let ν' be a weight of $L(\nu)$ such that $\lambda + \nu' \in W\mu$. Then $\nu' = \nu$.*

Proof. Let $C = C_+$ be the dominant chamber. Note that $\lambda, \lambda + \nu \in \overline{C}$. Let C' be a chamber such that $\lambda + \nu' \in \overline{C}'$. We can assume that $\nu' \neq \nu$ is choosen such that $d(C, C')$ is minimal. Let H_α be a wall of C' which sepparates C and C'. We can assume that $\langle C', \alpha^\vee\rangle > 0$, $\langle C, \alpha^\vee\rangle < 0$. Set $C'' := s_\alpha(C'')$. Set

$$\nu'' := s_\alpha(\lambda + \nu') - \lambda = s_\alpha(\nu') - \langle \lambda, \alpha^\vee\rangle\alpha = \nu' - \langle \lambda + \nu', \alpha^\vee\rangle\alpha \tag{2.44}$$

and note that $\lambda+\nu'' \in C''$. It follows from 2.44 that

$$s_\alpha(\nu') \leqslant \nu'' \leqslant \nu' \tag{2.45}$$

so ν'' is a weight of $L(\overline{\nu})$, hence, $\nu''=\nu$ is an extremal weight of $L(\overline{\nu})$. It now follows from 2.45 that $s_\alpha(\nu')=\nu=\nu''$, hence, $\langle\lambda,\alpha^\vee\rangle=0$. Recall that $\mu\in\overline{F}$ so $\langle\mu,\alpha^\vee\rangle=0$ and $\langle\nu,\alpha^\vee\rangle=0$, hence, $\nu'=s_\alpha(\nu)=\nu$. Contradiction. □

Theorem 2.46. *(1) Functor $T_{\lambda\to\mu}$ is biadjoint to $T_{\mu\to\lambda}$ and exact. It maps projectives (resp. injectives) to projectives (resp. injectives), standardly (resp. costandardly) filtered to standardly (resp. costandly) filtered.*

(2) We have a canonical isomorphism $T_{\lambda\to\mu}(\bullet^\vee)\simeq T_{\lambda\to\mu}(\bullet)^\vee$.

(3) If $\mu+\rho$ lies in the closure of the face of $\lambda+\rho$ (i.e. $\mathrm{Stab}_W(\lambda+\rho)\subset\mathrm{Stab}_W(\mu+\rho)$) then $T_{\lambda\to\mu}(M(w\cdot\lambda))=M(w\cdot\mu)$ (resp. $T_{\lambda\to\mu}(M^\vee(w\cdot\lambda))=M^\vee(w\cdot\mu)$) for any $w\in W$ and $T_{\mu\to\lambda}(M(w\cdot\mu))$ (resp. $T_{\mu\to\lambda}(M^\vee(w\cdot\mu))$) is filtered by modules of the form $M(w'\cdot\lambda)$ (resp. $M^\vee(w'\cdot\lambda)$), where $w'\in w\,\mathrm{Stab}_W(\mu+\rho)$.

(4) If $\mu+\rho$ lies in the closure of the face of $\lambda+\rho$ then $T_{\lambda\to\mu}(L(w\cdot\lambda))$ is $L(w\cdot\mu)$ if w is the longest element in coset $w\,\mathrm{Stab}_W(\mu+\rho)$ and is zero otherwise.

(5) Pick $\lambda_1,\lambda_2,\lambda_3\in\Lambda$. Denote by Γ_i the face containing $\lambda_i+\rho$. If $\overline{\Gamma}_3\subset\overline{\Gamma}_2\subset\overline{\Gamma}_1$ then

$$T_{\lambda_2\to\lambda_3}\circ T_{\lambda_1\to\lambda_2}\simeq T_{\lambda_1\to\lambda_3},\, T_{\lambda_2\to\lambda_1}\circ T_{\lambda_3\to\lambda_2}\simeq T_{\lambda_3\to\lambda_1}.$$

Proof. Part (2) follows from Proposition 2.28, Corollary 2.28 and the fact that $\bullet^\vee$ commutes with tensor products. Part (1) follows from part (2) and the fact that the functors T_V and T_{V^*}, ι_α and pr_α are biajoint for any finite dimensional $V\in\mathrm{Rep}(G)$ and $\alpha\in\mathfrak{h}^*$. First statement of part (3) follows from part (2) and Lemma 2.43 applied to $\lambda+\rho,\mu+\rho$. Second statement of (3) follows from the first statement using the adjunction of $T_{\lambda\to\mu}$ and $T_{\mu\to\lambda}$ and Corollary 2.56:

$$\begin{aligned}\mathrm{mult}(M(w'\cdot\lambda),T_{\mu\to\lambda}(M(w\cdot\mu)))=\dim\mathrm{Hom}(T_{\mu\to\lambda}(M(w\cdot\mu)),M^\vee(w'\cdot\lambda))=\\ =\dim\mathrm{Hom}(M(w\cdot\mu),M^\vee(w'\cdot\mu))=\delta_{w\cdot\mu,w'\cdot\mu}.\end{aligned}$$

To prove part (4) we consider surjection and injection:

$$M(w\cdot\lambda)\twoheadrightarrow L(w\cdot\lambda)\hookrightarrow M^\vee(w\cdot\lambda)$$

after applying $T_{\lambda\to\mu}$ and using (3) we obtain

$$M(w\cdot\lambda)\twoheadrightarrow T_{\lambda\to\mu}(L(w\cdot\lambda))\hookrightarrow M^\vee(w\cdot\lambda).$$

It follows that $T_{\lambda\to\mu}(L(w\cdot\lambda))=L(w\cdot\mu)$ or $T_{\lambda\to\mu}(L(w\cdot\lambda))=0$. It can be shown that $T_{\lambda\to\mu}(L(w\cdot\lambda))=L(w\cdot\mu)$ iff w is the longest in $w\,\mathrm{Stab}_W(\mu+\rho)$. □

Corollary 2.47. *Suppose $\lambda_1+\rho,\lambda_2+\rho\in\Lambda^+$ lie in the same face. Then there is an equivalence $\mathcal{O}_{\lambda_1}\xrightarrow{\sim}\mathcal{O}_{\lambda_2}$ that takes $M_{w\cdot\lambda_1}$ to $M_{w\cdot\lambda_2}$, $L_{w\cdot\lambda_1}$ to $L_{w\cdot\lambda_2}$, $M^\vee_{w\cdot\lambda_1}$ to $M^\vee_{w\cdot\lambda_2}$, $P_{w\cdot\lambda_1}$ to $P_{w\cdot\lambda_2}$.*

