Danny — Geometric rep. theory seminar – Ch. 1 Symplectic Geometry — Feb 5/20

§1.1 Symplectic manifolds:

X smooth manifold
$\mathcal{O}(X)$ algebra of smooth functions on X.

Def: A symplectic structure on X is a closed nondegenerate (smooth) 2-form ω.
(X, ω) is a symplectic manifold.

[Remark: Analogous definition for · X (smooth) algebraic variety: ω alg. 2-form, $\mathcal{O}(X)$ algebraic functions etc.
· X holomorphic variety]

ex) $X = \mathbb{C}^{2n}$ w/ coords $q_1, \dots, q_n, p_1 \dots p_n$

Careful: sign conventions

$$\omega_{std} = \sum_{i=1}^{n} dp_i \wedge dq_i$$

Thm (Darboux): (M, ω) symplectic manifold.
Near each point of M $\exists$ local coords s.t. ω is of above form.

ex) (Cotangent bundles):

(Base field $\mathbb{C}$, say)

M any manifold, $\begin{array}{c} T^*M \\ \pi \downarrow \\ M \end{array}$

Then $X := T^*M$ has canonical symplectic form ω, as follows:

First define tautological 1-form λ on T^*M by

$$\lambda_{(x,\alpha)} : \xi \longmapsto \langle \alpha, \pi_* \xi \rangle$$

pairing b/w T_x^*M, T_xM.

for $x \in M$, $\alpha \in T_x^*M$, $\xi \in T_{(x,\alpha)}(T^*M)$

Set $\omega := d\lambda$

<u>In coords</u>: $q_1, \ldots, q_n$ coords on M

[1-forms can be written $\sum p_i \, dq_i$]

$\rightsquigarrow$ Coords $q_1, \ldots, q_n, p_1, \ldots, p_n$ on T^*M

Write $\alpha = \sum p_i dq_i$, $\xi = \sum a_i \frac{\partial}{\partial q_i} + \sum b_i \frac{\partial}{\partial p_i}$

$\Rightarrow \pi_* \xi = \sum a_i \frac{\partial}{\partial q_i}$, $\lambda(\xi) = \sum a_i p_i$

$\Rightarrow \lambda = \sum p_i \, dq_i$

$\omega = d\lambda = \sum dp_i \wedge dq_i$ [cf. Standard form]

<u>ex</u>) (Coadjoint orbits):

G Lie group $\rightsquigarrow$ adjoint action Ad of G on $\mathfrak{g}$

[<u>Notation</u>: $g \in G$, $x \in \mathfrak{g}$ $\quad gxg^{-1} := Ad_g x$]

$\xrightarrow{\text{dualize}}$ Coadjoint G-action Ad^* on $\mathfrak{g}^*$.

<u>Claim</u>: Any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ has a natural symplectic form [Kirillov-Kostant-Souriau]

<u>Outline</u>: Let $\alpha \in \mathcal{O}$. Want alternating form on $T_\alpha \mathcal{O}$.

Note $\mathcal{O} \cong G/\underbrace{G^\alpha}_{\text{stabilizer of } \alpha}$ $\Rightarrow T_\alpha \mathcal{O} \cong \mathfrak{g}/\underbrace{\mathfrak{g}^\alpha}_{= Lie(G^\alpha)}$

Define $\omega_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$

$(x, y) \longmapsto \alpha([x,y])$

$\hookrightarrow$ descends to $\mathfrak{g}/\mathfrak{g}^\alpha = T_\alpha \mathcal{O}$

$\Rightarrow$ Gives 2-form on $\mathcal{O}$

<u>Check</u>: $d\omega = 0$. [Enough to check it vanishes on v.f. ξ_x coming from infinitesimal action of $x \in \mathfrak{g}$] $\square$

$\rightarrow$

§1.2. Poisson algebras:

<u>Def</u>: Let A be an associative $\mathbb{C}$-algebra.

A map $\{\cdot,\cdot\}: A \times A \to A$ is a <u>Poisson bracket</u> if:

(1) $\{,\}$ is a Lie bracket on A [antisymmetric, bilinear, Jacobi]

(2) $\{,\}$ satisfies Leibniz rule: $\{a, bc\} = \{a,b\}c + b\{a,c\}$

$(A, \{\cdot,\cdot\})$ is a <u>Poisson algebra</u>.

