Lectures on $\mathcal{D}$-modules.

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These are lecture notes of a course given at the University of Chicago in Winter 1998. The purpose of the lectures is to give an introduction to the theory of modules over the (sheaf of) algebras of algebraic differential operators on a complex manifold. This theory was created about 15-20 years ago in the works of Beilinson-Bernstein and Kashiwara, and since then had a number of spectacular applications in Algebraic Geometry, Representation theory and Topology of singular spaces.

We begin with defining some basic functors on D-modules, introduce the notion of characteristic variety and of a holonomic D-module. We discuss b-functions, and study the Riemann-Hilbert correspondence between holonomic D-modules and perverse sheaves. We then go on to some deeper results about D-modules with regular singularities. We discuss D-module aspects of the theory of vanishing cycles and Verdier specialization, and also the problem of "gluing" perverse sheaves. We also outline some of the most important applications to Representation theory and Topology of singular spaces. The contents of the lectures has effectively no overlapping with Borel's book "*Algebraic D-modules*".

These lectures can be divided into two parts. The reader who is mostly interested in $\mathcal{D}$-modules is advised to go directly to Part 2, and to return to results of Part 1 whenever a reference on such a result is made. There are only a few places where the results of Part 1 are used in Part 2 in an essential way.

The reader who is more tolerant to Abstract Algebra and is interested in some aspects of "non-commutative" Algebraic Geometry may find Part 1 interesting in its own right.

Table of Contents

1. Algebraic background.

The same way as an Algebraic Geometry course requires certain background in Commutative Algebra, a $\mathcal{D}$-module course requires some background in non-commutative algebra. Such a background is given below.

1.1. Filtered rings and modules.

Let A be an associative ring with unit. We call A a *filtered* ring if an increasing filtration $\ldots A_i \subset A_{i+1} \subset \ldots$ by additive subgroups is given such that

(i) $A_i A_j \subset A_{i+j}$,

(ii) $1 \in A_0$,

(iii) $\bigcup A_i = A$, i.e. the filtration is *exhausting.*

We will usually consider two choices for the range of the index i:

(a) $i \in \mathbb{N}$ (b) $i \in \mathbb{Z}$.

In case (a) we will call A positively filtered. Note that this case may be viewd as a special case of (b) if we set $A_{-1} = 0$. In the latter case we will consider the topology induced by the filtration (in which the $\{A_i\}_{i\in\mathbb{Z}}$ form a base of open subsets) and usually impose two extra conditions on the filtration:

(iv′) $\bigcap A_i = \{0\}$, i.e. the topology defined by A_i is *separating.*

(iv″) A is complete with respect to this topology.

Denote by $\operatorname{gr} A$ the associated graded ring $\bigoplus_i A_i/A_{i-1}$.

1.1.1 Notation. For any element $a \in A_i$ we denote by $\sigma_i(a)$ the image of a in $A_i/A_{i-1} \subset \operatorname{gr} A$. We call $\sigma_i(a)$ the *i-th principal symbol* of a. If $a \in A_i \setminus A_{i-1}$ then we write $\sigma(a)$ instead of $\sigma_i(a)$, and say that $\deg \sigma(a) = i$.

WARNING. The assignment $a \mapsto \sigma(a)$ does not give rise to an additive map $A \to \operatorname{gr} A$. Nonetheless, if $a_1, a_2 \in A$ and

$$\sigma(a_1)\sigma(a_2) \neq 0 \text{ then } \sigma(a_1 a_2) = \sigma(a_1)\sigma(a_2). \tag{1.1.2}$$

Definition 1.1.3 A ring A is called *almost commutative* if $\operatorname{gr} A$ is commutative.

We will deal mostly with almost-commutative algebras over a field of characteristic zero.

1.1.4 EXAMPLE Consider a Lie algebra $\mathfrak{g}$ over k, a ground field, and let $A = \mathcal{U}\mathfrak{g}$ be its universal enveloping algebra. By definition,

$\mathcal{U}\mathfrak{g}$ is a quotient of the tensor algebra $T\mathfrak{g}$, explicitly, we have $\mathcal{U}\mathfrak{g} := T\mathfrak{g}/\{x \otimes y - y \otimes x - [x,y]\}_{x,y\in\mathfrak{g}}$. The algebra $\mathcal{U}\mathfrak{g}$ inherits an increasing filtration $T\mathfrak{g}$ given by

$$A_i = \{\textit{span of monomials of degree } \leq i\}.$$

The Poincare-Birkhoff-Witt Theorem states that in this case $\mathrm{gr}\,\mathcal{U}\mathfrak{g}$ is isomorphic to the symmetric algebra $S\mathfrak{g}$. In particular, $\mathcal{U}\mathfrak{g}$ is almost commutative. The reason for this is that for $x, y \in \mathfrak{g} \subset \mathcal{U}_1\mathfrak{g}$ we have $x \cdot y - y \cdot x = [x,y] \in \mathfrak{g} \subset \mathcal{U}_1\mathfrak{g}$. $\Diamond$

Here is a partial converse for the previous example.

Proposition 1.1.5 *Let A be apositively filtered k-algebra such that*
(i) $A_0 = k$,
(ii) A is generated as a ring by A_1,
(iii) A is almost commutative.
Then A can be represented as a quotient of the universal enveloping algebra of some Lie algebra $\mathfrak{g}$:

$$A \simeq \mathcal{U}\mathfrak{g}/I\,.$$

Proof. Recall that $A_1 \cdot A_1 \subset A_2$. By almost commutativity for all $a, b \in A_1$, $ab = ba\,mod A_1$. Therefore $(ab - ba) \in A_1$ and A_1 acquires the structure of a Lie algebra. By the universal property of $\mathcal{U}(A_1)$ we have an algebra map $\mathcal{U}A_1 \to A$ which is necessarily surjective by (ii). $\square$

Proposition 1.1.6 *(i) If A is positively filtered and* $\mathrm{gr}\,A$ *is Noetherian,*
then A is itself Noetherian.
(ii) If A is $\mathbb{Z}$-filtered and complete and $\mathrm{gr}\,A$ *is Noetherian, then A is*
Noetherian.
(iii) If $\mathrm{gr}\,A$ *has no zero-divisors, then neither has A.*

1.1.7 REMARK If A is Noetherian, $\mathrm{gr}\,A$ does not have to be Noetherian. $\Diamond$

Proof. For any ideal $J \subset A$, the filtration on A induces a filtration $J_i := A_i \cap J$ on J, and we have an associated graded ideal $\mathrm{gr}\,J = \bigoplus J_i/J_{i-1} \subset \mathrm{gr}\,A$. If we have an increasing sequence $J \subset I \subset \ldots$ of ideals in A then the sequence $\mathrm{gr}\,J \subset \mathrm{gr}\,I \subset \ldots$ stabilizes, that is starting from some large enough ideal we have $\mathrm{gr}\,J = \mathrm{gr}\,I = \ldots$, since $\mathrm{gr}\,A$ is Noetherian.

We deduce from this that the sequence $J \subset I \subset \ldots$ stabilizes. To that end consider J_0 and I_0, the 0th terms of the filtration. Since $A_0 \hookrightarrow \operatorname{gr} A$, we have $J_0 = I_0$. This and $J_1/J_0 = \operatorname{gr}_1(J) = \operatorname{gr}_1(I) = I_1/I_0$ imply $J_1 = I_1$, which in turn, combined with $\operatorname{gr}_2(J) = \operatorname{gr}_2(I)$, implies $J_2 = I_2$, etc....