Proof. Functors $T_{\lambda_1\to\lambda_2}$, $T_{\lambda_2\to\lambda_2}$ are mutually inverse equivalences. □

Let us now consider the following important example of translation functors. Recall a standard parabolic subalgebra $\mathfrak{p}_I\subset\mathfrak{g}$ with Levi $\mathfrak{l}_I$. Set $\eta_I:=-\sum_{\alpha\in I}\omega_\alpha$. Note that $\mathrm{Stab}_W(\eta_I+\rho)=W_I$ so we have an identification $\mathrm{Irr}(\mathcal{O}_{\eta_I})\simeq W/W_I$ given by $w\mapsto L(w\cdot\eta_I)$. Recall also the identification $\mathcal{O}_0\simeq W$, $w\mapsto L(w\cdot 0)$.

Corollary 2.48. *The functor $T_{0\to\eta_I}$ sends w to $[w]$ if $w\in wW_I$ is the longest element and sends w to zero otherwise.*

2.4.2. *Reflection functors.* Take $\lambda = 0$ and $\mu = \mu_i$ such that $\mu_i + \rho$ is dominant and $\mathrm{Stab}_W(\mu + \rho) = \langle s_i \rangle$ i.e. $\langle \mu_i, \alpha_j^\vee \rangle > -1$ for any $j \neq i$ and $\langle \mu_i, \alpha_i^\vee \rangle = -1$. For example, we can take $\mu_i = -\omega_i$.

Definition 2.49. *We define $\Theta_i := T_{\mu_i \to 0} \circ T_{0 \to \mu_i}$. These functors are called reflection functors.*

Proposition 2.50. *We identify $W \xrightarrow{\sim} \mathcal{O}_0$ via $w \mapsto M(w \cdot 0)$. Then the map $[\Theta_i] \colon \mathbb{C}[W] = K_0(\mathcal{O}_0) \xrightarrow{\sim} \mathcal{O}_0 = \mathbb{C}[W]$ is given by $w \mapsto w(1 + s_i)$.*

Proof. It follows from Theorem 2.46 that $T_{0 \to \mu_i}(M(w \cdot 0)) = M(w \cdot \mu_i)$ and $T_{\mu_i \to 0}(M(w \cdot \mu_i))$ is filtered by modules $M(w \cdot 0), M(ws_i \cdot 0)$. Proposition follows. □

Remark 2.51. *We have the natural morphism of functors $\mathrm{Id} \to \Theta_i$ which comes from the adjointness. The cone of this morphism defines a derived self-equivalence of the category $D^b(\mathcal{O}_0)$ given by $w \mapsto ws_i$ on K_0. This autoequivalence is called wall-crossing functor.*

2.4.3. *Projective objects and* BGG*-reciprocity.* We say that an abelian category $\mathscr{C}$ has enough projectives if every object from $\mathscr{C}$ admits a surjection from the projective one.

Definition 2.52. *For $w \in W$ choose a reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_2}$. Set $\Theta_{\underline{w}} := \Theta_{i_l} \circ \dots \circ \Theta_{i_2} \circ \Theta_{i_1}$.*

Theorem 2.53. *(1) The category $\mathcal{O}_0$ has enough projectives.*

(2) For $w \in W$ we denote by $P(w \cdot 0)$ the projective cover of $L(w \cdot 0)$. Object $P(w \cdot 0)$ is standardly filtered and we have $[P(w \cdot 0)] \in M(w \cdot 0) + \sum_{w' \prec w} M(w' \cdot 0)$.

(3) The object $P_{\underline{w}} := \Theta_{\underline{w}}(\Delta(0))$ is isomorphic to the direct sum of $P(w \cdot 0)$ and $P(w' \cdot 0)$ with $w' \prec w$.

Proof. It follows from Theorem 2.46 that the objects $P_{\underline{w}}$ are projective and standardly filtered. By Proposition 2.50 we have $[P_{\underline{w}}] \in [M(w \cdot 0)] + \sum_{w' \prec w} \mathbb{Z}_{\geqslant 0} \cdot M(w' \cdot 0)$. It now follows from Proposition 2.12 that we have a surjection $P_{\underline{w}} \twoheadrightarrow M(w \cdot 0)$ so by Proposition 2.11 $\mathcal{O}_0$ has enough projectives and part (1) is proved.

It also follows that $P(w \cdot 0)$ is a direct summand of $P_{\underline{w}}$, hence is standardly filtered by Lemma 2.37 with possible associated graded of the form $M(w' \cdot 0)$ with $w' \preceq w$. The surjection $P(w \cdot 0) \twoheadrightarrow L(w \cdot 0)$ rises to a morphism $P(w \cdot 0) \to M(w \cdot 0)$ which is surjective by Theorem 2.7. Using Lemma 2.37 we deduce (2).

Now (3) follows from (2) and the fact that $[P_{\underline{w}}] \in [M(w \cdot 0)] + \sum_{w' \prec w} \mathbb{Z}_{\geqslant 0} \cdot M(w' \cdot 0)$. □

Corollary 2.54. *Every projective object of $\mathcal{O}$ admits a standard filtration.*

Proposition 2.55. *We have $\mathrm{Ext}^i(M(\lambda), M^\vee(\mu)) = 0$ for any $i > 0$.*

Corollary 2.56. *Let M be a standardly filtered module then $\mathrm{mult}(M_\lambda, M) = \dim(\mathrm{Hom}(M, M_\lambda^\vee))$.*

Proof. It follows from Proposition 2.55 that the functor $\mathrm{Hom}(-, M_\lambda^\vee)$ is exact being restricted to the category $\mathcal{O}^\Delta \subset \mathcal{O}$. Now the claim follows from Theorem 2.33 using the induction on the length. □

Corollary 2.57 (BGG resiprocity)**.** *We have*

$$\mathrm{mult}(M_\mu, P_\lambda) = \mathrm{mult}(L_\lambda, M_\mu^\vee),\ \mathrm{mult}(I_\lambda, M_\mu^\vee) = \mathrm{mult}(L_\lambda, M_\mu)$$

Proof. They are equal to $\dim \mathrm{Hom}(P(\lambda), M_\mu^\vee)$, $\dim \mathrm{Hom}(M_\mu, I_\lambda)$ respectively. □

2.5. **Highest weight structure and tilting objects.**

Definition 2.58 (HW categories)**.** *Let $\mathscr{C}$ be an abelian category with finite number of simple objects. Let Ξ be the parametrizing set for simples in $\mathscr{C}$. The highest weight structure on $\mathscr{C}$ is the pre-order on Ξ and a collection $\Delta(\lambda) \in \mathscr{C}$ of objects such that the following conditions hold:*

(1) We have the morphism $P(\lambda) \to \Delta(\lambda)$ such that the kernel of this morphism admits a filtration whose quotients are of the form $\Delta(\lambda')$, $\lambda' > \lambda$.

