1) Twisted equivariant $\mathcal{D}$-modules
2) Sheaves of twisted differential operators.

Motivation: We have used equivariant $\mathcal{D}$-modules on G/B to get info on categories like $U_0\text{-mod}^N$ & $U_0\text{-mod}^K$ that are of great importance. Can we use similar techniques when we replace U_0 with a more general central reduction of $U(\mathfrak{g})$: U_χ for $\chi \in \mathfrak{t}^*$. Can we study more general categories - with weaker equivariance conditions? These are two closely related questions and we start with the latter, which is more straightforward.

1) Let X be a smooth variety and $G \curvearrowright X$ be an algebraic action. Let $\chi \in (\mathfrak{g}^*)^G$.

Definition: Let $M \in QCoh(\mathcal{D}_X)$ be a weakly equivariant $\mathcal{D}$-module. We say that M is <u>(strongly) (G,χ)-equivariant</u> if $\xi_M m = (\xi_X - \langle \chi, \xi \rangle) m$ $\forall \xi \in \mathfrak{g}$ & local sections $m \in M$.

Example: 1) Let χ be the differential of a character $\chi: G \to \mathbb{C}^\times$. Let $\mathbb{C}_{-\chi}$ be the 1-dimensional representation of G, where the action is via $-\chi$ (the characters of G form an abelian group and it's common to use the additive notation: $(-\chi)(g) := \chi(g)^{-1}$). Then $\mathcal{O}_X \otimes \mathbb{C}_{-\chi}$ is (G,χ)-equivariant).

2) $R_X^\chi = \mathcal{D}_X / \mathcal{D}_X \operatorname{Span}(\xi_X - \langle \chi, \xi \rangle \mid \xi \in \mathfrak{g})$.

We write $Coh^{G,\chi}(X)$ for the cat'y of (G,χ)-equiv't coh't $\mathcal{D}_X$-modules.
Let us discuss the classification of twisted equivariant $\mathcal{D}$-modules on homogeneous spaces. Let $X = G/H$ & $\chi \in (\mathfrak{g}^*)^G$.

Important exercise 1: We have a category equivalence $Coh^{G,\chi}(G/H) = \{H\text{-reps, where } \mathfrak{h} \text{ acts via } -\chi|_{\mathfrak{h}}\}$.

Thanks to this exercise we can completely describe $Coh^{G,\chi}(X)$ if G acts on X w. finitely many orbits. Here is an important special case.

Let $X = G/B$, and we consider the action of N on X. Let us describe the space $(\mathfrak{n}^*)^N$. For a positive root α, let e_α denote the corresponding root vector in $\mathfrak{n}$. Since N is connected, we have $(\mathfrak{n}^*)^N = (\mathfrak{n}/[\mathfrak{n},\mathfrak{n}])^*$. So $\chi \in (\mathfrak{n}^*)^N \Leftrightarrow \chi(e_\alpha) = 0$ $\forall$ non-simple root α. We say χ is non-degenerate if $\chi(e_\alpha) \neq 0$ $\forall$ simple roots α.

Important exercise 2: Suppose $\chi \in (\mathfrak{n}^*)^N$ is non-degenerate. Then there is a unique irreducible module in $Coh^{N,\chi}(\mathcal{D}_{G/B})$, it's associated with the open N-orbit.

We have an equivalence $Coh^{N,\chi}(\mathcal{D}_{G/B}) \xrightarrow{\sim} \mathcal{U}_0\text{-mod}^{N,\chi}$. The image of the irreducible object of $Coh^{N,\chi}(\mathcal{D}_{G/B})$ in $\mathcal{U}_0\text{-mod}^{N,\chi}$ is the so called Whittaker module, it's $\mathcal{U}_0 \otimes_{\mathcal{U}(\mathfrak{n})} \mathbb{C}_\chi$, where $\mathbb{C}_\chi$ is the 1-dimensional $\mathfrak{n}$-module, where $\mathfrak{n}$ acts by χ.

Premium exercise: Classify the irreducibles in $Coh^{N,\chi}(\mathcal{D}_{G/B})$, where χ is an arbitrary element in $(\mathfrak{n}^*)^N$.