<u>Def</u>: A <u>Poisson manifold</u> is a mfld M equipped with a Poisson bracket $\{,\}$ on $C^\infty(M)$

[<u>Remark</u>: Poisson mfld $(M, \{,\})$ $\rightsquigarrow$ Bivector field $\pi \in \Gamma(\Lambda^2 TM)$ s.t. $\{f,g\} = \pi(df, dg)$

Conversely, a bivector field π defines a Poisson mfld (ie, satisfies Jacobi) iff $\underbrace{[\pi,\pi]}_{\text{Schouten bracket}} = 0$

Local coords: $\pi = \sum \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$; $\{f,g\} = \sum \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$

<u>Poisson mflds from symplectic mflds</u>:

(M, ω) symplectic

[ω nondegenerate $\rightsquigarrow$ gives bivector.
ω closed $\Rightarrow$ bivector is Poisson

ω nondegen $\Rightarrow$ $TM \xrightarrow{\sim} T^*M$

$\xi \mapsto \omega(\cdot, \xi)$

Sign conventions

$\rightsquigarrow$ $\mathcal{O}(M) \longrightarrow \mathrm{Vect}(M)$

$f \longmapsto \xi_f$ s.t. $\omega(\cdot, \xi_f) = df$

i.e. $\iota_{\xi_f} \omega = -df$

<u>Hamiltonian vector field</u> associated to f.

$\rightarrow$

Define bracket $\{f,g\} = \omega(\xi_f, \xi_g)$ $\left(= \xi_f g = -\xi_g f\right)$

[Note: In local Darboux coords,

$$\{f,g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right)$$

Lemma: $\forall f \in \mathcal{O}(M)$, ξ_f is symplectic, i.e. $L_{\xi_f}\omega = 0$

[Recall: • Lie derivative: $L_\xi \alpha = \frac{d}{dt}\big|_{t=0} (\psi_t)_* \alpha$, where ψ_t flow associated to ξ
ξ vector field, α k-form

• Cartan formula: $L_\xi \alpha = \iota_\xi d\alpha + d\iota_\xi \alpha$
[can use as definition]

Pf: Immediate by Cartan.

Lemma: $f \longmapsto \xi_f$ is a morphism of Lie algebras, i.e. $[\xi_f, \xi_g] = \xi_{\{f,g\}}$.

Important examples:

ex (T*X): $\mathcal{O}(T^*X)$ has Poisson bracket from canonical symplectic structure.

Note (Vector fields on T^*X):

(1) Given a diffeo $f: X \to X$, have cotangent lift $\tilde{f}: T^*X \to T^*X$

$$\langle \tilde{f}(\alpha), \xi \rangle = \langle \alpha, (df_x)^{-1}\xi \rangle \quad \text{for } \alpha \in T_x^*X, \ \xi \in T_{f(x)}X$$

↳
$$\begin{array}{ccc} T^*X & \xrightarrow{\tilde{f}} & T^*X \\ \pi\downarrow & & \downarrow\pi \\ X & \xrightarrow{f} & X \end{array}$$
commutes; $\tilde{f}$ diffeo (in fact symplectomorphism).

So, given vector field u on X, can canonically lift to vector field $\tilde{u}$ on T^*X
[lift corresponding flow, a diffeo.]

↳ for $\alpha \in T_x^*X$, $\pi_*(\tilde{u}_\alpha) = u_x$ →

(Cont'd): Claim: $\forall u$ on X, $\tilde{u}$ on T^*X is symplectic.

[Outline: Tautological 1-form λ invariant under automorphisms coming from automorphisms of X $\Rightarrow L_{\tilde{u}}\lambda = 0$.
But $L_{\tilde{u}}\omega = L_{\tilde{u}}d\omega = d L_{\tilde{u}}\lambda = 0$. $\square$

(2) vector field u on X $\rightsquigarrow$ linear function h_u on T^*X
$h_u(\alpha) = \langle \alpha, u_x \rangle$ for $\alpha \in T_x^*X$.

