1) Preservation of holonomicity.
2) Classification of irreducible $\mathcal{O}$-coherent $\mathcal{D}$-modules.

1) Our goal in this section is to prove the following theorem

Thm 1: Let $f: Y \to X$ be a morphism of smooth varieties. Assume that a) f is an open embedding, or
b) f is affine

Then $f_* Hol(\mathcal{D}_Y) \subset Hol(\mathcal{D}_X)$.

Here's a scheme of the proof:
1) Deduce a) from b)
2) Reduce b) to the case when $Y = \mathbb{A}^{n+1}$, $X = \mathbb{A}^n$ and f is the projection (to the 1st n coordinates).
3) Handle that case using the Bernstein filtration (recall there $\deg x^i = \deg \partial^i = 1$).

1.1) b) $\Rightarrow$ a). Recall that f_* is left exact for open embeddings & every holonomic $\mathcal{D}$-module has finite length. So it's enough to show that $f_*(\mathcal{F})$ is holonomic for a simple $\mathcal{F} \in Hol(\mathcal{D}_Y)$. We can pick an open affine subvariety $U \subset Y$ s.t. $\mathcal{F}|_U \neq 0$. Let $j: U \hookrightarrow Y$ be the inclusion. Since U is affine, both j & $f \circ j$ are both affine. Moreover, $(f \circ j)_* = f_* \circ j_*$. So $f_*(j_*(\mathcal{F}|_U))$ is holonomic. It remains to observe that $\mathcal{F} \subset j_*(\mathcal{F}|_U)$ & use that f_* is left exact for $f_*(\mathcal{F}) \subset f_* j_*(\mathcal{F}|_U)$, and that a submodule of a holonomic module is holonomic.

So it remains to prove a).

1.2) Reduction to $\mathbb{A}^{n+1} \twoheadrightarrow \mathbb{A}^n$: See Remark on page 11.

Every affine f factors as $\pi \circ i$ w. $i: Y \hookrightarrow X \times \mathbb{A}^m$ for some m, a closed embedding, and $\pi: X \times \mathbb{A}^m \twoheadrightarrow X$, the projection. Recall that push-forward for affine morphisms is transitive. So $f_* = \pi_* \circ i_*$ & it's enough to verify our claim for π & i separately.

We already know i_* preserves holonomicity (from Kashiwara's lemma). It remains to show π_* does.

Now we have a closed embedding $i': X \hookrightarrow \mathbb{A}^n$ & a commut. diagram:

$$\begin{tikzcd} X \times \mathbb{A}^m \arrow[r, hook, "\tilde{i}'"] \arrow[d, "\pi"] & \mathbb{A}^n \times \mathbb{A}^m \arrow[d, "\tilde{\pi}"] \\ X \arrow[r, hook, "i'"] & \mathbb{A}^n \end{tikzcd}$$

By Kashiwara's Lemma, $i'_*: Coh(\mathcal{D}_X) \xrightarrow{\sim} Coh_X(\mathcal{D}_{\mathbb{A}^n})$, $\tilde{i}'_*: Coh(\mathcal{D}_{X \times \mathbb{A}^m}) \xrightarrow{\sim} Coh_{X \times \mathbb{A}^m}(\mathcal{D}_{\mathbb{A}^n \times \mathbb{A}^m})$, restricting to equivalences between the holonomic subcategories. Also, by transitivity, $i'_* \circ \pi_* \simeq \tilde{\pi}_* \circ \tilde{i}'_*$. So, for $\mathcal{F} \in Hol(\mathcal{D}_{X \times \mathbb{A}^m})$: $\pi_* \mathcal{F}$ is holonomic $\Leftrightarrow$ $i'_* \circ \pi_*(\mathcal{F}) = \tilde{\pi}_* \circ \tilde{i}'_*(\mathcal{F})$ is holonomic. But we also know $\tilde{i}'_*(\mathcal{F}) \in Hol(\mathcal{D}_{\mathbb{A}^n \times \mathbb{A}^m})$. Therefore, once we know $\tilde{\pi}_*$ preserves holonomicity, we are done.