Now we bridge the discussion of twisted equivariant

$\mathcal{D}$-modules with our next topic. Let G denote an arbitrary algebraic group. Let $X \xrightarrow{\pi} X_0$ be a principal G-bundle. We have seen that $\pi_\cdot(R_X)^G \simeq \mathcal{D}_{X_0} \rightsquigarrow Coh^G(\mathcal{D}_X) \xrightarrow{\sim} Coh(\mathcal{D}_{X_0})$. What about $\pi_\cdot(R_X^\chi)^G$ & $Coh^{G,\chi}(\mathcal{D}_X)$, can we give a similar description? The answer is YES: $\pi_\cdot(R_X^\chi)^G$ is a sheaf of <u>twisted</u> differential operators on X_0.

2.1) Definition and basic example of TDO.

Recall that $\mathcal{D}_X$ is a quasi-coherent sheaf of algebras on X. It's filtered: $\mathcal{D}_X = \bigcup_{i=0}^\infty \mathcal{D}_{X,\leq i}$ and we have an isomorphism of sheaves of graded algebras $gr\, \mathcal{D}_X \xrightarrow{\sim} p_* \mathcal{O}_{T^*X}$, where $p: T^*X \twoheadrightarrow X$ is the projection. Both sheaves have Poisson brackets and the isomorphism is Poisson. These are the properties we'll be generalizing.

Definition: By a sheaf of twisted differential operators (TDO) one means a sheaf of filtered algebras $\mathcal{D} = \bigcup_{i\geq 0} \mathcal{D}_{\leq i}$ on X together with an isomorphism $gr\, \mathcal{D} \xrightarrow{\sim} p_* \mathcal{O}_{T^*X}$ of sheaves of graded Poisson algebras. (in particular, $\mathcal{O}_X \xrightarrow{\sim} \mathcal{D}_{\leq 0} \subset \mathcal{D}$).

Clearly, $\mathcal{D}_X$ is a sheaf of TDO. Here is a more general example.

Let $\mathcal{L}$ be a line bundle on X. We can define the sheaf $\mathcal{D}_X^\mathcal{L}$ of differential operators $\mathcal{L} \to \mathcal{L}$ using Grothendieck's definition: for an open affine subset $U \subseteq X$ we set $\mathcal{D}^\mathcal{L}_{X,\leq 0}(U) := \mathbb{C}[U] \subset End_\mathbb{C}(\mathcal{L}(U))$, $\mathcal{D}^\mathcal{L}_{X,\leq i}(U) := \{\varphi \in End_\mathbb{C}(\mathbb{C}[U]) \mid [f,\varphi] \in \mathcal{D}^\mathcal{L}_{X,\leq i-1}(U)\ \forall$

$f\in \mathbb{C}[U]\}$. Then $\mathcal{D}^{\mathcal{L}}_X(U) := \bigcup_{i\geqslant 0} \mathcal{D}^{\mathcal{L}}_{X,\leqslant i}(U)$ is a filtered algebra. Moreover, if $\mathcal{L}|_U$ is trivial, then a choice of an isomorphism $\mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}_U \rightsquigarrow$ filt'd algebra isomorphism $\mathcal{D}^{\mathcal{L}}_X(U) \xrightarrow{\sim} \mathcal{D}(U)$. As with $\mathcal{D}_X$, the filtered algebras $\mathcal{D}^{\mathcal{L}}_X(U)$ glue together to a sheaf of filtered algebras.

Proposition 1: $\mathcal{D}^{\mathcal{L}}_X$ is a sheaf of TDO.

Proof: We need to establish a graded Poisson iso $\operatorname{gr}\mathcal{D}^{\mathcal{L}}_X \xrightarrow{\sim} p_* \mathcal{O}_{T^*X}$.

Let U be such that $\mathcal{L}|_U \simeq \mathcal{O}_U$. Let us investigate how a change in trivialization affects an isomorphism $\mathcal{D}^{\mathcal{L}}_X(U) \xrightarrow{\sim} \mathcal{D}_X(U)$. Two parameterizations differ by an invertible function, say $f\in \mathbb{C}[U]^\times$. They give rise to an automorphism of $\mathcal{D}(U)$:

$$\begin{tikzcd} \mathcal{D}(U) \arrow[rr, bend right, "\text{our automorphism}"'] & \mathcal{D}^{\mathcal{L}}_X(U) \arrow[l, "\sim"', "\text{via param'n } \varphi"] \arrow[r, "\sim", "\text{via } f\varphi"'] & \mathcal{D}(U) \end{tikzcd}$$