Lemma (1.3.14): $\tilde{u} = \xi_{h_u}$, and $h_u = \lambda(\tilde{u})$
Hamiltonian wrt. symplectic structure.

Cor: $\{h_u, h_v\} = h_{[u,v]}$.

ex ($\mathfrak{g}^*$): Bracket of functions on $\mathfrak{g}^*$ [Lie-Poisson structure]
$f, g \in \mathbb{C}[\mathfrak{g}^*]$.
Define $\{f,g\}: \mathfrak{g}^* \to \mathbb{C}$
$\alpha \longmapsto \langle \alpha, [d_\alpha f, d_\alpha g] \rangle$.
(Note: $d_\alpha f, d_\alpha g \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$).

[Remark: If · $e_1, \dots, e_n$ basis of $\mathfrak{g}$, c_{ij}^k structure constants $[e_i, e_j] = \sum c_{ij}^k e_k$
· $t_1, \dots, t_n$ corresponding coordinate functions on $\mathfrak{g}^*$
then $\{f,g\} = \sum c_{ij}^k t_k \frac{\partial f}{\partial t_i} \frac{\partial g}{\partial t_j}$

Recall: Coadjoint orbit $\mathbb{O} \subset \mathfrak{g}^*$ has canonical symplectic structure
$\Rightarrow \mathbb{O}$ has Poisson structure.

Prop: $\mathbb{O}$ is a Poisson submfld of $\mathfrak{g}^*$:
$\forall f, g \in \mathbb{C}[\mathfrak{g}^*]$, $\{f,g\}|_{\mathbb{O}} = \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\}_{symp}$.
bracket on $\mathfrak{g}^*$ $\rightarrow$

Other example: B filtered noncommutative algebra s.t. $gr\, B$ commutative.

$$\{,\}: B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j-1}/B_{i+j-2}$$

by $\{\bar{a}_1, \bar{a}_2\} = a_1a_2 - a_2a_1 \pmod{B_{i+j-2}}$

§1.3 Symplectic submanifolds:

(V, ω) symplectic vector space

$U \subseteq V$ subspace

Symplectic complement $U^{\perp\omega} := \{v \in V : \omega(v, u) = 0 \;\; \forall u \in U\}$

i.e. $\omega(v, \cdot)|_U \equiv 0$.

[vs. $U^{\perp} := \{f \in V^* : f(u) = 0 \;\; \forall u \in U\}$ annihilator of U in V^*.

Def: The subspace U is called:

(1) isotropic if $\omega|_U \equiv 0$ $\quad (\Leftrightarrow U \subset U^{\perp\omega})$

(2) coisotropic if $U^{\perp\omega}$ is isotropic $\quad (\Leftrightarrow U^{\perp\omega} \subset U)$

(3) Lagrangian if it is isotropic & coisotropic $\quad (\Leftrightarrow U = U^{\perp\omega})$.

ex $V = \mathbb{C}^{2n}$ w/ basis $e_1, \ldots, e_n, f_1, \ldots, f_n$

$\omega(e_i, e_j) = 0 = \omega(f_i, f_j)$; $\omega(e_i, f_j) = \delta_{ij}$

For $k \leq n$:

- $U = \langle e_1, \ldots, e_k \rangle$ is isotropic
- $U^{\perp\omega} = \langle e_1, \ldots, e_n, f_{k+1}, \ldots, f_n \rangle$ is coisotropic
- $\langle e_1, \ldots, e_n \rangle$ & $\langle f_1, \ldots, f_n \rangle$ are Lagrangian.

Fact (linear algebra): isotropic subspace has $\dim \leq \frac{1}{2} \dim V$

coisotropic $\geq \frac{1}{2} \dim V$

Lagrangian $= \frac{1}{2} \dim V$

Def: • M symplectic manifold.

A submfld $N \subset M$ is isotropic if $\forall x \in N$, $T_xN \subset T_xM$ is an isotropic subspace

etc..