And, by transitivity, we can reduce to the case $m=1$.

1.3) Criteria for holonomicity

Here we are going to discuss equivalent conditions for a fin. generated $M \in \mathcal{D}(\mathbb{A}^n)$-mod to be holonomic. Consider the singular support $SS_{ord}(M)$ w.r.t. the filtration by the order of differential operator. Alternatively, we can consider the Bernstein filtration on $\mathcal{D}(\mathbb{A}^n)$, giving $SS_B(\mathcal{M}) \subset T^*\mathbb{A}^n = \mathbb{A}^{2n}$. Note that, in general, $SS_B(\mathcal{M}) \neq SS_{ord}(\mathcal{M})$.

Proposition 1: TFAE: a) $\mathcal{M}$ is holonomic

b) $\dim SS_{ord}(\mathcal{M}) \leq n$ ($\Leftrightarrow = n$)

c) $\dim SS_B(\mathcal{M}) \leq n$ ($\Leftrightarrow = n$).

d) $\mathrm{Ext}^i_{\mathcal{D}(\mathbb{A}^n)}(\mathcal{M}, \mathcal{D}(\mathbb{A}^n)) = 0 \ \forall i < n$.

e) For a good (w.r.t. Bernstein) filtr'n $\mathcal{M} = \bigcup_{i \geq 0} \mathcal{M}_{\leq i}$ $\exists F(x) \in \mathbb{Q}[x]$ w. $\deg F \leq n$ s.t. $\dim \mathcal{M}_{\leq i} = F(i) \ \forall i \gg 0$.

Proof: a) $\Leftrightarrow$ b), by def'n. In our discussion of the duality functor we've mentioned that b) $\Leftrightarrow$ c). But by the same argument, c) $\Leftrightarrow$ d). Equivalence c) $\Leftrightarrow$ e) is classical. $\square$

We will also need a version of e) which doesn't assume that $\mathcal{M}$ is finitely generated.

Lemma 1: Let $\mathcal{M} \in \mathcal{D}(\mathbb{A}^n)$-Mod. Suppose $\exists F(x) \in \mathbb{Q}[x]$, $F(x) = \frac{ax^n}{n!} +$ lower deg terms w. $a > 0$ s.t. $\dim \mathcal{M}_{\leq i} \leq F(i) \ \forall i \gg 0$. Then $\mathcal{M}$ is holonomic of multiplicity (of CC) $\leq a$ ($\Rightarrow$ of length $\leq a$)

Proof: It's enough to prove the claim for an arbitrary fin. gen'd submodule $\mathcal{N} \subset \mathcal{M}$. Equip $\mathcal{N}$ w. induced filtr'n, clearly $\dim \mathcal{N}_{\leq i} \leq F(i)$. Let $n_1, \dots n_\ell \in \mathcal{N}$ be generators & k be s.t. $n_i \in \mathcal{N}_{\leq k} \ \forall i$. Define a new filtr'n $\mathcal{N}_{\underline{\leq} j} := \mathcal{D}(\mathbb{A}^n)_{\leq j-k} \mathrm{Span}(n_1, \dots n_\ell)$. It's good & $\mathcal{N}_{\underline{\leq} \ell} \subset \mathcal{N}_{\leq \ell} \Rightarrow \dim \mathcal{N}_{\underline{\leq} i} \leq F(i) \ \forall i \gg 0$. So the Hilbert polynomial of the fin. gen'd $\mathbb{C}[\mathbb{A}^{2n}]$-module $\mathrm{gr}\,\mathcal{N}$ is $\leq F(i)$. It follows that $\dim \mathrm{Supp}\, \mathrm{gr}\,\mathcal{N} \leq n$ & $\mathrm{mult}(\mathrm{gr}\,\mathcal{N}) \leq a$ $\square$

1.4) f_* preserves holonomicity.