Important exercise 3: the automorphism $\mathcal{D}(U) \to \mathcal{D}(U)$ behaves as follows on the generators:

$$g \mapsto g \quad \forall g\in \mathbb{C}[U], \quad \xi \mapsto \xi + \frac{\xi f}{f}, \quad \forall \xi \in \operatorname{Vect}(U).$$

In particular, we see that it's the identity on the associated graded. It follows that $\operatorname{gr}\mathcal{D}^{\mathcal{L}}_X \xrightarrow{\sim} \operatorname{gr}\mathcal{D}_X (\xrightarrow{\sim} p_*\mathcal{O}_{T^*X})$. It's straightforward to see that this is a grading preserving isomorphism. To check that $\operatorname{gr}\mathcal{D}^{\mathcal{L}}_X \xrightarrow{\sim} \operatorname{gr}\mathcal{D}_X$ is Poisson is an exercise. $\square$

2.2) Classification of sheaves of TDO.

Let $\mathcal{D}, \mathcal{D}'$ be two sheaves on TDO. By their isomorphism one means an isomorphism $\iota: \mathcal{D} \xrightarrow{\sim} \mathcal{D}'$ of sheaves of filtered algebras s.t. $\mathrm{gr}\, \iota: p_* \mathcal{O}_{T^*X} \circlearrowleft$ is the identity (so ι is id on $\mathcal{O}_X$). Our goal now is to classify sheaves of TDO up to iso. We'll see that the iso classes are in nat'l bijection w. certain cohomology group.

We write Ω^i_X for the sheaf of i-forms on X. We have the de Rham complex $\Omega^\bullet_X = (\Omega^0_X \to \Omega^1_X \to \dots)$. Then we consider its truncation $\Omega^{\geq 1}_X = (\Omega^1_X \to \Omega^2_X \to \dots)$

Theorem 1: The iso classes of sheaves on TDO are parameterized by the hypercohomology group $H^1(X, \Omega^{\geq 1}_X)$, i.e. the cohomology of the total complex associated to the Čech complex of $\Omega^{\geq 1}_X$.

In more concrete terms, choose a cover $X = \bigcup_i U_i$ by open affines. Then the truncated Čech-de Rham complex we need looks as follows:

$$\begin{array}{ccccc}
\bigoplus_i \Omega^1(U_i) & \xrightarrow{d_{dR}} & \bigoplus_i \Omega^2(U_i) & \xrightarrow{d_{dR}} & \bigoplus_i \Omega^3(U_i) \\
\downarrow d_{\check{C}} & & \downarrow d_{\check{C}} & & \dots \\
\bigoplus_{i,j} \Omega^1(U_{ij}) & \xrightarrow{d_{dR}} & \bigoplus_{i,j} \Omega^2(U_{ij}) & & [U_{ij} := U_i \cap U_j] \\
\downarrow d_{\check{C}} & \dots & & & \\
\bigoplus_{i,j,k} \Omega^1(U_{ijk}) & & & &
\end{array}$$

the piece we care about.

So $H^1(X, \Omega^{\geq 1}_X)$ consists of $(\alpha_{ij} \in \Omega^1(U_{ij}), \omega_i \in \Omega^2(U_i) \mid \alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0,\ \omega_i - \omega_j = d\alpha_{ij},\ d\omega_i = 0)$ modulo the image of $\bigoplus_i \Omega^1(U_i)$, here as usual we assume $\alpha_{ji} := -\alpha_{ij}$.

Proof of Theorem 1: We need to explain how to pass from a class in $H^1(X, \Omega_X^{\geq 1})$ to a sheaf of TDO & back.

- From $H^1(X, \Omega_X^{\geq 1})$ to a sheaf of TDO:

Step 1: $X = U$ is affine. Here an element of $H^1(X, \Omega_X^{\geq 1})$ is a closed 2-form modulo exact form. So let $\omega \in \Omega^2(U)$ w. $d\omega = 0$. Consider the algebra $\mathcal{D}^\omega(U)$ generated by $\mathbb{C}[U]$ & $\operatorname{Vect}(U)$ w. rel'ns:

- $f, f' \in \mathbb{C}[U]$ are multiplied as in $\mathbb{C}[U]$.
- $f \cdot \xi = f\xi$, $\xi \cdot f = f\xi + \xi . f$, $f \in \mathbb{C}[U], \xi \in \operatorname{Vect}(U)$.
- $\xi \cdot \xi' - \xi' \cdot \xi = \underset{\operatorname{Vect}(U)}{\underset{\cap}{[\xi, \xi']}} + \underset{\mathbb{C}[U]}{\underset{\cap}{\omega(\xi, \xi')}}$, $\xi, \xi' \in \operatorname{Vect}(U)$.