• A (possibly singular) subvariety $Z \subset M$ is isotropic if $\forall$ smooth pt ...
→

[Later: Subvarieties of isotropic are isotropic. Nonobvious for subvarieties of singular locus

Conormal bundles:

X mfld, $Y \subset X$ submfld.

Def: The conormal bundle of Y is $T^*_Y X :=$ Covectors over Y that annihilate TY
$= \{(y,\alpha) \in T^*X : y \in Y, \alpha(v) = 0 \ \forall v \in T_y Y\}$

This is a bundle over Y:

$$\begin{array}{c} T^*_Y X \subset (T^*X)|_Y \\ \Downarrow \\ Y \end{array}$$

Prop: $T^*_Y X$ is a lagrangian submfld of T^*X,
and is a cone-subvariety of T^*X (ie stable under dilation along fibers)

[Reason: Linear algebra $\Rightarrow$ dim $= \frac{1}{2}$ dim T^*X.
Enough to show it is isotropic, ie. $\omega|_{T^*_Y X} = 0$.
But $\lambda|_{T^*_Y X} = 0$ by def, and $\omega = d\lambda$. $\square$

Lemma (1.3.27): (Characterization of lagrangian cone-subvarieties of cotangent bundle):
X smooth algebraic variety
$\Lambda \subset T^*X$ closed irreducible (possibly singular) $\mathbb{C}^*$-stable lagrangian subvariety
$\pi : T^*X \to X$, $Y :=$ smooth part of $\pi(\Lambda)$.

Then $\Lambda = \overline{T^*_Y X}$.

§1.4 Moment maps:

(M, ω) symplectic mfld.

Recall: $\mathcal{O}(M) \to$ Symplectic vector fields on M

$f \longmapsto \xi_f$

$\to$

Symplectic G-action

Suppose $G \circlearrowright M$ preserving symplectic form
Lie group
(i.e. $\omega(x,y) = \omega(gx, gy)$ $\forall\, m \in M$, $x,y \in T_m M$, $g \in G$)

Infinitesimal $\rightsquigarrow$ G-action: $\mathfrak{g} \to$ Symp. vector fields on M (Lie algebra homomorphism)

Def: The symplectic G-action is Hamiltonian if $\exists$ Lie alg. hom.
$$\begin{aligned} H : \mathfrak{g} &\to \mathcal{O}(M) \\ x &\mapsto H_x \end{aligned} \quad \text{s.t.}$$

$$\begin{tikzcd} & \mathcal{O}(M) \arrow[d] \\ \mathfrak{g} \arrow[ur, dashed] \arrow[r] & \text{Symp. vector fields} \end{tikzcd}$$

H is called the Hamiltonian.

Fix such H. [can view $H : M \times \mathfrak{g} \to \mathbb{C}$].

Def: The corresponding moment map $\mu : M \to \mathfrak{g}^*$ is defined by $\langle \mu(m), x \rangle = H_x(m)$.

Lemma:
(1) by def. — (1) $\forall\, x \in \mathfrak{g}$, $H_x = \mu^* x$ (denoting pullback of linear function from $\mathfrak{g}^*$ to M)

(2) $\mu^* : \mathbb{C}[\mathfrak{g}^*] \to \mathcal{O}(M)$ commutes w/ Poisson structure.

(3) G connected $\Rightarrow$ μ is G-equivariant (w.r.t. coadjoint action on $\mathfrak{g}^*$)

ex) $M = T^*X$, $G \circlearrowright X$ $\rightsquigarrow$ $G \circlearrowright T^*X$
$$\rightsquigarrow \quad \begin{aligned} \mathfrak{g} &\to \mathrm{Vect}(X) \to \mathrm{Vect}(T^*X) \\ x &\mapsto u_x \longmapsto \tilde{u}_x \end{aligned}$$
In fact, symplectic

Earlier discussion (Lemma 1.3.14) $\Rightarrow$ G-action is Hamiltonian, with Hamiltonian
$$x \longmapsto H_x = \lambda(\tilde{u}_x)$$

Recall: $\lambda(\tilde{v}) = h_v$, $\tilde{v} = \xi_{h_v}$

$\rightarrow$

ex) Coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$:

G-action on $\mathcal{O}$ is Hamiltonian, and the moment map is the inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$.