To prove part (ii), take an ideal $J \subset A$. Then $\operatorname{gr} J$ is finitely generated over $\operatorname{gr} A$ with generators $\bar{j}_1, \ldots \bar{j}_n$. Lift each $\bar{j}_i$ to $j_i \in J$, and show by going *downward* that $\{j_i\}$ generate J as follows. Let $b \in J_s \setminus J_{s-1}$ and write $\sigma(b) = \sum_i \sigma(a_{i1}) \cdot \bar{j}_i$ for some elements $a_{i1} \in A$, $i = 1, \ldots, n$. Then $b_1 = b - \sum_i a_{i1} j_i$ is an element of J_{s-1}, and we can choose elements $a_{i2} \in A$ such that $\sigma(b_1) = \sum_i \sigma(a_{i2}) \cdot \bar{j}_i$. Continuing in this way we construct $b_r \in J_{s-r}$ and $a_{ir} \in A$ such that $b_r = b_{r-1} - \sum_i a_{ir} j_i$ and $\sigma(b_{r-1}) = \sum_i \sigma(a_{ir}) \bar{j}_i$. By completeness of A the elements $a_i = a_{i1} + a_{i2} + a_{i3} + \ldots$ make sense. Using the separation property we obtain $(b - \sum_i a_i j_i) \in \bigcap_r J_{s-r} = \{0\}$.

Finally, (iii) follows from multiplicativity of the symbol map (1.1.2). $\square$

Corollary 1.1.8 *$\mathcal{U}\mathfrak{g}$ is Noetherian without zero divisors for any finite dimensional Lie algebra $\mathfrak{g}$.* $\square$

Modules over filtered rings. Let A be a filtered ring. An A-module M is said to be *filtered* if an increasing sequence of subgroups $\ldots \subset M_i \subset M_{i+1} \subset \ldots$ is given, such that:

$$A_i \cdot M_j \subset M_{i+j}, \qquad \forall i, j.$$

Again we usually consider two types of filtrations:

(a) If A is positively filtered then we require that the filtration on M is bounded below (that is, $M_{-n} = 0$ for $n \gg 0$). We do not require M to be positivly filtered since, for any filtration $F_\bullet M$, the shifted filtration, $F'_\bullet M := F_{\bullet-1} M$ makes M a filtered module again, and there is no reason why we should start at grade degree zero.

(b) If A is $\mathbb{Z}$-filtered then we usually require that the filtration is separating, $\bigcap M_i = 0$, and M is complete in the topology induced by the filtration (completeness will be important for us because of part (iii) of Lemma 1.1 below).

The associated graded space $\operatorname{gr} M = \bigoplus M_i/M_{i-1}$ has an obvious graded $\operatorname{gr} A$-module structure.

Definition 1.1.9 One defines the *Rees ring* of A by $\widehat{A} := \bigoplus_i A_i$.

Alternatively, we can embed A into the ring of Laurent polynomials $A[t, t^{-1}]$ and define $\widehat{A}$ as $\sum t^i A_i$. The two definitions are clearly equivalent. The element t in the latter definition corresponds in 1.1

to the unit of A, vieved as an element $1 \in A_1$ (not of A_0). Since t is invertible in $A[t, t^{-1}]$, the imbedding $\widehat{A} \hookrightarrow A[t, t^{-1}]$ gets identified naturally with the localization (with respect to t) map $\widehat{A} \hookrightarrow \widehat{A}_t$, The importance of the Rees ring $\widehat{A}$ can be best understood geometrically in the case where A is a commutative k-algebra. Then the imbedding $k[t] \hookrightarrow \sum t^i A_i = \widehat{A}$ induces a flat morphism of schemes: $\pi : Spec\widehat{A} \to \mathbb{A}^1 = \textit{Affine line over } k$. This morphism may be thought of as an explicit deformation of the ring A to $\mathrm{gr}A$ since we have, see e.g. [**CG**, ch.2]

$$\pi^{-1}(t) \simeq SpecA \;\; \forall t \neq 0 \quad \textit{while} \quad \pi^{-1}(0) \simeq Spec(\mathrm{gr}A)\,.$$

In exactly the same way, given a filtered A-module M we define $\widehat{M} = \bigoplus_i M_i = \sum t^i M_i \subset M[t, t^{-1}]$. Clearly $\widehat{A}$ is a graded ring, and $\widehat{M}$ is a graded $\widehat{A}$-module. It follows immediately from the definitions above that $\widehat{A}/t\widehat{A} \simeq \mathrm{gr}\, A$ and $\widehat{M}/t\widehat{M} \simeq \mathrm{gr}\, M$

Let A be a filtered ring, and M an A-module. We will assume that the filtration on A is given and fixed, but there is no apriori chosen filtration on M, and such a filtration is up to our choice.

Lemma 1.1.10 *The following two conditions on the filtration on M are equivalent:*

(i) $\widehat{M}$ is a finitely generated $\widehat{A}$-module

(ii) The filtration on M has the form

$$M_i = A_{i-r_1} m_1 + \ldots + A_{i-r_l} m_l$$

for some fixed $m_1, \ldots, m_l \in M$ and $r_1, \ldots, r_l \in \mathbb{Z}$.

If the filtration on A is complete, and M is finitely generated over A, then the above conditions are also equivalent to the following one:

(iii) $\mathrm{gr}\, M$ is a finitely generated $\mathrm{gr}\, A$-module.

Proof. (iii) follows from (i) if we choose a set of generators for $\widehat{M}$ and project them to $\widehat{M}/t\widehat{M} \simeq \mathrm{gr}\, M$. The implication (ii) $\Rightarrow$ (i) is trivial. To show that (iii) implies (ii), let $\{u_i\}$ be the finite set of generators of $\mathrm{gr}\, M$ over $\mathrm{gr}\, A$. We cam assume that each u_i is homogeneous. Let $m_i \in M$ be lifts of u_i to M. If A is positively filtered then (iii) implies that $M_i = 0$ for $i \ll 0$ and we can proceed by induction: the statement is true for $i \ll 0$ and if $m \in M_{i+1}$ then $\sigma(m) = \sum \sigma(a_i) u_i$, so $m = \sum a_i m_i (\mathrm{mod}\, M_i)$. If A is $\mathbb{Z}$-filtered then we proceed as in the proof of (1.1) using the completeness property. □

Definition 1.1.11 A filtration on M is called *good* if the equivalent conditions (i)-(ii) of Lemma 1.1 hold. If, in particular, A is positively filtered then the filtration on M is good iff $\operatorname{gr} M$ is a finitely generated $\operatorname{gr} A$-module.

From now on we will assume unless otherwise stated that

(1) A is an almost commutative algebra over a field $k\,(\subseteq A_0)$ of characteristic zero and, moreover,

(2) $\operatorname{gr} A$ is a finitely generated k-algebra.

Let M be a finitely generated A-module. Choose a good filtration on M. Then $\operatorname{gr} M$ is a module over $\operatorname{gr} A$, a commutative ring, so we can consider the support $Supp(\operatorname{gr} M) \subset Spec(\operatorname{gr} A)$ *with its reduced structure* so that $Supp(M)$ is given by the ideal $\sqrt{Ann(M)} \subset \operatorname{gr} A$.

Definition 1.1.12 **(Characteristic variety)** The support $Supp(\operatorname{gr} M)$ with its reduced scheme structure is called the *characteristic variety* (or *singular support*) of M and is denoted by $\operatorname{SS} M$. If S is an irreducible component of $\operatorname{SS} M$, and A' is its coordinate ring then the rank of the A'-module $\operatorname{gr} M \otimes_{\operatorname{gr} A} A'$ is called the *multiplicity* of M at S and is denoted by $mult(M, S)$.