Note that the condition that ω is closed translates to the Jacobi identity for the elements of $\operatorname{Vect}(U)$.

Step 2: Now let $\alpha \in \Omega^1(U)$ & $\omega' = \omega + d\alpha$. Consider the endomorphism of $\mathbb{C}[U] \oplus \operatorname{Vect}(U)$ given by $f \mapsto f$, $\xi \mapsto \xi + \langle \alpha, \xi \rangle$. This map extends to an algebra homomorphism $\mathcal{D}^\omega(U) \longrightarrow \mathcal{D}^{\omega'}(U)$ (exercise). This homomorphism is invertible - the inverse corresponds to $-\alpha$.

Step 3: The algebra $\mathcal{D}^\omega(U)$ comes w. a natural filtration (by degree w.r.t. $\operatorname{Vect}(U)$). The isomorphism $\mathcal{D}^\omega(U) \xrightarrow{\sim} \mathcal{D}^{\omega'}(U)$ from Step 2 is that of filtered algebras. Note that we get a natural epimorphism $\mathbb{C}[T^*U] = S_{\mathbb{C}[U]}(\operatorname{Vect}(U)) \twoheadrightarrow \operatorname{gr} \mathcal{D}^\omega(U)$. Let us show that it is an isomorphism. Pick $p \in U$. Let $\mathbb{C}[U]^{\wedge p}$ denote the formal completion of $\mathbb{C}[U]$ at p, in other words, the algebra of functions on the formal neighborhood of p. On that neighborhood, every closed form is exact. So $\mathbb{C}[U]^{\wedge p} \otimes_{\mathbb{C}[U]} \mathcal{D}^\omega(U)$

($=$"$\mathcal{D}^{\omega}(U^{\wedge p})$") is isomorphic to $\mathbb{C}[U]^{\wedge p} \otimes_{\mathbb{C}[U]} \mathcal{D}(U)$ as a filtered algebra. We use $\mathbb{C}[T^*U] \xrightarrow{\sim} \operatorname{gr}\mathcal{D}(U)$ to show that $\mathbb{C}[T^*U] \twoheadrightarrow \operatorname{gr}\mathcal{D}^{\omega}(U)$ becomes an iso after tensoring w. $\mathbb{C}[U]^{\wedge p}$. Since this is true for every pt $p \in U$, we get $\mathbb{C}[T^*U] \xrightarrow{\sim} \operatorname{gr}\mathcal{D}^{\omega}(U)$.

A preliminary conclusion: when U is affine an element of $H^1(U, \Omega_U^{\geq 1}) = H^2_{dR}(U)$ gives rise to an algebra of TDO.

Step 4: Now consider the case of general X. Let U_i, U_{ij}, U_{ijk}, α_{ij}, ω_i have the same meaning as before the statement of the theorem. Each ω_i gives rise to the filtered algebra $\mathcal{D}^{\omega_i}(U_i)$, Step 1. Each α_{ij} gives rise to a filtered algebra isomorphism $\mathcal{D}^{\omega_i}(U_{ij}) \xrightarrow{\sim} \mathcal{D}^{\omega_j}(U_{ij})$ which the id on the associated graded, $\mathbb{C}[T^*U_{ij}]$. The claim that these data form a sheaf is equivalent to the isomorphisms $\mathcal{D}^{\omega_i}(U_{ij}) \xrightarrow{\sim} \mathcal{D}^{\omega_j}(U_{ij})$ satisfying the cocycle condition on the triple intersections. The cocycle condition is equivalent to $\alpha_{ij}+\alpha_{jk}+\alpha_{ki}=0$ (exercise). The resulting sheaf is that of TDO.

So given a cocycle (ω_i, α_{ij}) we get a sheaf of TDO. To check that cohomologous cocycles give rise to isomorphic sheaves of TDO is an exercise.

Step 5: Let us now sketch how to get in the opposite direction: from a sheaf of TDO $\mathcal{D}'$ to a cohomology class. Consider the subsheaf $\mathcal{D}'_{\leq 1}$. Thanks to a graded Poisson isom'm $\operatorname{gr}\mathcal{D}' \xrightarrow{\sim} p_* \mathcal{O}_{T^*X}$, $\mathcal{D}'_{\leq 1}$ is a coherent sheaf & a Lie algebra, and we get a SES: $0 \to \mathcal{O}_X \to \mathcal{D}'_{\leq 1} \longrightarrow \mathrm{Vect}_X \to 0$ (of coherent sheaves and of Lie algebras).