(2) $\mathrm{Hom}(\Delta(\mu), \Delta(\lambda)) \neq 0$ implies $\mu \leqslant \lambda$.

(3) $\mathrm{End}(\Delta(\lambda)) = \mathbb{C}$.

Theorem 2.59. *Category $\mathcal{O}_0$ is HW with standards $M(w \cdot (-2\rho))$. The order is the Bruhat order $\prec$.*

Let us list the main properties of HW-categories.

Proposition 2.60. *Let $\mathscr{C}$ be a HW category then the following holds.*

a) Fix $\lambda, \mu \in \Xi$ then $L(\lambda)$ occurs in $\Delta(\mu)$ only if $\lambda \leqslant \mu$. Moreover the multiplicity of $L(\lambda)$ in $\Delta(\lambda)$ is one, $\Delta(\lambda) \twoheadrightarrow L(\lambda)$ and $\mathrm{Hom}(\Delta(\lambda), L(\mu)) = \delta_{\lambda,\mu}$.

b) If $\mathrm{Ext}^i(\Delta(\lambda), \Delta(\mu)) \neq 0$ for some $i > 0$ then $\lambda < \mu$.

c) If $\mathrm{Ext}^i(\Delta(\lambda), L(\mu)) \neq 0$ for some $i > 0$ then $\lambda < \mu$.

d) Fix $\lambda \in \Xi$. Consider the Serre subcategory $\mathscr{C}_{\leqslant\lambda}$ (resp. $\mathscr{C}_{\not>\lambda}$) spanned by $L(\mu)$ with $\mu \leqslant \lambda$ (resp. $\mu \not> \lambda$). Then $\Delta(\lambda)$ is the projective cover of $L(\lambda)$ in $\mathscr{C}_{\leqslant\lambda}$ (resp. $\mathscr{C}_{\not>\lambda}$).

Definition 2.61. *By the definition, $\nabla(\lambda)$ is the injective envelope of $L(\lambda)$ in $\mathscr{C}_{\leqslant\lambda}$ or in $\mathscr{C}_{\not>\lambda}$.*

For $\mathscr{C} = \mathcal{O}_0$ we have $\Delta(w) = M^\vee(w \cdot (-2\rho))$.

Lemma 2.62 (c.f. Theorem 2.33 and Proposition 2.55)**.** $\dim \mathrm{Hom}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda,\mu}$ *and* $\mathrm{Ext}^i(\Delta(\lambda), \nabla(\mu)) = 0$ *for $i > 0$.*

Proposition 2.63. *Object $M \in \mathscr{C}$ is standardly (resp. costandardly) filtered iff $\mathrm{Ext}^i(M, \nabla(\lambda)) = 0$ (resp. $\mathrm{Ext}^i(\Delta(\lambda), M) = 0$) for any $i > 0$.*

Definition 2.64. *An object in $\mathscr{C}$ is called* tilting *if it is both standardly and costandardly filtered.*

Let us point out that by Proposition 2.63 for any tilting object T we have $\mathrm{Ext}^i(T, T) = 0$ for $i > 0$. Note also that if T is tilting and $T \simeq T_1 \oplus T_2$ then both T_1 and T_2 are also tilting. It follows that each tilting is a direct sum of indecomposable tilting objects. We describe indecomposable tiltings in the following proposition.

Proposition 2.65. *For each $\lambda \in \Xi$ there exists an indecomposable tilting object $T(\lambda)$ uniquely determined by the following property: $T(\lambda) \in \mathscr{C}_{\leqslant\lambda}$, $[\Delta(\lambda) : T(\lambda)] = 1 = [\nabla(\lambda) : T(\lambda)]$ and we have $\Delta(\lambda) \hookrightarrow T(\lambda) \twoheadrightarrow \nabla(\lambda)$.*

Proof. Fix $\lambda \in \Xi$ and order linearly elements of $\{\mu \in \Xi \,|\, \mu \leqslant \lambda\}$ refining the original poset structure on Ξ. Say $\lambda = \lambda_1 > \lambda_2 > \ldots > \lambda_k$. Let us construct the object $T^i(\lambda), i = 1, \ldots, k$ inductively as follows. Set $T^1(\lambda) = \Delta(\lambda)$. Further, if $T^{i-1}(\lambda)$ is already defined let $T^i(\lambda)$ be the extension of $\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda)) \otimes \Delta(\lambda_i)$ by $T^{i-1}(\lambda)$ corresponding to the unit endomorphism of $\mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}(\lambda))$ i.e. we have the following short exact sequence

$$0 \to T^{i-1}(\lambda) \to T^i(\lambda) \to \mathrm{Ext}^1(\Delta(\lambda_i), T^{i-1}) \otimes \Delta(\lambda_i) \to 0. \tag{2.66}$$

Object $T(\lambda) := T^k(\lambda)$ is tilting and satisfies all the desired properties. □

Example 2.67. *Let us consider the example $\mathscr{C} = \mathcal{O}_0(\mathfrak{sl}_2)$. We see that $T(-2) = \Delta(-2) = L(-2) = \nabla(-2)$. It is also easy to see that $T(0) = P(-2)$.*

Example 2.68. *More general* $T(0) = P(w_0\cdot 0) = P_{\min}$, $T(w_0\cdot 0) = T^1(w_0\cdot 0) = \Delta(w_0\cdot 0) = L_{\min}$. *Actually if we order linearly elements of* $W\cdot 0$: $0 = \lambda_1 > \lambda_2 > \ldots > \lambda_k = w_0\cdot 0$ *then* $T^1(0) = \Delta(0), T^k(0) = P_{\min}$. *To see this we note that* $P_{\min}$ *is both injective and projective (because* $P_{\min} = T_{-\rho\to 0}(M_{-\rho})$*), hence, tilting, we also now that* $P_{\min}$ *is indecomposable and that* $[P_{\min}] = \sum_{w\in W}[\Delta(w\cdot 0)]$. *It follows that we have an embedding* $\Delta(0) \hookrightarrow P_{\min}$ *(because* $\operatorname{Ext}^1(\Delta(0), \Delta(w\cdot 0)) = 0$ *for any* $w\in W$*). Now it follows that* $P_{\min} = T(0)$.