Theorem 1.1.13 (J. Bernstein)

(i) $Supp(\operatorname{gr} M)$ does not depend on the choice of a good filtration on M.

(ii) For any irreducible component S of $\operatorname{SS} M$, the multiplicity $mult(M, S)$ does not depend on the choice of a good filtration.

Moreover, the multiplicity function $mult(\bullet, S)$ is additive on short exact sequences, that is given

$$0 \to M' \to M \to M'' \to 0$$

one has: $mult(M, S) = mult(M', S) + mult(M'', S)$ whenever $mult(M, S)$ is defined (if S is not an irredicible component of $\operatorname{SS} M'$ or $\operatorname{SS} M''$ we set the corresponding multiplicity to be zero).

1.1.14 Remark The theorem fails if we would not take the *reduced* structure on $Supp(\operatorname{gr} M)$. $\Diamond$

Lattices. Let A be a (not necessarily commutative) Noetherian ring, and $t \in A$ a central non-zero divisor. Then one defines, in a standard way, see §1.3 below, the localization, A_t, of A with respect to the multiplicative subset $\{t^n\}_{n\in\mathbb{Z}}$.

Let M be a finitely generated A_t-module.

Definition 1.1.15 A subgroup $L \subset M$ is called a *lattice* if L is a finitely generated A-submodule of M such that $\bigcup_k t^{-k} L = M$.

The following properties of lattices are immediate:
(1) For an exact sequence of modules

$$0 \to M' \to M \to M'' \to 0$$

and a lattice $L \subset M$, the subgroup $L' = M' \cap L$ (resp. $L'' = \{$ *image of* L *in* $M''\}$) is a lattice in M' (resp. M'') and one has an exact sequence

$$0 \to L' \to L \to L'' \to 0$$

(2) For any two lattices L_1, L_2 in M there exist big enough integers, $k, l \gg 0$, such that

$$t^k L_2 \subset L_1 \subset t^{-k} L_2$$

(of course, one can switch the roles of L_1 and L_2).

1.1.16 Notation Given a ring B, write $K^+(B)$ for Grothendieck *semigroup* of finitely generated B-modules, that is the abelian semigroup freely generated by symbols $[N]$, for all B-modules N, modulo relations $[N] = [N'] + [N'']$ for any short exact sequence $0 \to N' \to N \to N'' \to 0$.

Clearly, for any lattice L, the quotient L/tL is a module over $\bar{A} = A/tA$.

Theorem 1.1.17 *For any two lattices L, L', the classes $[L/tL]$ and $[L'/tL']$ are equal in $K^+(\bar{A})$.*

Corollary 1.1.18 *The assignment $[M] \mapsto [L]$ gives rise to a well-defined semigroup homomorphism $K^+(A_t) \to K^+(\bar{A})$.* $\Box$

1.1.19 Remark This is similar to the situation one encounters in representation theory of p-adic groups: one has natural maps

$$\mathbb{Q}_p \hookleftarrow \mathbb{Z}_p \to \mathbb{F}_p.$$

For any $G(\mathbb{Q}_p)$-module M we can choose a lattice L (i.e. a $G(\mathbb{Z}_p)$-submodule such that $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = M$. Then L/pL is a module over $G(\mathbb{F}_p)$ which depends on the choice of the lattice L, but whose class in $K^+(G(\mathbb{F}_q))$ does not depend on this choice. $\Diamond$

Proof of Theorem 1.1. First consider a special case when L is *adjacent* to L', i.e. $tL' \subset tL \subset L' \subset L$. The natural short exact sequence

$$0 \to L'/tL \to L/tL \to L/L' \to 0$$

induces the equality $[L/tL] = [L'/tL] + [L/L']$ of classes in $K^+(\bar{A})$. Another exact sequence

$$0 \to tL/tL' \to L'/tL' \to L'/tL \to 0$$

implies that $[L'/tL'] = [tL/tL'] + [L'/tL]$. But $tL/tL \simeq L/L'$ as $\bar{A}$-modules (since t is invertible, multiplication by t is an isomorphism). Hence the two equalities imply $[L/tL] = [L'/tL']$.

In the general case of an arbitrary pair of lattices, L, L', we introduce a sequence of lattices $L^j = L + t^j L'$, $j \in \mathbb{Z}$. One can easily prove that L^j is adjacent to L^{j+1} for all $j \in \mathbb{Z}$. Moreover, $L_j = L$ for $j \gg 0$, while $L_j = t^j L'$ for $j \ll 0$. Since

$$\ldots = [L^{j-1}/tL^{j-1}] = [L^j/tL^j] = [L^{j+1}/tL^{j+1}] = \ldots$$

by the first part of the proof, and multiplication by t^j is an isomorphism, the theorem follows. □

Lemma 1.1.20 *Let A $\mathbb{Z}$-filtered complete ring. If* gr$A$ *is Noetherian then $\widehat{A}$ is Noetherian.*

Proof. We will find a filtration on $\widehat{A}$ such that associated graded will be Noetherian. Put

$$F_j\widehat{A} = \sum_{i \le j} t^i A_i + \sum_{i > j} t^i A_j \;\; (= \sum_{i \le j} t^i A_i + A_j t^{j+1}[t]).$$

Then $\mathrm{gr}^F(\widehat{A}) \simeq (\mathrm{gr}\, A)[t, t^{-1}]$. Since $\mathrm{gr}\, A$ is Noetherian, $\mathrm{gr}\, A[t, t^{-1}]$ is also Noetherian. Hence, $\mathrm{gr}^F(A)$ and $\widehat{A}$ are both Noetherian, due to Lemma 1.1. □

Now let A be a filtered ring and M a filtered A-module. From the previous lemma and Lemma 1.1 we obtain the following relationship between filtrations and lattices.

Lemma 1.1.21 *A filtration on M is good iff $\widehat{M}$ is an $\widehat{A}$-lattice in the $A[t, t^{-1}]$-module $M[t, t^{-1}]$.* □

Therefore the choice of a good filtration corresponds precisely to a choice of a lattice.

Proof of theorem 1.1. For any two good filtrations on M, their respective Rees modules are two lattices in $M[t, t^{-1}]$ which have the same class in $K^+(\mathrm{gr}\, A)$ by (1.1). Hence not only their supports are equal but also they have equal multiplicities along each irreducible component of the support which has maximal dimension. The additivity of the multiplicity with respect to short exact sequances folows from its definition. □

1.1.22 Remark It is important for the proof to use K^+, not the K-group, since $K(\mathrm{gr}\, A)$ is less friendly with supports and multiplicities. ◊

1.1.23 Elementary properties of characteristic varieties.

Keep the setup of Theorem 1.1. The general properties of lattices mentioned above imply the following
(1) Suppose $M' \subset M$ is an A-submodule. If a good filtration $\{M_i\}$ on M is chosen then the induced filtration $M'_i = M_i \cap M'$ on M' is also good.
(2) The induced filtration on the quotient module M/M' is good and we have a short exact sequence

$$0 \to \mathrm{gr}\, M' \to \mathrm{gr}\, M \to \mathrm{gr}(M/M') \to 0$$

(3) The exact sequence above and standard properties of supports of modules over a commutative algebra yield

$$\mathrm{SS}(M) = \mathrm{SS}(M') \cup \mathrm{SS}(M/M')$$

(4) If $M \simeq A/J$ then $\mathrm{SS}(M)$ is the zero variety of $\mathrm{gr}\, J$.