Since Vect_X is a vector bundle, we have that each $\mathrm{Vect}(U_i)$ is a projective $\mathbb{C}[U_i]$-module. So we can (non-canonically) split the exact sequence: $\mathcal{D}'_{\leq 1}(U_i) \simeq \mathbb{C}[U_i] \oplus \mathrm{Vect}(U_i)$. We define ω_i by $\omega_i(\xi,\xi') = \xi\cdot\xi' - \xi'\cdot\xi - [\xi,\xi']$. We emphasize that ω_i depends on the choice of the splitting, hence isn't canonical (and it should not be). We get $\mathcal{D}'(U_i) = \mathcal{D}^{\omega_i}(U_i)$: the relation for two vector fields is the definition of ω_i, $f\cdot\xi = f\xi$ follows from the splitting, $f\cdot\xi - \xi\cdot f = \xi.f$ follows from the isomorphism $\mathrm{gr}\,\mathcal{D}'(U_i) \simeq \mathbb{C}[T^*U_i]$ being Poisson.

The sheaf structure on $\mathcal{D}'$ gives rise to an isomorphism $\mathcal{D}^{\omega_i}(U_{ij}) \xrightarrow{\sim} \mathcal{D}^{\omega_j}(U_{ij})$ of filtered algebras that is the identity on the common associated graded, $\mathbb{C}[T^*U_{ij}]$. Such an isomorphism must have the form $f \mapsto f$, $\xi \mapsto \xi + \langle \alpha_{ji}, \xi\rangle$ for $\alpha_{ji} \in \Omega^1(U_{ij})$ w. $\omega_j - \omega_i = d\alpha_{ji}$ (exercise). The cocycle condition for isomorphisms translates to $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0$.

So from $\mathcal{D}'$ we produce a 1-cocycle (ω_i, α_{ij}). The ambiguity in (ω_i, α_{ij}) is precisely controlled by 1-coboundaries (exercise). So from $\mathcal{D}'$ we get an element in $H^1(X, \Omega_X^{\geq 1})$. Moreover, the construction implies that the maps between $H^1(X, \Omega_X^{\geq 1})$ & the iso classes of sheaves of TDO are mutually inverse. $\square$

Example: Take $\mathcal{D} = \mathcal{D}_X^{\mathcal{L}}$. Pick an open affine cover $X = \bigcup_i U_i$ s.t. $\mathcal{L}|_{U_i}$ trivializes. The $f_{ij} \in \mathbb{C}[U_{ij}]^\times$ be the transition functions. Then a cocycle giving $\mathcal{D}_X^{\mathcal{L}}$ is $\omega_i = 0$, $\alpha_{ij} = d\log f_{ij}$, this follows from our computation in the proof of Proposition 1. The cohom. class of this (ω_i, α_{ij}) is known as the <u>1st Chern class</u>, $c_1(\mathcal{L})$.

2.3) TDO via quantum Hamiltonian reduction.

Let G be an algebraic group and $\pi: X \to X_0$ be a principal G-bundle. Fix $\chi \in (\mathfrak{g}^*)^G$ and set $R_G^\chi = \mathcal{D}_X / \mathcal{D}_X\{\xi_X - \langle \chi, \xi \rangle \mid \xi \in \mathfrak{g}\}$. Note that $\varphi: \mathfrak{g} \to \Gamma(\mathcal{D}_X)$, $\xi \mapsto \xi_X - \langle \chi, \xi \rangle$, is a quantum moment map, so we are in the framework of quantum Hamiltonian reduction that we have discussed in a previous lecture.

We get a quasi-coherent sheaf of algebras $\pi_\bullet(R_X^\chi)^G$ on X_0.

Theorem 2: $\pi_\bullet(R_X^\chi)^G$ is a sheaf of TDO.

In the proof we will need the concept of the <u>classical Hamiltonian reduction</u> and its connection to the quantum Hamiltonian reduction.

Let A be a commutative Poisson algebra equipped w. a rational G-action by Poisson algebra automorphisms. By a <u>comoment map</u> for this action we mean a G-equivariant linear map $\varphi: \mathfrak{g} \to A$ s.t. $\xi_A = \{\varphi(\xi), \cdot\}$ $\forall \xi \in \mathfrak{g}$ (may not exist).