Lemma 2.69. *An object* $M \in \mathcal{O}_0$ *is tilting iff it is standardly (resp. costandardly) filtered and selfdual i.e.* $M \simeq M^\vee$.

Proof. Follows from Proposition 2.65 and the equivalence $\bullet^\vee : \mathcal{O}_0 \xrightarrow{\sim} \mathcal{O}_0^{\mathrm{opp}}$. □

Let us now restrict ourselves to the category $\mathcal{O}_0$. Recall the reflection functors Θ_i and their compositions $\Theta_{\underline{w}} := \Theta_{i_l}\circ \ldots \circ \Theta_{i_2}\circ\Theta_{i_1}$ which depened on a reduced decomposition $w = s_{i_1}s_{i_2}\ldots s_{i_2}$. Note that by Theorem 2.46 reflection functors send tiltings to tiltings. The following proposition describes indecomposable tiltings using functors $\Theta_{\underline{w}}$ (c.f. Theorem 2.53).

Theorem 2.70 (c.f. Theorem 2.53)**.** *The object* $T_{\underline{w}} := \Theta_{\underline{w}}(L_{-2\rho})$ *is the direct sum of* T_w *and* $T_{w'}$ *with* $w \prec w'$.

2.6. **Parabolic categories $\mathcal{O}$: parabolic Verma modules, structure of K_0.** We now fix a standard parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_I$ which is determined by the subset $I \subset J$ of the set of simple roots of $\mathfrak{g}$.

Definition 2.71. *We define the parabolic category* $\mathcal{O}^{\mathfrak{p}}$ *as a full subcategory of* $\operatorname{Mod}\mathcal{U}(\mathfrak{g})$ *consisting of modules* M *such that*

(1) M *is a finitely generated* $\mathfrak{g}$*-module.*

(2) Levi $\mathfrak{l} = \mathfrak{l}_I$ *acts locally finitely and semi-simply (i.e. its action integrates to the action of the group* L*).*

(3) Weights of M *are bounded from the above.*

Remark 2.72. *For* $I = \varnothing$ *we have* $\mathfrak{p} = \mathfrak{b}$ *and* $\mathcal{O}^{\mathfrak{b}} = \mathcal{O}$. *For* $I = J$ *we have* $\mathfrak{p} = \mathfrak{g}$ *and* $\mathcal{O}^{\mathfrak{g}} = \operatorname{Rep}_{f.d.}\mathfrak{g}$. *In general we have a full embedding* $\mathcal{O}^{\mathfrak{p}} \subset \mathcal{O}$.

Definition 2.73. *Let* Λ_I^+ *be the set of* $\lambda\in\Lambda$ *such that* $\langle\lambda, \alpha^\vee\rangle \geqslant 0$ *for any* $\alpha \in I$. *Note that* $\Lambda_S^+ = \Lambda^+$ *and* $\Lambda_\varnothing^+ = \Lambda$.

Definition 2.74. *For* $\lambda\in\Lambda_I^+$ *we define the parabolic Verma module* $M_I(\lambda) := \mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p})} L_I(\lambda)$, *where* $L_I(\lambda)$ *is the irreducible representation of* $\mathfrak{l}$ *with highest weight* λ. *Note that* $\dim L_I(\lambda) < \infty$.

Recall the embedding $\mathcal{O}^{\mathfrak{p}} \subset \mathcal{O}$.

Proposition 2.75. *(1)* $\mathcal{O}^{\mathfrak{p}}$ *is closed under direct sums, submodules, quotients, and extensions in* $\mathcal{O}$, *as well as tensoring with finite dimensional* $\mathcal{U}(\mathfrak{g})$*-modules.*

(2) For $\lambda\in\Lambda_I^+$ *the module* $M_I(\lambda)$ *belongs to the category* $\mathcal{O}^{\mathfrak{p}}$, *we have canonical surjection* $M_I(\lambda) \twoheadrightarrow L(\lambda)$.

(3) Simple object $L(\lambda)$ *lies in* $\mathcal{O}^{\mathfrak{p}}$ *iff* $\lambda\in\Lambda_I^+$. *Moreover category* $\mathcal{O}^{\mathfrak{p}}\subset\mathcal{O}$ *the Serre span of* $L(\lambda)$, $\lambda\in\Lambda_I^+$ *i.e. an object* $M\in\mathcal{O}$ *lies in* $\mathcal{O}^{\mathfrak{p}}$ *iff all its composition factors* $L(\lambda)$ *satisfy* $\lambda\in\Lambda_I^+$.

(4) If $M\in\mathcal{O}^{\mathfrak{p}}$ *decomposes as* $M = \bigoplus M^\chi$ *with* $M^\chi\in\mathcal{O}_\chi$ *then each* M^χ *lies in* $\mathcal{O}^{\mathfrak{p}}$*; this gives a decomposition* $\mathcal{O}^{\mathfrak{p}} = \bigoplus_\chi \mathcal{O}^{\mathfrak{p}}_\chi$.

(5) Translation functors preserve $\mathcal{O}^{\mathfrak{p}}$.

(6) Functor $\bullet^\vee : \mathcal{O}\to\mathcal{O}$ *restricts to the contravariant self-equivalence of the category* $\mathcal{O}^{\mathfrak{p}}$.