Lemma 2: For $\mathcal{M} \in \mathcal{D}(\mathbb{A}^n)$-mod, we have $f_*(\mathcal{M}) = \mathcal{M}/x^{n+1}\mathcal{M}$

Proof: We've seen that $\mathcal{D}_{\mathbb{A}^n \leftarrow \mathbb{A}^{n+1}} = \mathbb{C}[x^1, .. x^n][\partial^1, ..., \partial^{n+1}] =$
$= \mathcal{D}(\mathbb{A}^{n+1})/x^{n+1}\mathcal{D}(\mathbb{A}^{n+1})$. The claim follows $\square$

Now let $\mathcal{M} \in Hol(\mathcal{D}(\mathbb{A}^{n+1}))$. We want to show $\mathcal{M}/x\mathcal{M} \in Hol(\mathcal{D}(\mathbb{A}^n))$.
Here and below we set $x := x^{n+1}$.

Case 1: x is not a zero divisor in $\mathcal{M}$. Equip $\mathcal{M}$ w. a good filtration. We know $\exists$ deg $n+1$ polynomial F s.t. $\dim \mathcal{M}_{\leq i} = F(i)$ for $i \gg 0$. We have an induced filtration on $f_*\mathcal{M} = \mathcal{M}/x\mathcal{M}$, it's compatible w. Bernstein filtration on $\mathcal{D}(\mathbb{A}^n)$. We have $\dim (f_*\mathcal{M})_{\leq i} = \dim \mathcal{M}_{\leq i} - \dim (x\mathcal{M} \cap \mathcal{M}_{\leq i}) \leq [x\mathcal{M}_{\leq i-1} \subset x\mathcal{M} \cap \mathcal{M}_{\leq i}] \dim \mathcal{M}_{\leq i} - \dim \mathcal{M}_{\leq i-1} = F(i) - F(i-1)$. Now we use Lemma 1 to deduce that $f_*\mathcal{M}$ is holonomic.

Case 2: x acts locally nilpotently on $\mathcal{M} \Leftrightarrow \mathrm{Supp}_{\mathbb{A}^{n+1}}(\mathcal{M}) \subset (x=0)$. In our proof of Kashiwara's lemma we have seen that $\mathcal{M} \simeq \mathcal{M}^x \otimes \mathbb{C}[\partial]$ ($\curvearrowleft \mathcal{D}(\mathbb{A}^{n+1}) = \mathcal{D}(\mathbb{A}^n) \otimes \mathcal{D}(\mathbb{A}^1)$). Note that $x\mathbb{C}[\partial] = \mathbb{C}[\partial] \Rightarrow x\mathcal{M} = \mathcal{M} \Rightarrow f_*\mathcal{M} = \{0\}$.

Case 3 : general. Note that $\mathcal{M}_0 := \bigcup_{i \geq 0} \ker_{\mathcal{M}}(x^i) \subset \mathcal{M}$ is a $\mathcal{D}(\mathbb{A}^{n+1})$-submodule. By the construction, x is a nonzero divisor in $\mathcal{M}/\mathcal{M}_0$. So we have an exact sequence $0 \to \underset{0}{\underset{\parallel}{f_*\mathcal{M}_0}} \to f_*\mathcal{M} \to f_*(\mathcal{M}/\mathcal{M}_0)$
$\Rightarrow f_*\mathcal{M} \subset f_*(\mathcal{M}/\mathcal{M}_0)$ - holonomic!

This finishes the proof of Theorem 1.

1.5) More on preservation of holonomicity.*

In fact, preservation of holonomicity holds for both <u>derived</u>

pullbacks & push-forwards. The derived pull-back functor $Lf^*: D^b(QCoh\, \mathcal{D}_X) \longrightarrow D^b(QCoh\, \mathcal{D}_Y)$ is defined as $\mathcal{D}_{Y\to X} \otimes^L_{f^\bullet \mathcal{D}_X} f^\bullet(?)$ - i.e. as the derived functor of the right exact functor.