Example 1: Let G act on an affine variety Y. Set $A = \mathbb{C}[T^*Y]$, $\varphi(\xi) = \xi_Y \in \mathrm{Vect}(Y) \subset \mathbb{C}[T^*Y]$. This is a comoment map.

Example 2: Let $\mathcal{A}$ be a filtered associative algebra w. $\deg[\cdot,\cdot] \leq -1$ so that $A = \operatorname{gr} \mathcal{A}$ becomes a graded Poisson algebra. Assume that G acts on $\mathcal{A}$ rationally by filtered algebra automorphisms. The action descends to a rational action on A by graded Poisson algebra automorphisms. Assume that there is a quantum comoment map $\Phi: \mathfrak{g} \to \mathcal{A}$ w. $\operatorname{im}\Phi \subset \mathcal{A}_{\leq 1}$. It gives rise to $\varphi: \mathfrak{g} \to A_1$, the top degree term. Tracking the definition of the Poisson bracket on A, one sees that φ is a comoment map.

In particular, if $\mathcal{A} = \mathcal{D}(Y)$ w. $G \curvearrowright Y$, we can fix $\chi \in (\mathfrak{g}^*)^G$ and set $\Phi(\xi) = \xi_Y - \langle \chi, \xi \rangle$. Then we recover Example 1.

Let's get back to the general setting of $G \curvearrowright A$ w. comoment map $\varphi: \mathfrak{g} \to A$.

Important exercise 4: The algebra $[A/A\varphi(\mathfrak{g})]^G$, the classical Hamiltonian reduction of A, has a well-defined Poisson bracket given by $\{a + A\varphi(\mathfrak{g}), b + A\varphi(\mathfrak{g})\} := \{a, b\} + A\varphi(\mathfrak{g})$.

Important exercise 5: Suppose we are in the general setting of Example 2. Then

- $\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g})$, $(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}))^G$ inherit filtrations from $\mathcal{A}$.
- $A/A\varphi(\mathfrak{g}) \twoheadrightarrow \operatorname{gr}(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}))$, a natural epimorphism
- If $A/A\varphi(\mathfrak{g}) \xrightarrow{\sim} \operatorname{gr}(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}))$, then $\operatorname{gr}[(\mathcal{A}/\mathcal{A}\Phi(\mathfrak{g}))^G] \hookrightarrow (A/A\varphi(\mathfrak{g}))^G$, a monomorphism of graded Poisson algebras.

Sketch of proof of Theorem 2: The proof uses the same ideas as that of $\pi_\cdot(R_X)^G \simeq \mathcal{D}_{X_0}$, so we will be brief.

Step 1: For the same reason as in that proof we have $[\pi_\cdot(p_*\mathcal{O}_{T^*X}/(p_*\mathcal{O}_{T^*X})\varphi(\mathfrak{g}))]^G \xrightarrow{\sim} p_{0,*}\mathcal{O}_{T^*X_0}$, an isomorphism of sheaves of graded Poisson algebras.

Step 2: We have a nat'l epimorphism $\pi_\cdot(p_*\mathcal{O}_{T^*X}/p_*\mathcal{O}_{T^*X}\varphi(\mathfrak{g})) \twoheadrightarrow gr\, \pi_\cdot(R_X^\chi)$, see Important exercise 5. We prove that it is an isomorphism: we first do this in the case when $X = G \times X_0$ – both sheaves are $\mathcal{O}_G \boxtimes p_{0,*}\mathcal{O}_{T^*X_0}$ – and then reduce to this case using etale base change, as in the proof of $\pi_\cdot(R_X)^G \xrightarrow{\sim} \mathcal{D}_{X_0}$.

Step 3: Thanks to Important Exercise 5 and the previous two steps we get a graded Poisson algebra embedding $gr[\pi_\cdot(R_X^\chi)^G] \hookrightarrow p_{0*}\mathcal{O}_{T^*X_0}$. We argue as in Step 2 to prove that this embedding is an isomorphism.