Proof. Part (1) is obvious. Part (2) is an exersise. Part (3) follows from part (2) and representation theory of $\mathfrak{sl}_2$. Part (4) is an exercise. Part (5) follows from parts (1), (4). Part (6) follows from part (3). □

Proposition 2.76. *(1) The category $\mathcal{O}^{\mathfrak{p}}$ has enough projectives.*

(2) Every projective in $\mathcal{O}^{\mathfrak{p}}$ is a direct sum of indecomposables $P_I(\lambda)$ indexed by Λ_I^+, where $P_I(\lambda)$ is a projective cover of $L(\lambda)$.

(3) Every $P_I(\lambda)$ can be filtered with associated graded $M_I(\mu)$

Corollary 2.77. *Every block $\mathcal{O}^{\mathfrak{p}}_\chi$ is* HW *with respect to the Bruhat order and with standard objects $M_I(\mu)$.*

Corollary 2.78. *The analogue of* BGG *reciprocity holds in $\mathcal{O}^{\mathfrak{p}}$: for $\lambda, \mu \in \Lambda_I^+$ we have*

$$\mathrm{mult}(M_I(\mu), P_I(\lambda)) = \mathrm{mult}(L(\lambda), M_I^\vee(\mu)).$$

It will be important for the latter to have a description of irreducible objects in the category $\mathcal{O}_0^{\mathfrak{p}}$. We identify $W \xrightarrow{\sim} \mathrm{Irr}(\mathcal{O}_0)$ via $w \mapsto L(w \cdot 0)$.

Lemma 2.79. *Let $\lambda \in \Lambda$ be a dominant weight. Then $w\lambda \in \Lambda_I^+$ iff w is the shotest element in the coset $W_L w$.*

Proof. Exercise. □

Corollary 2.80. *The set $\mathrm{Irr}(\mathcal{O}_0^{\mathfrak{p}}) \subset W$ is in bijection with the set of shortest representatives in cosets $W_L \backslash W$.*

We finish this section with a theorem which expresses $\mathrm{ch}\, M_I(\lambda)$, $\lambda \in \Lambda_I^+$ in terms of $\mathrm{ch}\, M(w \cdot \lambda)$, $w \in W_I$.

Theorem 2.81. *For $\lambda \in \Lambda_I^+$ we have*

$$\mathrm{ch}\, M_I(\lambda) = \sum_{w \in W_I} (-1)^{l(w)} \mathrm{ch}\, M(w \cdot \lambda)$$

Proof. Recall that $M_I(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} L_I(\lambda)$. It follows that $\mathrm{ch}\, M_I(\lambda) = \mathrm{ch}\, \mathcal{U}(\mathfrak{u}_{-,I}) \cdot \mathrm{ch}\, L_I(\lambda)$. By the (version of) Weyl character formula for $\mathfrak{l}_I$ we have $\mathrm{ch}\, L_I(\lambda) = \sum_{w \in W_I} (-1)^{l(w)} \mathrm{ch}\, M_I^{\mathfrak{l}}(w \cdot_I \lambda)$ where $M_I^{\mathfrak{l}}(w \cdot_I \lambda)$ are Verma modules in the category $\mathcal{O}$ for $\mathfrak{l}$ and $w \cdot_I \lambda = w(\lambda + \rho_I) - \rho_I$. Note that for $w \in W_I$ we have $w \cdot_I \lambda = w \cdot \lambda$ so we can drop subscript I. We have $\mathrm{ch}\, L_I(\lambda) = \sum_{w \in W_I} \mathrm{ch}\, \mathcal{U}(\mathfrak{l} \cap \mathfrak{u}_-) e(w \cdot \lambda)$. So we have $\mathrm{ch}\, M_I(\lambda) = \sum_{w \in W} \mathrm{ch}\, \mathcal{U}(\mathfrak{u}_{-,I}) \cdot \mathrm{ch}\, \mathcal{U}(\mathfrak{l} \cap \mathfrak{u}_-) e(w \cdot \lambda)$. Note now that $\mathcal{U}(\mathfrak{u}_{-,I}) \cdot \mathrm{ch}\, \mathcal{U}(\mathfrak{l} \cap \mathfrak{u}_-) = \mathrm{ch}(\mathcal{U}(\mathfrak{u}_{-,I}) \otimes \mathcal{U}(\mathfrak{l} \cap \mathfrak{u}_-)) = \mathrm{ch}(\mathcal{U}(\mathfrak{u}_-))$. Theorem follows. □

3. Kazhdan-Lusztig theory

3.1. Hecke algebras of Coxeter groups.

Definition 3.1. *Coxeter group W is a group with the presentation*

$$W = \langle s_1, \ldots, s_r \,|\, (s_i s_j)^{m_{ij}} \rangle,$$

where $m_{ii} = 1, m_{ij} > 2$ for $i \neq j$ and m_{ij} is allowed to be ∞. Pair (W, S) where W is a Coxeter group with and generators $S = \{s_1, \ldots, s_r\} \subset W$ is called a Coxeter system.

Example 3.2. *The simplest example is S_r with $S = \{(i i{+}1) \,|\, i = 1, \ldots, r-1\}$. More generally for any reductive group G the pair (W, S), where W is the Weyl group of G and $S = \{s_\alpha \,|\, \alpha \in J\}$ is the Coxeter system.*

Definition 3.3. *We denote by $\preceq$ the Bruhat order (see Remark 1.90 for the definition convenient for us). We also set $\mathcal{L} := \mathbb{Z}[v, v^{-1}]$. For $w \in W$ we say that the decomposition $w = s_{i_1} \dots {}_{i_k}$ is reduced if k is minimal. We recall the length function $l\colon W \to \mathbb{Z}_{\geqslant 0}$, $w \mapsto k$.*

Definition 3.4. *The Hecke algebra $\mathcal{H} = \mathcal{H}(W, S) = \bigoplus_{x \in W} \mathcal{L} H_x$ is an associative $\mathcal{L}$-algebra generated by H_x, $x \in W$ subject to relations*

$$H_x H_y = H_{xy} \text{ if } l(xy) = l(x) + l(y), \tag{3.5}$$

$$H_s^2 = 1 + (v^{-1} - v) H_s \Leftrightarrow H_s^{-1} = H_s + v - v^{-1} \Leftrightarrow (H_s + v)(H_s - v^{-1}) = 0. \tag{3.6}$$

Let us list some properties of Hecke algebras.