Given a filtered ring A such that $\mathrm{gr}A$ is a commutative Noetherian ring, we have the scheme $\mathit{Spec}(\mathrm{gr}A)$. The grading on $\mathrm{gr}A$ corresponds geometrically to an algebraic $\mathbf{G_m}$-action on $\mathit{Spec}(\mathrm{gr}A)$. If A is positively filtered, then the projection

$$\mathrm{gr}A \twoheadrightarrow A_0 = \mathrm{gr}A/\bigoplus_{i>0} \mathrm{gr}_i A \qquad \textit{gives an inculsion} \quad \mathit{Spec}(A_0) \subset \mathit{Spec}(\mathrm{gr}A)\,.$$

The subscheme $\mathit{Spec}(A_0)$ is precisely the fixed point scheme of the $\mathbf{G_m}$-action on $\mathit{Spec}(\mathrm{gr}A)$. Moreover, $\mathit{Spec}(\mathrm{gr}A)$ is a *cone-scheme* over $\mathit{Spec}(A_0)$, i.e. there is a projection $\mathit{Spec}(\mathrm{gr}A) \to \mathit{Spec}(A_0)$ induced by the imbedding $A_0 \hookrightarrow A$. Furthermore, the $\mathbf{G_m}$-action contracts $\mathit{Spec}(\mathrm{gr}A)$ to the fixed point variety, $\mathit{Spec}(A_0)$, along the fibers of this projection.

The following simple criterion is quite useful.

Lemma 1.1.24 *Let A be a positively filtered algebra such that* $\mathrm{gr}A$ *is a finitely generated A_0-algebra, and M a finitely generated A-module. Then*

$$\mathrm{SS}M = \mathit{Spec}A_0 \subset \mathit{Spec}(\mathrm{gr}A)$$

iff M is finitely generated over A_0. $\square$

1.2. Three theorems of Gabber.

Recall the definition of a Poisson structure.

Definition 1.2.1 A *Poisson algebra* consists of the following data:
(1) a commutattive (associative) algebra $(B, \cdot)$ with unit
(2) a Lie bracket $(B, \{\ ,\ \})$.

(3) These two structures are related by the Leibniz identity:

$$\{a_1 \cdot a_2, b\} = a_1 \cdot \{a_2, b\} + a_2\{a_1, b\}\,.$$

Proposition 1.2.2 *If A is a filtered almost commutative algebra then* $\mathrm{gr}A$ *has a canonical structure of a Poisson algebra.*

Proof. Let $\bar{a}_i \in A_i/A_{i-1}$ and $\bar{a}_j \in A_j/A_{j-1}$. We will construct an element $\{\bar{a}_i, \bar{a}_j\} \in A_{i+j-1}/A_{i+j-2}$ such that the operation $\{,\}$ together with multiplication in $\mathrm{gr}A$ will satisfy the definition of a Poisson structure.

To that end, choose a lift $a_i \in A_i$ (resp. $a_j \in A_j$) of $\bar{a}_i$ (resp. $\bar{a}_j$). Form an element $a_ia_j - a_ja_i$. Apriori, this is an element of A_{i+j}. But since $\mathrm{gr}A$ is commutative, $(a_ia_j - a_ja_i) \in A_{i+j-1}$. We define $\{\bar{a}_i, \bar{a}_j\}$ to be the image of $(a_ia_j - a_ja_i)$ in A_{i+j-1}/A_{i+j-2}. One can show the class of $(a_ia_j - a_ja_i$ in A_{i+j-1}/A_{i+j-2} does not depend on the choice of lifts a_i and a_j (while, of course, $(a_ia_j - a_ja_i)$ itself depends on this lift). □

1.2.3 Example Let $\mathfrak{g}$ be a Lie algebra over a field k, and $\mathfrak{g}^*$ the dual space. Consider the enveloping algebra $A = \mathcal{U}\mathfrak{g}$ equipped with the standard filtration. Then $\mathrm{gr}\mathcal{U}\mathfrak{g} = S\mathfrak{g} = k[\mathfrak{g}^*]$, see Example 1.1. We will give three equivalent formulas for the Poisson structure on $\mathrm{gr}\mathcal{U}\mathfrak{g}$ arising from Proposition 1.2:

(1) For $x, y \in \mathfrak{g}$ one has $\{x^n, y^m\} = (mn)x^{n-1}y^{m-1}[x, y]$, where $[x, y]$
is the Lie bracket in $\mathfrak{g}$.

(2) Choose a base $x_1, \dots, x_r$ of $\mathfrak{g}$. Each x_i gives a linear function on $\mathfrak{g}^*$,
so $\{x_i\}_{i=1,\dots,n}$ is a coordinate system on $\mathfrak{g}^*$. In these coordinates we have

$$\{P, Q\} = \sum_{i,j,k} c_{ij}^k \cdot x_k \cdot \frac{\partial P}{\partial x_i}\frac{\partial Q}{\partial x_j} \quad , \quad P, Q \in k[\mathfrak{g}^*]\,,$$

where $c_{ij}^k \in k$ are the structure constants of the Lie algebra, defined by $[x_i, x_j] = \sum_k c_{ij}^k x_k$.

(3) For any $\lambda \in \mathfrak{g}^*$ one has

$$\{P, Q\}(\lambda) = \langle\lambda, \big[dP(\lambda), dQ(\lambda)\big]\rangle \quad , \quad P, Q \in k[\mathfrak{g}^*]\,,$$

where $dP(\lambda), dQ(\lambda) \in \mathfrak{g}$ and $\langle\ ,\ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to k$ is the natural pairing. ♢

Definition 1.2.4 Let B be a Poisson algebra. We say that a subvariety $V \subset Spec(B)$ is *coisotropic* if the ideal I_V of functions vanishing on V satisfies

$$\{I_V, I_V\} \subset I_V.$$

Theorem 1.2.5 (Involutivity of characteristic variety) *Let A be an almost commutative algebra such that* $\mathrm{gr}A$ *is Noetherian, and M be a finitely generated A-module. Then the characteristic variety* $\mathrm{SS}(M) \subset$ *Spec*$(\mathrm{gr}A)$ *is coisotropic with respect to the Poisson structure on* $\mathrm{gr}A$.

1.2.6 COMMENT. In a special case when $M = A/J$ for some left ideal $J \subset A$ we have $I_{\mathrm{SS}(M)} = \sqrt{\mathrm{gr}J}$. This special case actually implies the Theorem. Note further that is easy to prove that $\{\mathrm{gr}(J), \mathrm{gr}(J)\} \subset \mathrm{gr}(J)$. However, in general, given an ideal I in a Poisson algebra, we have

$$\{I,\, I\} \subset I \quad \not\Rightarrow \quad \{\sqrt{I},\, \sqrt{I}\} \subset \sqrt{I}\,.\lozenge$$

1.2.7 REMARK One recent application of the Involutivity of Characteristic Variety is the proof by Beilinson-Drinfeld (cf. [**BeFM**]) of the fact that conformal blocks for the Virasoro Lie algebra are finite-dimensional, based on Lemma (1.1). $\lozenge$

The original proof of the Involutivity of characteristics theorem by Gabber [Ga] was quite tricky. We present here a simplified version of the argument due to F. Knop.

Consider the dual numbers $\mathbb{D} := \mathbb{C}[\epsilon]/\epsilon^2$. For each $\mathbb{D}$-module M put $\overline{M} := M/\epsilon M$. For $m \in M$ let $\overline{m}$ be its image in $\overline{M}$. Multiplication by ϵ induces a map $\overline{M} \to \epsilon M$. Then M is $\mathbb{D}$-free if and only if this map is an isomorphism.