With the derived push-forward, it's more tricky: we define $f_*: D^b(QCoh\, \mathcal{D}_Y) \longrightarrow D^b(QCoh\, \mathcal{D}_X)$ as $Rf_\bullet(\mathcal{D}_{X\leftarrow Y} \otimes^L_{\mathcal{D}_Y} ?)$. So for affine f, the derived version of f_* is the left derived functor of the abelian version, while for open embeddings (and, more generally, for etale morphisms) we get the right derived functor. In general, only the derived version makes sense.

We will not need this stuff in what follows.

2) Classification of irreducible $\mathcal{O}$-coherent $\mathcal{D}$-modules.

Slogan: (Nice) $\mathcal{O}$-coherent $\mathcal{D}$-modules should be closely related to finite dimensional reps of the fundamental group

2.1) Complex analytic setting.

So far we have talked about algebraic differential operators - and therefore about algebraic $\mathcal{D}$-modules. The similar story makes sense for complex analytic manifolds. In this section we are going to work with a complex analytic manifold X^{an} and the category of $\mathcal{O}_{X^{an}}$-coherent $\mathcal{D}_{X^{an}}$-modules - a.k.a. vector bundles w. flat connection, denote it by $Flat(X^{an})$.

The main result of this section is the following:

Theorem 2: There is an equivalence $Flat(X^{an}) \longrightarrow Rep_{fd}(\pi_1(X))$ of abelian categories (where Rep_{fd} stands for the category of finite dimensional representations).

Sketch of proof:

Case 1: X^{an} is an open disk. Then every $\mathcal{O}$-coherent $\mathcal{D}_{X^{an}}$-module is the direct sum of several copies of $\mathcal{O}_{X^{an}}$. More canonically, for a $\mathcal{D}_{X^{an}}$-module $\mathcal{M}$ we write $\mathcal{M}^\nabla$ for the subspace of flat (=killed by Vect_X) sections of $\mathcal{M}$. Then we have a natural isomorphism $\mathcal{O}_{X^{an}}\otimes_{\mathbb{C}}\mathcal{M}^\nabla \to \mathcal{M}$. Compare with the case of formal disc, $\mathrm{Spec}\,\mathbb{C}[[x_1,\dots x_n]]$.

Case 2: X^{an} is simply connected. We claim that still $\mathcal{O}_{X^{an}}\otimes_{\mathbb{C}}\mathcal{M}^\nabla \xrightarrow{\sim} \mathcal{M}$. Pick $x\in X$ & let U_x be a disc around x so that $\mathcal{O}_{U_x^{an}}\otimes(\mathcal{M}|_{U})^\nabla \xrightarrow{\sim} \mathcal{M}|_{U_x}$. We claim we can extend sections from $(\mathcal{M}|_{U_x})^\nabla$ to $\mathcal{M}^\nabla$. Namely, let y be another point. Connect x & y by a path. For a point p in the path choose an open disc U_p around p so that $\mathcal{O}_{U_p^{an}}\otimes_{\mathbb{C}}(\mathcal{M}|_{U_p})^\nabla \xrightarrow{\sim} \mathcal{M}|_{U_p}$. If $U_p\cap U_{p'}\neq\emptyset$ we can find an open disc $U_{pp'}\subset U_p\cap U_{p'}$ $\rightsquigarrow$ $(\mathcal{M}|_{U_p})^\nabla \xleftrightarrow{\sim} (\mathcal{M}|_{U_{pp'}})^\nabla \xleftrightarrow{\sim} (\mathcal{M}|_{U_{p'}})^\nabla$. This leads to a composite identification $(\mathcal{M}|_{U_x})^\nabla \xleftrightarrow{\sim} (\mathcal{M}|_{U_y})^\nabla$. And, since X is simply connected, any two paths between x & y are homotopic. This implies that the identification $(\mathcal{M}|_{U_x})^\nabla \xleftrightarrow{\sim} (\mathcal{M}|_{U_y})^\nabla$ doesn't depend on the choice of a path. By the construction, the images of $m\in(\mathcal{M}|_{U_x})^\nabla$ in various $(\mathcal{M}|_{U_y})^\nabla$ glue together giving a map $(\mathcal{M}|_{U_x})^\nabla \to \mathcal{M}^\nabla$. This map is two-sided inverse of the restriction map $\mathcal{M}^\nabla \to (\mathcal{M}|_{U_x})^\nabla$. So indeed $\mathcal{M}^\nabla \xrightarrow{\sim} (\mathcal{M}|_{U_x})^\nabla$.