Proposition 3.7. *(1) $\mathcal{H}$ is a free module of rank $|W|$ over $\mathcal{L}$.*

(2) Set $\mathcal{HC}_{\mathbb{C}} := \mathcal{HC} \otimes_{\mathcal{L}} \mathbb{C}[v, v^{-1}]$. If W is finite then for generic $v_0 \in \mathbb{C}^\times$ the specialization of $\mathcal{HC}_{\mathbb{C}}$ at $v = v_0$ is isomorphic to the group algebra $\mathbb{C}[W]$.

(3) For $x \in W$ and $s \in S$ we have $H_x H_s = H_{xs}$ if $l(xs) = l(x) + 1$ and $H_x H_s = H_{xs} + (v^{-1} - v) H_x$ if $l(xs) = l(x) - 1$.

(4) For $u \in \{-v, v^{-1}\}$ the map $H_s \mapsto u$ defines a surjection of algebras $\mathcal{H} \twoheadrightarrow \mathcal{L}$. In this way $\mathcal{L}$ becomes an $\mathcal{H}$-bimodule to be denoted $\mathcal{L}(u)$.

Proof. We only check (3) and (4). First part of (3) is clear. If $l(xs) = l(x) - 1$ then $H_x H_s = H_{xs} H_s H_s = H_{xs} + (v^{-1} - v) H_x$. Part (4) follows from the relation $(H_s + v)(H_s - v^{-1})) = 0$. □

Remark 3.8. *Representation $\mathcal{L}(v^{-1})$ is an analogue of a trivial representation, $\mathcal{L}(-v)$ is a sign representation.*

3.2. **Bar involution and Kazhdan-Lusztig basis.**

Lemma 3.9 (Kazdan-Lusztig involution)**.** *There exists the unique involution $d\colon \mathcal{H} \to \mathcal{H}$, $H \mapsto \overline{H}$ such that $\overline{v} = v^{-1}$ and $\overline{H}_x = (H_{x^{-1}})^{-1}$.*

Proof. Exersise. □

Definition 3.10. *We call $H \in \mathcal{H}$ self-dual if $\overline{H} = H$.*

Theorem 3.11 (Kazdan-Lusztig basis)**.** *For all $x \in W$ there exists a unique self-dual element $\underline{H}_x \in \mathcal{H}$ such that $\underline{H}_x \in H_x + \sum_y v\mathbb{Z}[v] H_y$. Moreover we have $\underline{H}_x \in H_x + \sum_{y \prec x} v\mathbb{Z}[v] H_y$.*

Proof. Set $C_s = H_s + v$. We see that $\overline{C}_s = H_s^{-1} + v^{-1} = H_s + v = C_s$. So $\underline{H}_s = C_s$. By Proposition 3.7 $H_x C_s = H_{xs} + v^{l(xs) - l(x)} H_x$ (note that $l(xs) - l(x) \in \{\pm 1\}$).

We prove existence of $\underline{H}_x$ by induction on $l(x)$. For $x \neq e$ there exists $s \in S$ such that $l(xs) = l(x) - 1$ so by our induction hypothesis we have $\underline{H}_{xs} C_s = H_x + \sum_{y \prec x} h_y H_y$ for some $h_y \in \mathbb{Z}[v]$. We set $\underline{H}_x := \underline{H}_{xs} C_s - \sum_{y \prec x} h_y(0) \underline{H}_y$.

The unicity of $\underline{H}_x$ follows from:

Claim 3.12. *For $H \in \sum_y v\mathbb{Z}[v] H_y$, $\overline{H} = H$ implies $H = 0$.*

To prove the claim we observe that $H_z \in \underline{H}_z + \sum_{y \prec z} \mathcal{L} H_y$, hence, $\overline{H}_z \in H_z + \sum_{y \prec z} \mathcal{L} H_y$. If we write $H = \sum_y h_y H_y$ and choose z maximal such that $h_z \neq 0$ then $\overline{H} = H$ implies $h_z = \overline{h}_z$ contradicting $h_z \in v\mathbb{Z}[v]$. □

Let us list some properties of the elements $\underline{H}_x$.

Proposition 3.13. *(1) $H_x C_s = H_{xs} + v^{l(xs) - l(x)} H_x$.*

(2) If $(W', S') \subset (W, S)$ is a Coxeter subsystem then for any $x' \in W'$ we have $\underline{H}_{x'} \in \mathcal{H}(W', S')$ and $\{H_{x'} \,|\, x' \in W'\}$ is the KL *basis of $\mathcal{H}(W', S')$.*

Proof. Part (1) follows from Proposition 3.7, part (2) follows from the uniqueness of $\underline{H}_{x'}$. □

Example 3.14. *Let us start from the simplest case $W = S_3 = \langle s_1, s_2\rangle$, where $s_1 = (12)$, $s_2 = (23)$. We have $\underline{H}_1 = 1$, $\underline{H}_{s_1} = H_{s_1} + v$, $\underline{H}_{s_2} = H_{s_2} + v$. We see that $\underline{H}_{s_1s_2} = (H_{s_1} + v)(H_{s_2} + v)$, $\underline{H}_{s_2s_1} = (H_{s_2} + v)(H_{s_1} + v)$. It remains to compute H_{w_0}. We have $C_{s_1}C_{s_2}C_{s_1} = H_{w_0} + vH_{s_1s_2} + vH_{s_2s_1} + v^2H_{s_1} + v^2H_{s_2} + H_{s_1} + v^3 + v$. We should now substract C_s and get*

$$\underline{H}_{w_0} = H_{w_0} + v(H_{s_1s_2} + H_{s_2s_1}) + v^2(H_{s_1} + H_{s_2}) + v^3.$$

Proposition 3.15. *Let W be finite, $w_0 \in W$ the longest element. Then we have $\underline{H}_{w_0} = \sum_{x\in W} v^{l(w_0)-l(x)}H_x =: R$. Moreover $\underline{H}_{w_0}\mathcal{H} \simeq \mathcal{L}(v^{-1})$ as a rigth $\mathcal{H}$-module.*

Proof. It follows from Proposition 3.13 that $RC_s = (v + v^{-1})R$, hence, $RH_s = v^{-1}R$ and $R\mathcal{H} \simeq \mathcal{L}(v^{-1})$. We also have $\overline{R}C_s = (v + v^{-1})\overline{R}$. It easily follows from Proposition 3.13 that $\overline{R} \in \mathcal{L}R$. Note now that $R \in \underline{H}_{w_0} + \sum_{y\prec w_0} \mathcal{L}\underline{H}_y$ so $R = \overline{R}$. □