Let A be a finitely generated $\mathbb{D}$-algebra and M a finitely generated A-module. Assume that both A and M are $\mathbb{D}$-free. Assume moreover that $\overline{A}$ is commutative. Consider $I := \sqrt{\|Ann|_{\overline{A}}\overline{M}}$. Then $\{I, I\} \subseteq I$.

Let A be a $\mathbb{D}$-algebra. Assume that $\overline{A}$ is commutative and that A is $\mathbb{D}$-free. Then for each $\bar{a}, \bar{b} \in \overline{A}$ one can define a Poisson product $\{\bar{a}, \bar{b}\} \in \overline{A}$ by the formula $[a, b] = \epsilon\{\bar{a}, \bar{b}\}$. The theorem of Gabber follows from:

Theorem 1.2.8 *Let A be a finitely generated $\mathbb{D}$-algebra and M a finitely generated A-module. Assume that both A and M are $\mathbb{D}$-free. Assume moreover that $\overline{A}$ is commutative, and put $I := \sqrt{\|Ann|_{\overline{A}}\overline{M}}$. Then $\{I, I\} \subseteq I$.*

Proof of the Involutivity of characteristics theorem. The ideal I can also be defined as the intersection of all minimal associated prime ideals $\mathfrak{p}$ of $\overline{M}$. Thus it suffices to show $\{\mathfrak{p},\mathfrak{p}\} \subseteq \mathfrak{p}$ for each of these $\mathfrak{p}$.

Choose $x_1, \ldots, x_l \in A$ such that the $\overline{x}_i + \mathfrak{p}$ form a maximal algebraically independent set in $\overline{A}/\mathfrak{p}$. Let $R \subseteq \overline{A}$ be the subalgebra generated by the $\overline{x}_i$. Then there is an $0 \neq f \in R$ such that $B := \overline{A}_f/\mathfrak{p}_f$ is a free R_f-module (of finite rank). Since $\mathfrak{p}$ is a minimal associated prime of $\overline{M}$ one can find f and $s > 0$ such that $\mathfrak{p}^s\overline{M}_f = 0$ and each of $\mathfrak{p}^i\overline{M}_f/\mathfrak{p}^{i+1}\overline{M}_f$ is a free B-module. Thus, one can find $m_1, \ldots, m_s \in M$ such that the $\overline{m}_i$ form an R_f-basis of $\overline{M}_f$ with the property $\mathfrak{p}\overline{m}_i \subseteq \sum_{j\prec i} R\overline{m}_j$. Here $j \prec i$ means $\overline{m}_j \in \mathfrak{p}^t\overline{M}_f$ and $\overline{m}_i \notin \mathfrak{p}^t\overline{M}_f$ for some t.

Lemma 1.2.9 *For all $a, b \in A$ with $\bar{a}, \bar{b} \in \mathfrak{p}$ one can find integers $n_1, n_2, n_3 \geq 0$ and elements $e_{ij} \in A$ with $\overline{e}_{ij} \in R$ such that $f^{n_1}[f^{n_2}a, f^{n_2}b]m_i = \epsilon \sum_j e_{ij}m_j$ and $\sum_i \overline{e}_{ii} = 0$.*

We show first that this Lemma implies the Theorem. We have $f^{n_1}[f^{n_2}a, f^{n_3}b] = \epsilon c$ where $c \in f^t\{\bar{a},\bar{b}\} + \mathfrak{p}$ with $t = n_1 + n_2 + n_3$. The action of c on the free R_f-module $\overline{M}_f$ is given by the matrix $(\overline{e}_{ij})$ whose trace is zero. Since $\mathfrak{p}$ acts nilpotently, we conclude that the trace of $\{\bar{a},\bar{b}\}$ on $\overline{M}_f$ is zero. Apply this to $\bar{a}$ replaced by $x\bar{a}$ for any $x \in \overline{A}_f$. From $\{x\bar{a},\bar{b}\} \in x\{\bar{a},\bar{b}\} + \mathfrak{p}_f$ we conclude that $\|tr|_{R_f}(x\{\bar{a},\bar{b}\} : \overline{M}_f) = 0$ for all $x \in \overline{A}$.

On the other hand, the trace of $y := x\{\bar{a},\bar{b}\}$ can be calculated as the trace on $\oplus_i\mathfrak{p}^i\overline{M}_f/\mathfrak{p}^{i+1}\overline{M}_f \cong B^\ell$ for some $\ell > 0$. Thus $\|tr|_{R_f}(y : \overline{M}_f) = \ell\|tr|_{R_f}(y : B)$. The extension $B|R_f$ is (generically) separable, hence its trace form is non-degenerate. Thus the image of $\{\bar{a},\bar{b}\}$ in B is zero, i.e., $\{\bar{a},\bar{b}\} \in \mathfrak{p}$. □

Proof of the Lemma: Let $A' \subseteq A$ be the set of a with $\bar{a} \in R$. By construction, one can find $n_2 \geq 0$ and $u_{ij}^{(0)}, u_{ij}^{(1)} \in A'$ such that

$$f^{n_2}am_i = \sum_{j\prec i} u_{ij}^{(0)}m_j + \epsilon\sum_j u_{ij}^{(1)}m_j.$$

Similarily, we obtain

$$f^{n_3}bm_i = \sum_{j\prec i} v_{ij}^{(0)}m_j + \epsilon\sum_j v_{ij}^{(1)}m_j.$$

Define the matrices $U^{(0)} = (u_{ij}^{(0)}), \ldots, V^{(1)} := (v_{ij}^{(1)})$. Then

$$f^{n_2}af^{n_3}bm_i = \sum_{i\prec j}(v_{ij}^{(0)}f^{n_2}a + \epsilon\{f^{n_2}a, v_{ij}^{(0)}\})m_j + \epsilon\sum_j v_{ij}^{(1)}f^{n_2}am_j =$$

$$= \sum_{i \prec j} ((V^{(0)}U^{(0)})_{ij} + \epsilon\{f^{n_2}a, v_{ij}^{(0)}\})m_j + \epsilon \sum_j \big((V^{(0)}U^{(1)})_{ij} + (V^{(1)}U^{(0)})_{ij}\big)m_j.$$

Thus

$$[f^{n_2}a, f^{n_3}b]m_i = \sum_{j \prec i} c_{ij}^{(0)} m_j + \epsilon \sum_j c_{ij}^{(1)} m_j.$$

where

$$c_{ij}^{(0)} = [V^{(0)}, U^{(0)}]_{ij} + \epsilon(\{f^{n_2}a, v_{ij}^{(0)}\} - \{f^{n_3}b, u_{ij}^{(0)}\})$$
$$c_{ij}^{(1)} = [V^{(0)}, U^{(1)}]_{ij} + [V^{(1)}, U^{(0)}]_{ij}.$$

Since $\bar{a}$ and $\bar{b}$ commute we have $c_{ij}^{(0)} \in \epsilon A$. Thus one can find $n_1 \geq 0$ and $d_{ij} \in R$ such that $f^{n_1} \sum_{j \prec i} c_{ij}^{(0)} m_j = \epsilon \sum_{j \prec i} d_{ij} m_j$. Now we define $e_{ij} = d_{ij} + f^{n_1} c_{ij}^{(1)}$. The trace of the matrix $\overline{e}_{ij}$ is zero since it it the sum of a strictly upper triangular matrix and two commutators. □

The most important application is as follows: let D be a filtered $\mathbb{C}$-algebra whose associated graded algebra $\tilde{D}$ is commutative. Then the commutator induces a Poisson product on $\tilde{D}$. Let X be a D-module with compatible filtration. Then the associated graded object $\tilde{X}$ is a $\tilde{D}$-module.