Now consider the map $\mathcal{O}_{X^{an}}\otimes_{\mathbb{C}}\mathcal{M}^\nabla \to \mathcal{M}$. The above argument shows this map is an isomorphism after restriction to a small neighborhood of each point. Hence it's an isomorphism.

Case 3: general. Let $\tilde{X}^{an} \xrightarrow[\pi]{} X^{an}$ be the simply connected cover. Let $\Gamma := \pi_1(X^{an})$ so that Γ freely acts on $\tilde{X}^{an}$ & π is the quotient map for the Γ-action. We can talk about Γ-equivariant $\mathcal{D}_{\tilde{X}^{an}}$-modules (a $\mathcal{D}_{\tilde{X}^{an}}$-module & a Γ-module $\tilde{M}$ s.t. the action map $\mathcal{D}_{\tilde{X}^{an}} \otimes_{\mathbb{C}} \tilde{M} \to \tilde{M}$ is Γ-equivariant), leading to the category $Flat^{\Gamma}(\tilde{X}_{an})$. An example of an object in $Flat^{\Gamma}(\tilde{X}_{an})$ is provided by $\mathcal{O}_{\tilde{X}^{an}}$ w. its natural Γ-action. The functor $\mathcal{M} \mapsto \mathcal{M}^{\nabla}: Flat(\tilde{X}_{an}) \xrightarrow{\sim} Vect_{fd}$ upgrades to $\bullet^{\nabla}: Flat^{\Gamma}(\tilde{X}_{an}) \xrightarrow{\sim} Rep_{fd}(\Gamma)$. On the other hand, have the pullback functor $\pi^*: Flat(X^{an}) \longrightarrow Flat^{\Gamma}(\tilde{X}^{an})$. Its quasi-inverse is $\pi_*(\bullet)^{\Gamma}: Flat^{\Gamma}(\tilde{X}^{an}) \longrightarrow Flat(X^{an})$, where the superscript Γ in the functor stands for taking Γ-invariants.

So we have the composite equivalence $\pi^*(\bullet)^{\nabla}: Flat(X^{an}) \xrightarrow{\sim} Rep_{fd}(\Gamma)$ $\square$

Remarks: 1) There's an important intermediate category between $Flat(X^{an})$, $Rep_{fd}(\pi_1(X^{an}))$. This is the category of <u>local systems</u>, i.e. of locally constant sheaves of finite dimensional vector spaces, denote it by $Loc(X^{an})$. Then the equivalence in the theorem is $Flat(X^{an}) \xrightarrow[\underline{Hom}_{\mathcal{D}_{X^{an}}}(\mathcal{O}_{X^{an}}, \bullet)]{\sim} Loc(X^{an}) \xrightarrow[\text{monodromy}]{\sim} Rep_{fd}(\pi_1(X^{an}))$.

Note that $\bullet^{\nabla} = Hom_{\mathcal{D}_{X^{an}}}(\mathcal{O}_{X^{an}}, \bullet) = \Gamma(\underline{Hom}_{\mathcal{D}_{X^{an}}}(\mathcal{O}_{X^{an}}, \bullet))$.

2) Our equivalence $Flat(X^{an}) \to Rep_{fd}(\pi_1(X))$ depends on choosing $x \in X$.