(Twisted) cotangent bundles to homogeneous G-spaces:

G Lie group, $P \subset G$ Lie subgroup
$\mathfrak{p} \subset \mathfrak{g}$ Lie algebras

[Recall: $\mathfrak{p}^\perp \subset \mathfrak{g}^*$ annihilator of $\mathfrak{p}$]

G-action on G/P $\rightsquigarrow$ Hamiltonian G-action on $T^*(G/P)$

Q: What is the moment map $\mu: T^*(G/P) \to \mathfrak{g}^*$?

Lemma: $\exists$ natural G-equivariant isom
$$T^*(G/P) \cong G \times_P \mathfrak{p}^\perp$$ (where P acts on $\mathfrak{p}^\perp$ by coadjoint action)

[$G \times_P \mathfrak{p}^\perp := (G \times \mathfrak{p}^\perp)/P$; $(gp, \alpha) \sim (g, p\alpha)$ for $g \in G$, $\alpha \in \mathfrak{p}^\perp$, $p \in P$]

Note: Tangent vectors to $T^*(G/P)$ are of the form:

(1) vertical vectors (i.e. tangent to fibres of $T^*(G/P) \to G/P$)
↳ can identify w/ elements of fibres [vector space]

(2) ξ_x coming from action of $x \in \mathfrak{g}$
↳ [Note: Stabilizer of $\bar{g} \in G/P$ has Lie alg. $g\mathfrak{p}g^{-1}$
$\Rightarrow$ If $x \in g\mathfrak{p}g^{-1}$, then ξ_x is a vertical vector]

Prop: Under iso. $T^*(G/P) \to G \times_P \mathfrak{p}^\perp$, μ is given by
$$(g, \alpha) \longmapsto g\alpha g^{-1} \quad \text{for } g \in G, \alpha \in \mathfrak{p}^\perp.$$
(Note: well-defined on quotient)

→

x [Prop (1.4.11) Canonical ω on $T^*(G/P)$ is given by:
(1) $\omega(\alpha_1, \alpha_2) = 0$ for vertical α_1, α_2
(2) $\omega(\xi_x, \xi_y)|_\alpha = \alpha\left(g[x,y]g^{-1}\right)$ for $x, y \in \mathfrak{g}$, $\alpha \in T^*_g(G/P)$
(3) $\omega(\beta, \xi_x)|_\alpha = \beta(gxg^{-1})$ for vertical $\beta \in T^*_g(G/P)$ viewed as tangent to $T^*(G/P)$ at $\alpha \in T^*_g(G/P)$

__Generalize__: $G \supset P$ closed connected Lie subgroup
λ linear function on $\mathfrak{g}$ s.t. $\lambda|_{[\mathfrak{p},\mathfrak{p}]} = 0$
[Not tautological 1-form]

Prop (1.4.14): (1) $\lambda + \mathfrak{p}^\perp \subset \mathfrak{g}^*$ (affine linear subspace) is stable under coadjoint P-action

(2) $G \times_P (\lambda + \mathfrak{p}^\perp)$ has natural G-invariant symplectic structure ω:

Analogous to 1.4.11 x [
(1) $\omega(\alpha_1, \alpha_2) = 0$ for vertical α_1, α_2
(2) $\omega(\xi_x, \xi_y)|_\alpha = \alpha(g[x,y]g^{-1})$ for $(g, \alpha) \in G \times_P (\lambda + \mathfrak{p}^\perp)$, $x \in \mathfrak{g}$
(3) $\omega(\beta, \xi_x) = \beta(gxg^{-1})$ for β tangent to $gP \times_P (\lambda + \mathfrak{p}^\perp)$.

Note: Fibres of $G \times_P (\lambda + \mathfrak{p}^\perp) \to G/P$ are Lagrangian submanifolds.

Call $G \times_P (\lambda + \mathfrak{p}^\perp)$ a __twisted cotangent bundle__ on G/P.