Corollary 1.2.10 Assume $\tilde{A}$ is a finitely generated commutative $\mathbb{C}$-algebra and $\tilde{X}$ a finitely generated $\tilde{D}$-module. Let $I := \sqrt{\|Ann|_{\tilde{D}}\tilde{X}}$. Then $\{I, I\} \subseteq I$.

Proof: Let $(D_n)_{n\in\mathbb{Z}}$ and $(X_n)_{n\in\mathbb{Z}}$ denote the filtrations of D and X repectively. Then apply [**?**] to $A := \oplus_{n\in\mathbb{Z}} D_n/D_{n-2}$, $M := \oplus_{n\in\mathbb{Z}} X_n/X_{n-2}$, and $\epsilon := 1 + D_{-1} \in D_1/D_{-1}$. □

Let A be as before and assume in addition that gr$A$ is the coordinate ring of a *smooth* connected affine variety, $char(k) = 0$. For any finitely generated (hence Noetherian) A-module M define a finite filtration of M *by A-submodules*, called the *Gabber filtration*, by setting

$$G_i(M) = \{\text{ largest } A\text{-submodule } N \subset M \text{ such that } \dim(\mathrm{SS}N) \leq i\}$$

("largest" makes sense by the Noetherian property). Alternatively we may define $G_i(M)$ as follows

$$G_i(M) \;=\; \{m \in M \mid \dim\ \mathrm{SS}(A \cdot m) \leq i\}.$$

Theorem 1.2.11 (Equi-dimensionality) *Assume* gr$A = k[X]$ *is the regular ring of a* **smooth** *affine algebraic variety over k, $char(k) = 0$. Then, for any $i = 0, 1, \ldots$, the characteristic variety of $G_iM/G_{i-1}M$ is of pure dimension i, provided $G_iM/G_{i-1}M$ is non-zero.* □

Corollary 1.2.12 *If Spec*$(\mathrm{gr}A)$ *is smooth and equidimensional, then the characteristic variety of any irreducible* A*-module is of pure dimension.* $\Box$

1.2.13 Remark The characteristic variety of an irreducible module need not be irreducible. For example, let $A = \mathbb{C}[z, \frac{d}{dz}]$ be the ring of polynomial differential operators in one variable with the standard filtration by the order of differential operators, see ch.2 below. Then we have $\mathrm{gr}A = \mathbb{C}[z, \xi]$, so that $Spec(\mathrm{gr}A) = \mathbb{C}^2$. Fix $\lambda \in \mathbb{C}$ and put $M_\lambda := \mathbb{C}[z, \frac{d}{dz}]{\cdot}z^\lambda = A/A{\cdot}(z\frac{d}{dz} - \lambda)$. Then, $\mathrm{SS}(M_\lambda) = \{z \cdot \xi = 0\} \subset \mathbb{C}^2$ is the "coordinate cross" consisting of the two coordinate lines. Now, it is easy to verify, that if λ is not an integer, the A-module M_λ is simple. We see that the characteristic variety of this simple module has two irreducible components of the same dimension. $\Diamond$

We now state the third theorem due to O.Gabber. Let $\mathfrak{r}$ be a solvable finite dimensional Lie algebra over a field k of characteristic zero. Suppose further that $\mathfrak{r}$ can be represented as a sum $\mathfrak{n} \oplus k \cdot \delta$ such that

(1) $\mathfrak{n}$ is a nilpotent Lie ideal and

(2) The adjoint action of δ on $\mathfrak{n}$ is semisimple with strictly positive rational eigenvalues.

Let M be a finitely generated $\mathcal{U}\mathfrak{r}$-module. Assume that there is filtration $\{M_i\}_{i\in\mathbb{N}}$ on M compatible with $\mathcal{U}\mathfrak{r}$-action such that $\mathrm{gr}M$ is finitely generated *over* $\mathrm{gr}\big(\mathcal{U}\mathfrak{n}\big) = S\mathfrak{n}$ (hence, over $S\mathfrak{r}$, in particular).

Theorem 1.2.14 (Separation theorem) *If* M *and* $\mathfrak{n}$ *are as above, one has* $\mathfrak{n} \cdot M \neq M$*, or equivalently,* $\bigcap_i \mathfrak{n}^i M = 0$.

1.2.15 Remarks. (1) If $\mathcal{U}_+ := \mathfrak{n} \cdot \mathcal{U}\mathfrak{n}$ denotes the augmentation ideal in $\mathcal{U}\mathfrak{n}$, then the theorem above can be restated as $\mathcal{U}_+ M \neq M$ and by the Artin-Rees Lemma one has

$$\bigcap \mathcal{U}_+^i M = 0,$$

i.e. the augmentation filtration on M is separating. This explains why two claims of the Separation Theorem are equivalent.

(2) Nilpotency of $\mathfrak{n}$ is essential: if $\mathfrak{n} = \mathfrak{sl}_2$, then any non-trivial finite dimensional simple $\mathfrak{n}$-module M satisfies $\mathfrak{n}M = M$.

(3) If $\mathfrak{n}$ is abelian, then $\bigcap \mathcal{U}_+^i M = 0$ is a standard fact in Commutative Algebra. Assume first that the point $0 \in Spec(S\mathfrak{n})$ does not belong to $Supp(M)$. Then there is a polynomial $P \in S\mathfrak{n}$ that vanishes on $Supp\ M$ and such that $P(0) \neq 0$. Replacing P by its high enough

power we may achieve that P annihilates M, i.e. $P \in Ann(M) \subset S\mathfrak{n}$. But the space $Ann(M)$ is clearly stable under the adjoint δ-action on $S\mathfrak{n}$. Moreover, since $P = P(0) + P_1$, where $P_1 \in \mathfrak{n}\cdot S\mathfrak{n}$, and all weights of $ad\delta$ on $\mathfrak{n}\cdot S\mathfrak{n}$ are strictly positive, we deduce from $P \in Ann(M)$ that $P(0)$, $P_1 \in Ann(M)$. Since $P(0) \neq 0$ this yields $1 \in Ann(M)$, a contradiction. Thus we have proved $0 \in Supp\ M$.

Now we can localize M at $0 \in Spec(S\mathfrak{n})$ to get a non-zero module $M_{(0)}$. But then Nakayama's lemma yields, $M_{(0)} = \mathfrak{n}M_{(0)}$, hence $M_{(0)} = 0$, a contradiction. $\square$

1.2.16 Casselman Theorem in Representation Theory.

The Separation Theorem was discovered as an attempt to find a purely algebraic proof of a theorem of Casselmann. The latter is a rather deep result in Representation theory originally proved by Casselmann using analytic methods. To state the Casselman Theorem we need some notation.

Let G be a real semisimple Lie group with Lie algebra $\mathfrak{g}$ (over $\mathbb{R}$). Let $K \subset G$ be a maximal compact subgroup. We have an Iwasawa decomposition

$$G = N \cdot A \cdot K\,,$$

where N is a unipotent subgroup, and A is isomorphic to a product of several copies of $\mathbb{R}^{>0}$. For example

$$G = SL_n(\mathbb{R}), \quad K = SU_n, \quad A = \{\, diag(\alpha_1, \ldots \alpha_n), \quad \alpha_i > 0\,\}\,,$$

$$N = \{\text{upper-triangular matrices with 1 on the diagonal}\}$$

Writing $\mathfrak{n} = Lie\ N$, $\mathfrak{a} = Lie\ A$, $\mathfrak{k} = Lie\ K$, we have the corresponding Lie algebra direct sum decomposition (as vector spaces, not as Lie algebras)

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k} \tag{1.2.17}$$

Now let M be an "admissible", e.g. irreducible unitary, representation of G in a complex (infinite-dimensional) topological vector space. Write

$$M^{\text{fin}} = \{m \in M \mid m \textit{ belongs to a finite-dimensional } K\textit{-stable subspace}\}$$

By a deep theorem of Harish-Chandra one knows, [**Wa**], that

(i) M^{fin} is dense in M, and $M \mapsto M^{\text{fin}}$ is an exact functor.