Corollary 3.16. *Take $w \in W$ which is the longest element with respect to some Coxeter subsystem $(W', S') \subset (W, S)$, $|W'| < \infty$. Then $\underline{H}_w = \sum_{x'\in W'} v^{l(w)-l(x')}H_{x'}$. We have an isomorphism $\mathcal{L}(v^{-1}) \otimes_{\mathcal{H}'} \mathcal{H} \xrightarrow{\sim} \underline{H}_w\mathcal{H}$ given by $1 \otimes H \mapsto \underline{H}_wH$.*

Definition 3.17. *For $x, y \in W$ we define $h_{y,x} \in \mathbb{Z}[v]$ by the equality $\underline{H}_x = \sum_y h_{y,x}H_y$. Polynomials $h_{y,x}$ are called Kazdan-Lusztig polynomials.*

Definition 3.18. *We have two anti-authomorphisms a and i of $\mathcal{H}$ given by $a(v) = v, a(H_x) = (-1)^{l(x)}H_x^{-1}$, $i(v) = v, i(H_x) = H_{x^{-1}}$. Both of them commute with the* KL*-involution d.*

Lemma 3.19. *For any $x, y \in W$ we have $h_{y,x} = h_{y^{-1},x^{-1}}$.*

Proof. Apply the anti-automorphism i. □

Theorem 3.20. *For all $x \in W$ there exists a unique self-dual $\underline{\tilde{H}}_x \in \mathcal{H}$ such that $\underline{\tilde{H}}_x = H_x + \sum_{y\prec x} v^{-1}\mathbb{Z}[v^{-1}]H_y$.*

Proof. Consider the authomorphism $dia\colon \mathcal{H} \xrightarrow{\sim} \mathcal{H}$, $v \mapsto v^{-1}, H_x \mapsto (-1)^{l(x)}H_x$. It commutes with d and maps $v^{-1}\mathbb{Z}[v^{-1}]$ to $v\mathbb{Z}[v]$ so we are allowed and forced to take $\underline{\tilde{H}}_x := (-1)^{l(x)}dia(\underline{H}_x)$. □

Proposition 3.21 (c.f. Proposition 3.15)**.** *We have $\underline{\tilde{H}}_{w_0} = \sum_{x\in W}(-v)^{l(x)-l(w_0)}H_x$ and $\underline{\tilde{H}}_{w_0}\mathcal{H} \simeq \mathcal{L}(-v)$.*

3.3. **Variations: spherical and anti-spherical modules.** We fix a subset $S_f \subset S$ and the corresponding Coxeter group $W_f \subset W$ and denote by $W^f \subset W$ the set of minimal length representatives of the right cosets $W_f\backslash W$. So we have a bijection $W^f \times W_f \xrightarrow{\sim} W$, $(x, y) \mapsto xy$. Set $\mathcal{H}_f := \mathcal{H}(W_f, S_f)$ and consider the induced modules

$$\mathcal{M} = \mathcal{M}^f = \mathcal{L}(v^{-1}) \otimes_{\mathcal{H}_f} \mathcal{H},\ \mathcal{N} = \mathcal{N}^f = \mathcal{L}(-v) \otimes_{\mathcal{H}_f} \mathcal{H}.$$

Mosules $\mathcal{M}, \mathcal{N}$ will be called spherical and anti-spherical respectively.

Proposition 3.22. *(1) Elements $M_x = 1 \otimes H_x \in \mathcal{M}$ (resp. $N_x = 1 \otimes H_x \in \mathcal{N}$) $x \in W^f$ form an $\mathcal{L}$-basis in $\mathcal{M}$ (resp. $\mathcal{N}$).*

(2) We have

$$M_xC_s = \begin{cases} M_{xs} + v^{l(xs)-l(x)}M_x & \text{if } xs \in W^f \\ (v + v^{-1})M_x & \text{otherwise} \end{cases} \qquad N_xC_s = \begin{cases} N_{xs} + v^{l(xs)-l(x)}N_x & \text{if } xs \in W^f \\ 0 & \text{otherwise}. \end{cases}$$

Proof. We prove (2). $(1 \otimes H_x)C_s = 1 \otimes (H_{xs} + v^{l(xs)-l(x)}H_x)$. So we are done if $xs \in W^f$. Otherwise there exists some $r \in W^f$ such that $rxs \prec xs$. We also have $x \prec rx$. Now part (2) follows from:

Claim 3.23. *If $x, y \in W$, $s \in S$ are such that $x \prec y$ and $ys \prec xs$ then $y = xs, x = ys$.*

Which is left as an exersise. □

We now generalize the KL involution $H \mapsto \overline{H}$ (see Lemma 3.9) to the parabolic case.

Definition 3.24. *We define involutions $\mathcal{M} \xrightarrow{\sim} \mathcal{M}$, $\mathcal{N} \xrightarrow{\sim} \mathcal{N}$ by $a \otimes H \mapsto \overline{a} \otimes \overline{H}$.*

Exersise 3.25. *Check that the involutions of Definition 3.24 are correctly defined.*

Theorem 3.26. *For all $x \in W^f$ there exists a unique self-dual $\underline{M}_x \in \mathcal{M}$ (resp. $\underline{N}_x \in \mathcal{N}$) such that $\underline{N}_x \in N_x + \sum_{y \prec x} v\mathbb{Z}[v]N_y$.*

Proof. With Proposition 3.22 in hand proof of this theorem is the same as the one of Theorem 3.11. □

Definition 3.27. *For $x, y \in W^f$ we define $m_{y,x} \in \mathbb{Z}[v]$ (resp. $n_{y,x} \in \mathbb{Z}[v]$) by $\underline{M}_x = \sum_y m_{y,x} M_y$ (resp. $\underline{N}_x = \sum_y n_{y,x} N_y$). Polynomials $m_{y,x}, n_{y,x}$ are called parabolic Kazdan-Lusztig polynomials.*

Let us now describe the relation between parabolic KL polynomials $m_{y,x}, n_{y,x}$ and ordinary KL polynomials $h_{y,x}$.