Stopped

§1.5 Coisotropic Subvarieties:

(M, ω) symplectic w/ $\{,\}$ on $\mathcal{O}(M)$.
$\Sigma \subset M$ subvariety w/ defining ideal I_Σ
[Recall: Σ coisotropic $:\Leftrightarrow$ $\forall$ smooth $m \in \Sigma$, $T_m\Sigma \supseteq (T_m\Sigma)^{\perp\omega}$

Prop (1.5.1): Σ coisotropic $\Leftrightarrow$ $\{I_\Sigma, I_\Sigma\} \subset I_\Sigma$
i.e. I_Σ is a Lie subalgebra.

$\to$

Pf: ($\Leftarrow$): $f, g \in \mathcal{I}_\Sigma$. Then $\{f,g\}(m) = \omega(\xi_f, \xi_g)(m) \equiv 0$ for $m \in \Sigma^{reg.}$ (*)

Let $f \in \mathcal{I}_\Sigma$, $W := T_m\Sigma$, $V := T_m M$ for $m \in \Sigma^{reg.}$

$df = 0$ on W $\Rightarrow$ $df \in W^\perp \subset V^*$
$\Rightarrow$ $\xi_f \in W^{\perp\omega} \subset V$

Further, $W^{\perp\omega}$ spanned by ξ_f, $f \in \mathcal{I}_\Sigma$.

Hence by (*), $\omega(W^{\perp\omega}, W^{\perp\omega}) \equiv 0$
$\Leftrightarrow$ $W^{\perp\omega}$ isotropic $\Leftrightarrow$ W coisotropic.

Reverse for ($\Rightarrow$). $\square$

Prop (1.3.30): M smooth alg. symplectic variety.
Z (possibly singular) isotropic (reduced) alg subvariety

Then any subvariety of Z is again isotropic.
[Nonobvious for subvariety of singular locus of Z]

Pf: Z isotropic, $N \subset Z$ (reduced) subvariety
Induction on codimension.
- $\dim Z = \dim N$ clear.
- $\dim N = \dim Z - 1$:
Since claim is local, can assume $\exists f \in \mathcal{O}(Z)$ (nonconstant)
s.t. $N = f^{-1}(0)$.

Let $t \in N$. Want: $T_t N$ isotropic in $T_t M$.

Technical lemma (1.5.12) $\Rightarrow$ $\exists$ sequence t_i of regular points of Z and sequence of vector spaces $W_i \subset T_{t_i} Z$
s.t. $t_i \to t$, $W_i \to T_t N$ (in appropriate Grassmannian)

Z isotropic $\Rightarrow$ Each W_i isotropic $\overset{\text{continuity}}{\Rightarrow}$ $T_t N$ isotropic.

- $\dim N < \dim Z - 1$: Shrinking N if needed, choose codim one $Z' \subset Z$ containing N. By above, Z' isotropic.
Finally, note $\mathrm{codim}_{Z'} N < \mathrm{codim}_Z N$; conclude by induction. $\square$

Thm (1.5.7): A solvable algebraic group w/ Hamiltonian action on symplectic alg variety M.

$\mu: M \to \mathfrak{a}^*$ moment map $\quad (\mathfrak{a} = Lie(A))$

For any coadjoint orbit $\mathbb{O} \subset \mathfrak{a}^*$, the set $\mu^{-1}(\mathbb{O})$ is either empty or a coisotropic subvariety of M.

Remark: For any $\mathfrak{a}$, the defining ideal $\mathcal{I}_{\mathbb{O}} \subset \mathbb{C}[\mathfrak{a}^*]$ of a coadjoint orbit $\mathbb{O} \subseteq \mathfrak{a}^*$ is stable under natural Poisson structure:

↳ If f, g vanish on $\mathbb{O}$ then $\{f,g\}|_{\mathbb{O}} = \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\}_{symp} = 0$.

It follows that the ideal $\mathcal{O}(M) \cdot \mu^* \mathcal{I}_{\mathbb{O}} \subset \mathcal{O}(M)$ is also stable under bracket [Recall Lemma: properties of μ]

Thm 1.5.7 $\Longleftrightarrow$ the radical $\sqrt{\mathcal{O}(M) \cdot \mu^* \mathcal{I}_{\mathbb{O}}}$ is also stable, in the solvable case.