This gives a graded Poisson algebra isomorphism $gr[\pi_\cdot(R_X^\chi)^G] \xrightarrow{\sim} p_{0*}\mathcal{O}_{T^*X_0}$ so $\pi_\cdot(R_X^\chi)^G$ is indeed a sheaf of TDO $\square$

Rem: Similarly to the case of $\chi=0$, we also get that the functors $\pi^*, \pi_\cdot(?)^G$ are mutually quasi-inverse equivalences between $Coh^{G,\chi}(\mathcal{D}_X)$ & $Coh(\pi_\cdot(R_X^\chi)^G)$.

Example 1: Assume that $G=\mathbb{C}^\times$. To give a principal G-bundle X over X_0 is the same as to give a line bundle $\mathcal{L}$ on X_0: X is the complement to the zero section in the total space of $\mathcal{L}$.

Let us write $\mathbb{1}$ for the identity character of $\mathcal{L}$. The group

$\mathbb{C}^\times$ acts on $\pi_\bullet(\mathcal{O}_X)$ and we can decompose $\pi_\bullet(\mathcal{O}_X)$ into the direct sum of "eigen subsheaves" according to characters of $\mathbb{C}^\times$. Then $\mathcal{L}$ is identified with the $\mathbb{1}$ eigensheaf. In other words $\mathcal{L} = \pi_\bullet(\mathcal{O}_X \otimes \mathbb{C}_{-\mathbb{1}})^{\mathbb{C}^\times}$. Note that $\mathcal{O}_X \otimes \mathbb{C}_{-\mathbb{1}} \in Coh^{G,\mathbb{1}}(\mathcal{D}_X)$. So $\pi_\bullet(R_X^{\mathbb{1}})^G$ acts on $\mathcal{L}$. This action is by differential operators and hence gives a homomorphism $\pi_\bullet(R_X^{\mathbb{1}})^{\mathbb{C}^\times} \longrightarrow \mathcal{D}_{X_0}^{\mathcal{L}}$ of sheaves of algebras, and in fact, of sheaves of filtered algebras that is the identity on the common associated graded. So $\pi_\bullet(R_X^{\mathbb{1}})^{\mathbb{C}^\times} \simeq \mathcal{D}_X^{\mathcal{L}}$.

Example 2: The previous example admits several generalizations. For example, in the setting of Example 1 take $\chi = z\cdot\mathbb{1}$, where $z \in \mathbb{C}$. We have the sheaf of TDO $\pi_\bullet(R_X^{z\mathbb{1}})^G$. The corresponding cohomology class can be shown to depend linearly on z so it equals $z c_1(\mathcal{L})$.

Even more generally, consider the case $G = (\mathbb{C}^\times)^n$. To give a principal G-bundle over X_0 is the same thing as to give n line bundles, $\mathcal{L}_1, \ldots, \mathcal{L}_n$. An element $\chi \in \mathfrak{g}^* = \mathbb{C}^n$ is written as $\sum_i z_i \mathbb{1}_i$. The cohomology class corresponding to $\pi_\bullet(R_X^\chi)^G$ depends linearly on χ hence is given by $\sum_i z_i c_1(\mathcal{L}_i)$.

Rem: Let's explain how to compute the cohomology class of $\pi_\bullet(R_X^\chi)^G$ in general. For a smooth variety w. G-action (at least when X can be covered w. open G-stable affines) one can define the equivariant Čech-de Rham complex, $\Omega^\bullet_{X,G}$. We have a natural map $(\mathfrak{g}^*)^G \longrightarrow H^1(X, \Omega^{\geq 1}_{X,G})$. It's known that for principal G-bundles, we have $H^i(X, \Omega^{\geq 1}_{X,G}) \xrightarrow{\sim} H^i(X_0, \Omega^{\geq 1}_{X_0})$. The composed map $(\mathfrak{g}^*)^G \longrightarrow H^i(X_0, \Omega^{\geq 1}_{X_0})$ sends χ to the cohomology class corresponding to $\pi_\bullet(\mathcal{R}^\chi_X)^G$.

2.4) Sheaves of TDO on G/B, and localization theorems.

Now consider the special case of $X = G/B$. Note that, by Hodge theory, $H^2_{dR}(X) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ & $H^1(X, \Omega^{\geq 1}_X) = H^{2,0}(X) \oplus H^{1,1}(X)$. For $X = G/B$, we have $H^{2,0}(X) = \{0\}$: there are no G-invariant holomorphic 2-forms on G/B. Alternatively, we can use that $X = G/B$ has a stratification by affine spaces to conclude that $H^2_{dR}(X) = H^{1,1}(X)$ and also that this space is isomorphic to $Pic(X) \otimes_{\mathbb{Z}} \mathbb{C}$, the isomorphism $Pic(X) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow H^{1,1}(X)$ is given by $\mathcal{L} \mapsto c_1(\mathcal{L})$. The Pickard group $Pic(X)$ is identified with the character lattice $\mathfrak{X}(T)$, where $T \subset B$ is a max torus if G is simply connected. From here we see that every sheaf of TDO on X is isomorphic to $\pi_\bullet(\mathcal{R}^\chi_{G/N})^T$, where $N \subset B$ is max'l unipotent.