Proposition 3.28 (c.f. Theorem 2.81)**.** *(1) We have $n_{y,x} = \sum_{z \in W_f} (-v)^{l(z)} h_{zy,x}$.*

(2) If W_f is finite and $w_f \in W_f$ is its longest element then we have $m_{y,x} = h_{w_f y, w_f x}$.

Proof. To prove (2) recall the isomorphism $\mathcal{M} \xrightarrow{\sim} \underline{H}_{w_f}\mathcal{H}$. So we have the embedding $\iota\colon \mathcal{M} \hookrightarrow \mathcal{H}$ of right $\mathcal{H}$-modules which is compatible with dualities. It follows from Corollary 3.16 that $\iota(M_x) = \underline{H}_{w_f} H_x = \sum_{z \in W_f} v^{l(w_f)-l(z)} H_{zx}$. It now follows from the uniqueness part of Theorem 3.11 that $\iota(\underline{M}_x) = \underline{H}_{w_f x}$.

To prove (1) consider the canonical surjection $T\colon \mathcal{H} \twoheadrightarrow \mathcal{N}, H \mapsto 1 \otimes H$. It commutes with the dualities and $T(H_{zx}) = T(H_z)T(H_x) = (-v)^{l(z)} N_x$ for all $z \in W_f, x \in W^f$. It follows from the uniqueness part of Theorem 3.11 that $T(\underline{H}_x) = \underline{N}_x$ for $x \in W^f$ and is zero otherwise. □

Remark 3.29. *Note that we have a canonical map $R\colon \mathcal{H} \twoheadrightarrow \mathcal{M}$ and $R(H_{zx}) = v^{-l(z)} N_x$ but we can not deduce from this anything about the images of $\underline{H}_x$.*

We finish with the following theorem (c.f. Theorem 3.20).

Theorem 3.30. *For all $x \in W^f$ there exists a unique self-dual $\underline{\tilde{M}}_x \in \mathcal{M}$ (resp. $\underline{\tilde{N}}_x \in \mathcal{N}$) such that $\underline{\tilde{N}}_x \in N_x + \sum_{y \prec x} v^{-1}\mathbb{Z}[v^{-1}]N_y$.*

3.4. **Application: multiplicities in category $\mathcal{O}$.** In this section W is a Weyl group of a reductive Lie algebra $\mathfrak{g}$ and $S = \{s_\alpha \,|\, \alpha \in J\}$ is the set of simple reflections of W. Let $\mathfrak{p} \subset \mathfrak{g}$ be a standard parabolic subalgebra of $\mathfrak{g}$ which corresponds to a Coxeter subsystem $(W_f, S_f) \subset (S, W)$. We denote by $I \subset J$ the corresponding sybset of simple roots. Recall that $W^f \subset W$ is the set of minimal representatives of the right cosets $W_f \backslash W$. We denote by ${}^f W \subset W$ the set of minimal representatives of the left cosets W/W_f. We fix $x, y \in W$.

Theorem 3.31 (Kazdan-Lusztig conjecture)**.** *We have*

$$\mathrm{mult}(L(x \cdot 0), M(y \cdot 0)) = \mathrm{mult}(M(y \cdot 0), P(x \cdot 0)) = h_{y,x}(1).$$

Theorem 3.32 (Parabolic version of KL conjecture)**.** *For $x, y \in W^f$ we have*

$$\mathrm{mult}(L(x\cdot 0), M_I(y\cdot 0)) = \mathrm{mult}(M_I(y\cdot 0), P_I(x\cdot 0)) = n_{y,x}(1), \tag{3.33}$$

For $x, y \in {}^fW$ we have

$$\mathrm{mult}(L(x\cdot(-\rho(\mathfrak{l}))), M(y\cdot(-\rho(\mathfrak{l})))) = \mathrm{mult}(M(x\cdot(-\rho(\mathfrak{l}))), P(y\cdot(-\rho(\mathfrak{l})))) = h_{yw_f,xw_f}(1) = m_{y^{-1},x^{-1}}. \tag{3.34}$$

Proof. Let us prove the equality 3.33. For $x, y \in W^f$ set $q_{y,x} := \mathrm{mult}(L(x\cdot 0), M_I(y\cdot 0))$. It follows from Theorem 2.81 that

$$\mathrm{ch}\, M_I(y\cdot 0) = \sum_{z\in W_f} (-1)^{l(z)}\, \mathrm{ch}\, M(zy\cdot 0). \tag{3.35}$$

By Theorem 3.31 we have

$$\mathrm{ch}\, M(zy\cdot 0) = \sum_{x\in W} h_{zy,x}(1)\, \mathrm{ch}\, L(x\cdot 0). \tag{3.36}$$

Combining 3.35 and 3.36 we obtain

$$\sum_{z\in W_f, x\in W} (-1)^{l(z)} h_{zy,x}(1)\, \mathrm{ch}\, L(x\cdot 0) = \mathrm{ch}\, M_I(y\cdot 0) = \sum_{x\in W^f} q_{y,x}\, \mathrm{ch}\, L(x\cdot 0). \tag{3.37}$$

It follows from 3.37 that for $x \in W^f$ we have $q_{y,x} = \sum_{z\in W_f} (-1)^{l(z)} h_{zy,x}(1)$. Recall now that by Proposition 3.28, we have $n_{y,x} = \sum_{z\in W_f} (-v)^{l(z)} h_{zy,x}$, hence, $n_{y,x}(1) = q_{y,x}$.

Let us prove the equality 3.34. Fix $x, y \in {}^fW$. It follows from Corollary 2.48 that $T_{0\to -\rho(\mathfrak{l})}(M(xw_f\cdot 0)) = M(x\cdot -\rho(\mathfrak{l}))$, $T_{0\to-\rho(\mathfrak{l})}(L(yw_f\cdot 0)) = L(y\cdot -\rho(\mathfrak{l}))$. Functor $T_{0\to-\rho(\mathfrak{l})}$ is exact, hence, $\mathrm{mult}(L(x\cdot -\rho(\mathfrak{l}), M(y\cdot -\rho(\mathfrak{l})) = \mathrm{mult}(L(xw_f\cdot 0, M(yw_f\cdot 0)) = h_{yw_f,xw_f}(1)$. Recall now that by Lemma 3.19 and Proposition 3.28 we have $h_{yw_f,xw_f} = h_{w_fy^{-1},w_fx^{-1}} = m_{y^{-1},x^{-1}}$. ∎