(ii) The Lie algebra action of any $x \in \mathfrak{g}$ on any $m \in M^{\text{fin}}$ is well-defined,
i.e. $\forall m \in M^{\text{fin}}$, the map $g \mapsto g \cdot m$, $G \to M$ is differentiable,
hence M^{fin} is a $\mathcal{U}\mathfrak{g}$-module.

(iii) For any "admissible" representation M, see e.g. [**Wa**], the $\mathcal{U}\mathfrak{g}$-module M^{fin} is finitely generated, and if M is irreducible then M^{fin} is a simple $\mathcal{U}\mathfrak{g}$-module.

Assume M^{fin} is simple. Then a version of Schur Lemma yields:

(iv) $Z(\mathfrak{g}) = (\textit{center of } \mathcal{U}\mathfrak{g})$ acts on M by scalars.

Further, from (iii) we deduce

(v) M^{fin} is generated as $\mathcal{U}\mathfrak{g}$-module by a finite dimensional $\mathcal{U}\mathfrak{k}$-stable subspace M_0, i.e. $M^{\text{fin}} = \mathcal{U}\mathfrak{g} \cdot M_0$.

Define a filtration $\{M_i, i \in \mathbb{N}\}$ on M^{fin} by $M_i = \mathcal{U}_i\mathfrak{g} \cdot M_0$ where $\{\mathcal{U}_i\mathfrak{g}\}$ is the standard increasing filtration on $\mathcal{U}\mathfrak{g}$.

Lemma 1.2.18 $\mathrm{gr}(M^{\text{fin}})$ *is a finitely generated* $\mathrm{gr}(\mathcal{U}\mathfrak{n})$*-module.*

Proof. By (1.2.17) we have

$$\mathcal{U}\mathfrak{g} \simeq \mathcal{U}\mathfrak{n} \otimes \mathcal{U}\mathfrak{a} \otimes \mathcal{U}\mathfrak{k}$$

Hence by (v) we get $M = \mathcal{U}\mathfrak{g} \cdot M_0 = \mathcal{U}\mathfrak{n} \cdot \mathcal{U}\mathfrak{a} \cdot M_0$. Further we have a Harish-Chandra algebra homomorphism $Z(\mathfrak{g}) \to \mathcal{U}\mathfrak{a}$. It is compatible with filtrations and $\mathrm{gr}(\mathcal{U}\mathfrak{a})$ is a finite module over the image of $\mathrm{gr}Z(\mathfrak{g})$. The claim can be derived from this and (iv). $\square$

Next we find $\delta \in \mathfrak{a}$, a generic $\mathbb{Q}$-rational linear combination of simple coroots in $\mathfrak{a}$, such that $ad_{\mathfrak{g}}\delta$ is a diagonalizable diagonalizable, its eigenvalues on $\mathfrak{n}$ are positive integers, and such that $Ker\ ad_{\mathfrak{g}}\delta$, the centralizer of δ has minimal possible dimension. Then $\mathfrak{l} = Ker\ ad_{\mathfrak{g}}\delta$ is a Levi subalgebra in $\mathfrak{g}$, and one has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n}$$

where $\mathfrak{n}^-$ is the Lie subalgebra spanned by the negative weight spaces for $ad_{\mathfrak{g}}\delta$. Thus, $\mathfrak{p} := \mathfrak{l} + \mathfrak{n}^-$ and $\mathfrak{l} + \mathfrak{n}^-$ are the opposite parabolic subalgebras of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{l}$.

Write $\mathcal{U}_+ = \mathfrak{n} \cdot \mathcal{U}\mathfrak{n}$ for the augmentation ideal. The Gabber theorem yields

Theorem 1.2.19 (Casselman Theorem) $\bigcap \mathcal{U}_+^i M^{\text{fin}} = 0\,.$ $\square$

Let $\mathcal{O}$ be the abelian category of finitely generated $\mathcal{U}\mathfrak{g}$-modules V such that $\mathcal{U}\mathfrak{p}$-action (recall $\mathfrak{p} := \mathfrak{l} + \mathfrak{n}^-$) on V is locally finite, i.e.:

$$\dim\ \mathcal{U}\mathfrak{p} \cdot v < \infty \quad \forall v \in V.$$

We define an *exact faithful* functor (called Jacquet functor)

$$J : \textit{Admissible G-representations} \ \to \ \mathcal{O}$$

as follows. First we introduce a naive functor

$$\widehat{J}: \; M \;\mapsto\; \varprojlim_i M^{\text{fin}}/\mathcal{U}_+^i M^{\text{fin}}$$

This functor is faithful by 1.2 and exact (since completion is exact) but the $\mathcal{U}\mathfrak{g}$-module $\widehat{J}(M)$ is too large to be an object of $\mathcal{O}$, it is not even finitely generated as an $\mathcal{U}\mathfrak{g}$-module.

We may do better. Notice that $M^{\text{fin}}/\mathcal{U}_+M^{\text{fin}}$ is finite dimensional. It follows that $\forall i$, the space $M^{\text{fin}}/\mathcal{U}_+^i M^{\text{fin}}$ is finite dimensional (since $\text{gr}M^{\text{fin}}$ is finitely generated over gr $\mathcal{U}\mathfrak{n}$). The action of $\delta \in \mathfrak{a}$ on M^{fin} induces a δ-action on each finite dimensional space in the following inverse system

$$M^{\text{fin}}/\mathcal{U}_+M^{\text{fin}} \leftarrow M^{\text{fin}}/\mathcal{U}_+^2 M^{\text{fin}} \leftarrow M^{\text{fin}}/\mathcal{U}_+^3 M^{\text{fin}} \leftarrow \dots \tag{1.2.20}$$

One deduces from the positivity of $ad\delta$-eigenvalues on $\mathfrak{n}$ that, for each $\lambda \in \mathbb{C}$, the generalized λ-eigenspaces, $(M^{\text{fin}}/\mathcal{U}_+^j M^{\text{fin}})_\lambda$, (= all Jordan blocks with eigenvalue λ) of δ in (1.2.20) *stabilize*, i.e., the projection give isomorphisms $(M^{\text{fin}}/\mathcal{U}_+^{j+1} M^{\text{fin}})_\lambda \xrightarrow{sim} (M^{\text{fin}}/\mathcal{U}_+^j M^{\text{fin}})_\lambda$, for all j sufficiently large. Let $J(M)$ be the direct sum of all such "stable" generalized eigenspaces of δ. It is clear that

(i) $J(M)$ is an $\mathcal{U}(\mathfrak{g})$-submodule in $\widehat{J}(M)$ and $J(M)$ is dense in $\widehat{J}(M)$ in the $\mathfrak{n}$-adic topology, hence

$$M \neq 0 \;\;\Rightarrow\;\; \widehat{J}(M) \neq 0 \text{ (by Casselman Theorem) } \;\;\Rightarrow\;\; J(M) \neq 0.$$

(ii) J is exact, since taking (generalized) eigenvalues is an exact functor;

(iii) The δ-action on $J(M)$ is locally finite and each generalized eigenspace is finite-dimensional.