2*) One can ask a question whether we can extend the equivalence $Flat(X^{an}) \xrightarrow{\sim} Loc(X^{an})$ to $Hol(\mathcal{D}_{X^{an}})$. The answer is yes: one replaces $Loc(X^{an})$ with the category of "perverse sheaves" & $\underline{Hom}_{\mathcal{D}_{X^{an}}}(\mathcal{O}_{X^{an}}, \cdot)$ with $R\underline{Hom}_{\mathcal{D}_{X^{an}}}(\mathcal{O}_{X^{an}}, \cdot)$.

Ex: Let $A \in Mat_n(\mathbb{C})$, $X = \mathbb{C}^\times$. Consider the $\mathcal{D}$-module $V = \mathcal{O}_{X^{an}} z^A$ $:= \{f(z) z^A \mid f(z) = (f_1(z), \ldots f_n(z))$, a column of holomorphic functions$\}$ & $\partial(f(z)z^A) = (f'(z) + f(z)\frac{A}{z}) z^A$. Over a simply connected open subspace in $\mathbb{C}^\times$, the space of flat sections is of the form $\{v z^{-A} \mid v \in \mathbb{C}^n\}$. So when we go around a clockwise loop around 0, $t \mapsto \exp(-2\pi\sqrt{-1}t)$, $t \in [0,1]$, the space of flat sections gets transformed by $\exp(2\pi\sqrt{-1}A)$. So the monodromy representation associated with V sends a generator of $\pi_1(X^{an}) = \mathbb{Z}$ to $\exp(2\pi\sqrt{-1}A)$.

2.2) Algebraic setting: issues.

Every smooth algebraic variety can be viewed as a complex analytic manifold. Every algebraic $\mathcal{D}$-module gives rise to a complex analytic $\mathcal{D}$-module via the analytification functor: $M \mapsto \mathcal{O}_{X^{an}} \otimes_{\mathcal{O}_X} M$. So we still have a functor $Flat(X) (\to Flat(X^{an})) \to Rep_{fd}(\pi_1(X))$. This functor is exact & faithful, but it's NOT full.

For example, take $X = \mathbb{A}^1$, here $\pi_1(X) = \{1\}$. However, not every $\mathcal{D}$-module has nonzero algebraic flat sections; a basic example is $\mathbb{C}[x]e^x$ - w. obvious action of ∂. The monodromy functor sends both $\mathbb{C}[x]$, $\mathbb{C}[x]e^x$ to the 1 dim'l space but $\mathbb{C}[x] \not\cong \mathbb{C}[x]e^x$.

Still, when X is projective, we have $Flat(X) \to Rep_{fd}(X)$ (the "GAGA principle"). More generally, can take proper X.

2.3) Algebraic setting: fix.

The category $Flat(X)$ is too big (too many non-isomorphic objects) to be equivalent to $Rep_{fd}(\pi_1(X))$. It turns out, there is a full subcategory $Flat_{reg}(X)$ of "regular" vector bundles w. flat connection in $Flat(X)$ s.t. the analytification functor restricts to an equivalence $Flat_{reg}(X) \xrightarrow{\sim} Flat(X^{an})$. This is a very non-trivial result due to Deligne.

We start by explaining what "regular" means for curves.

Case 1: $\dim X = 1$. There's a unique smooth projective curve $\bar{X}$ w. $j: X \hookrightarrow \bar{X}$. We note that any vector bundle V on X extends to a vector bundle $\bar{V}$ on $\bar{X}$ (non-canonically). Pick $p \in \bar{X} \setminus X$ and choose a disc $U^{an} \subset \bar{X}$ around p. Let z be a coordinate on U^{an} w $z(p)=0$. We can trivialize V on $U^{\times, an} = U^{an} \setminus \{p\}$. Then $V|_{U^{\times,an}} \simeq \mathcal{O}_{U^{\times,an}} \otimes \mathbb{C}^m$. The choice of trivialization gives us a connection matrix $A(z)$ s.t. $\nabla(f \otimes v) = df \otimes v + f dz \otimes A(z) v$. Note that $A(z)$ depends on the trivialization: if we twist the trivialization by $g(z) \in GL_m(\mathcal{O}_{U^{\times,an}})$, then we need to replace $A(z)$ w. $g(z) A(z) g(z)^{-1} + g(z)^{-1} dg(z)$ (exercise).