The sheaf $\mathcal{D}_{G/B}$ is closely related to the representation theory of $\mathcal{U}_0$, the quotient of $\mathcal{U}(\mathfrak{g})$ by a suitable maximal ideal in the center. Namely, we have stated that:

(I) $\Gamma(\mathcal{D}_{G/B}) \xleftarrow{\sim} \mathcal{U}_0$

(II) $\Gamma: Coh(\mathcal{D}_{G/B}) \xrightarrow{\sim} \mathcal{U}_0\text{-mod}: Loc$

It turns out that an analog of (I) holds for all X, while an analog of (II) needs some additional assumptions on X.

Namely, we have the identification between the center of

$\mathcal{U}(\mathfrak{g})$ and $\mathbb{C}[\mathfrak{t}^*]^{(W,\cdot)}$. Let m_χ be the maximal ideal in $\mathbb{C}[\mathfrak{t}^*]^{(W,\cdot)}$ consisting of all polynomials vanishing at $\chi\in\mathfrak{t}^*$. Then set $\mathcal{U}_\chi = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\, m_\chi$. Also set $\mathcal{D}^\chi_{G/B} := \pi_\cdot(R^\chi_{G/N})^T$.

The actions of G & T on G/N commute. In particular, the elements $\xi_{G/N}$ in $\mathcal{D}_{G/N}$ are T-invariant. Therefore they survive in $\pi_\cdot(R^\chi_{G/N})^T$. Abusing the notation, we denote the element of $\Gamma(\mathcal{D}^\chi_{G/B})$ corresponding to $\xi\in\mathfrak{g}$ by $\xi_{G/B}$. The map $\xi\mapsto\xi_{G/B}$ is still a Lie algebra homomorphism $\mathfrak{g}\longrightarrow\Gamma(\mathcal{D}^\chi_{G/B})$.

Theorem 3: The algebra homomorphism $\mathcal{U}(\mathfrak{g})\longrightarrow\Gamma(\mathcal{D}^\chi_{G/B})$ factors through an isomorphism $\mathcal{U}_\chi\xrightarrow{\sim}\Gamma(\mathcal{D}^\chi_{G/B})$.

Thanks to this isomorphism we can consider an adjoint pair of functors: $\Gamma_\chi : Coh(\mathcal{D}^\chi_{G/B}) \rightleftarrows \mathcal{U}_\chi\text{-mod} : Loc_\chi$. We want to understand when these functors are mutually inverse. For this, we need a definition:

Definition: We say that χ is:

- dominant if for all positive coroots $\alpha^\vee$ we have $\langle\alpha^\vee,\chi\rangle \notin \mathbb{Z}_{<0}$
- regular if for all positive coroots $\alpha^\vee$ we have $\langle\alpha^\vee,\chi\rangle \neq 0$.

For example, for $\mathfrak{g}=\mathfrak{sl}_2$, "dominant" means "not a negative integer" and "regular" means "$\neq 0$."

The following is the most general version of the Beilinson-Bernstein Theorem for G/B (more precisely of its abelian version, there's also a derived version).

Theorem 4: • Suppose that $\chi+\rho$ is dominant. Then $H^i(G/B, \mathcal{F}) = 0$ $\forall$ $i>0$ & $\mathcal{F} \in \mathrm{Coh}(\mathcal{D}^\chi_{G/B})$.

• If $\chi+\rho$ is, in addition, regular, then $\mathcal{F}$ is generated by its global sections. Moreover, Γ_χ, Loc_χ are mutually quasi-inverse equivalences.

Rem: Note that the conditions on χ in both parts of the theorem are necessary: for $G = SL_2$ & $G/B = \mathbb{P}^1$, we have $H^1(\mathbb{P}^1, \mathcal{O}(j)) \neq 0$ for $j \leq -2$ & $\Gamma(\mathbb{P}^1, \mathcal{O}(-1)) = \{0\}$.