Definition: V is called regular at p if $\exists$ a trivialization at p s.t. all entries of $A(z)$ have poles of order ≤ 1. V is called regular if it's regular $\forall p$.

Remark: If $A(z)$ is holomorphic, it means that $j_{!*}(V)$

is $\mathcal{O}$-coherent around p.

Examples: 1) If $m=1$, then the condition on the order of poles is independent of trivialization $\rightsquigarrow$ examples of non-regular modules.

2) Consider the $\mathcal{O}_{\mathbb{C}^\times}$-coherent $\mathcal{D}$-module $\mathbb{C}[x^{\pm 1}]x^A$. We claim it has regular singularities. Namely, include $\mathbb{C}^\times \hookrightarrow \mathbb{P}^1$. Then we need to test at two points: 0 and ∞. Take $p=0$, $z=x$. Then $\partial(f\otimes v) = \partial f\otimes v + f\otimes \frac{A}{z}v$. The connection matrix $A(z)=\frac{A}{z}$ - poles of order 1. Now consider $p=\infty$ w. $z=x^{-1} \Rightarrow \partial_z = -z^{-2}\partial_x$. Then $\partial_z(f\otimes v) = -z^{-2}(\partial_x f\otimes v + f\otimes \frac{A}{x}v)$ $= \partial_z f\otimes v - f\otimes \frac{A}{z}v$, and $A(z) = -\frac{A}{z}$, again poles of order 1.

3) Consider the trivial vector bundle $\mathcal{O}_X\otimes_{\mathbb{C}}\mathbb{C}^2$ on a curve X & let the connection be given by the matrix $A=\begin{pmatrix} 0 & f\\ 0 & 0\end{pmatrix}$, $f\in\mathbb{C}[X]$. We claim that it's regular. Namely, we can find a meromorphic function F on a disc s.t. $u'(z)+f(z)$ has poles of order ≤ 1. Set $g=\begin{pmatrix} 1 & u\\ 0 & 1\end{pmatrix}$. Then $gAg^{-1}+g^{-1}dg = A + \begin{pmatrix} 1 & -u\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & u'\\ 0 & 0\end{pmatrix} = \begin{pmatrix} 0 & f+u'\\ 0 & 0\end{pmatrix}$, has poles of order ≤ 1.

In general, an extension of two regular $\mathcal{O}$-coherent $\mathcal{D}_X$-modules is regular. The same applies to subs & quotients, duals & tensor products.

Case 2: general. Assume now $\dim X > 1$.

Definition: Let V be an $\mathcal{O}$-coherent $\mathcal{D}_X$-module. We say V is regular if i^*V is regular for any inclusion i of a smooth curve.

Thm (Deligne): The analytification functor $Flat_{reg}(X) \longrightarrow Flat(X^{an})$ is a category equivalence

This is a hard theorem. And even once we know the theorem classifying irreducible regular holonomic modules (= intermediate extensions of irreducible regular $\mathcal{O}$-coherent modules) is difficult. It turns out that in the presence of a large symmetry group classifying irreducible <u>equivariant</u> $\mathcal{D}$-modules is easy, and doesn't require appealing to regularity. Equivariant $\mathcal{D}$-modules is what we are going to study next.

Remark for page 2: We can reduce to the case when X is affine: for $U \subset X$ an open subset, let π_U denote the projection $U \times \mathbb{A}^m \to U$. Note that $(\pi_* \mathcal{F})|_U \simeq \pi_{U*}(\mathcal{F}|_{U\times\mathbb{A}^m})$. So π_* preserves holonomicity $\Leftrightarrow$ $\forall x \in X$ $\exists$ open neighborhood U of x s.t. π_{U*} preserves holonomicity.