(iv) All the eigenvalues are bounded from below by some $\lambda = \lambda(M) \in \mathbb{C}$.

Property (iii) follows from the stabilization of eigen-spaces in the inverse system (1.2.20). To prove (iv) observe that, since the action of $\mathfrak{l} \subset \mathfrak{g}$ commutes with that of δ, it takes each generalized eigenspace of δ into itself. Hence $\mathcal{U}\mathfrak{l}$-action is locally finite by (iii). The $\mathfrak{n}^-$-action *strictly decreases* the eigenvalue of δ, hence, $\mathcal{U}\mathfrak{n}^-$-action on $J(M)$ is locally finite by (iv). It follows that $J(M) \in \mathcal{O}$, as promised. $\square$

1.2.21 REMARK An argument similar to the one used above will be used again, in chapter 4, in the construction of the second microlocalization functor Φ. $\Diamond$

1.3. Non-commutative localization and microlocalization.

Definition 1.3.1 Let A be an associative ring with unit $1 \in A$. A subset $S \subset A$ is called *multiplicative* if
(1) $1 \in S$; (2) $0 \notin S$; (3) $s_1, s_2 \in S \implies s_1 s_2 \in S$.

In the commutative situation these conditions are enough to localize A at S. In non-commutative situation we encounter the following obstacles:

(a) One can form both $s^{-1}a$ and as^{-1} and it is not clear which to choose.

(b) It is hard to say when $s_1^{-1}a_1 = s_2^{-1}a_2$.

(c) It is not clear how to multiply $s_1^{-1}a_1$ by $s_2^{-1}a_2$.

(d) We don't have a common denominator for $s_1^{-1}a_1 + s_2^{-1}a_2$.

To remove these obstacles one has to impose *Ore conditions* on S. There are two left Ore conditions and two right Ore conditions:

($\mathbf{1_{left}}$) Every left fraction can be written as a right fraction: $\forall s \in S, a \in$
A, $\exists t \in S, b \in A$ such that $at = sb$ (informally, this means: $s^{-1}a = bt^{-1}$)

($\mathbf{1_{right}}$) Every right fraction can be written as a left fraction.

($\mathbf{2_{left}}$) If $s \in S, a \in A$ and $sa = 0$ then $\exists t \in S$ such that $at = 0$.

($\mathbf{2_{right}}$) If $t \in S, a \in A$ and $at = 0$ then $\exists s \in S$ such that $sa = 0$

Below, we will usually try to escape from having to verify the second condition by requiring that: no element of S is a zero divisor.

Consider the category whose objects are ring homomorphisms $f : B \to A$ such that

(i) B is a ring with unit and $f(1) = 1$.

(ii) All the elements of $f(S)$ are invertible in B,

and whose morphisms are obvious commutative triangles.

Theorem 1.3.2 *(Ore, see* [**?**, ch.3.6]*) If Ore's conditions are satisfied then there exists the universal object $A \to S^{-1}A$ in this category, i.e. for any morphism $f : A \to B$ satisfying (i)-(ii) there is a canonical commutative diagram*

$$\begin{tikzcd} & S^{-1}A \arrow[d, dashed] \\ A \arrow[ur] \arrow[r] & B. \end{tikzcd}$$

Idea of Proof. One can easily see that the Ore conditions remove obstacles (a) - (d) mentioned above. Obstacle (a) is handled by the conditions $(1_{left}) - (1_{right})$. Obstacle (b) is removed by declaring two left fractions to be equal if they are equal to the same right fraction. Further, one defines multiplication of left fractions as follows:

$$s_1^{-1}a_1 \cdot s_2^{-1}a_2 \;=\; s_1^{-1}\,(a_1s_2^{-1})\,a_2 \;=\; s_1^{-1}\,s^{-1}b\,a_2 \;=\; (s\,s_1)^{-1}(b\,a_2)\,,$$

where we have used the Ore conditions to write : $a_1s_2^{-1} = s^{-1}b$. Similarly, the Ore conditions ensure that, given $s_1,\, s_2 \in S$, one can find t_1, t_2 such that $t_1s_1 = t_2s_2 = t$, and such that $t_1 \in S$. It follows that $t \in S$ is a common denominator for $s_1^{-1}a_1 + s_2^{-1}a_2$. □

1.3.3 Digression: Localization of categories. A similar localization techniques applies for additive categories instead of rings (note that giving an additive category $\mathcal{C}$ with one object, X, amounts to giving a ring $A = Hom_{\mathcal{C}}(X, X)$; thus rings are just categories with one object).

Let $\mathcal{C}$ be an additive category and Φ a family of morphisms in $\mathcal{C}$. Assume that Φ is closed under composition and contains the identity maps for all objects. Motivated by the ring case, we say that the category $\mathcal{C}_\Phi$ is a *localization of* $\mathcal{C}$ *with respect to* Φ if a functor $\alpha_\Phi : \mathcal{C} \to \mathcal{C}_\Phi$ is given, satisfying the following universal property:

For any category D and a functor $F : \mathcal{C} \to D$ such that $F(\phi)$ is an isomorphism in D for every $\phi \in \Phi$, there exists a unique functor $F_\Phi : \mathcal{C}_\Phi \to D$ such that F is naturally isomorphic to $F_\Phi \circ \alpha_\Phi$.

We claim that if the multiplicative family Φ satisfies obvious analogues of Ore conditions, then the localized category, $\mathcal{C}_\Phi$, exists. Specifically, given $(\mathcal{C}, \Phi)$ we construct $\mathcal{C}_\Phi$ as follows. Put $Ob(\mathcal{C}_\Phi) = Ob(\mathcal{C})$. Define an element of $Hom_{\mathcal{C}_\Phi}(X, Z)$ to be a diagram of morphisms $X \xrightarrow{\alpha} Y \xleftarrow{\phi} Z$, where $\phi \in \Phi$. It is easy to see as in the proof of Theorem 1.3 that the Ore conditions for Φ ensure the possibility of composing morphisms thus defined.

The most important example of such a situation is the construction of the derived category of an abelian category $\mathcal{A}$. Given such an $\mathcal{A}$, let $\mathcal{C} = C(\mathcal{A})$ be the category of complexes of objects of $\mathcal{A}$. Recall that a morphism of complexes is called a *quasi-isomorphism* provided it induces isomorphisms on the cohomology. We would like to declare all quasi-isomorphisms to be invertible, i.e. we would like to localize the category of complexes with respect to the family, Φ, of all quasi-isomorphisms. This family *does not* satisfy the Ore conditions, however. To fix the situation, one has first to pass from $C(\mathcal{A})$ to the homotopy category.

In more detail. Let $X^\bullet \in C(\mathcal{A})$ be a complex. Then $Cone(X^\bullet)$ is a complex such that $Cone(X^\bullet)^i = X^i \oplus X^{i+1}$ with the differential given by the differential of X. One has the short exact sequence of complexes:

$$0 \to X^\bullet \to Cone(X^\bullet) \to X^\bullet[1] \to 0.$$

Definition 1.3.4 A morphism of complexes $\phi : X^\bullet \to Y^\bullet$ is said to be *homotopic to 0* if it factors as $X^\bullet \to Cone(X^\bullet) \to Y^\bullet$. In other words, one should be able to define morphisms $h^i : X^{i+1} \to Y^i$ such that $\phi_i = d_Y h^i \pm h^i d_X$. We write that $\phi \overset{hot}{\sim} 0$.

Definition 1.3.5 The homotopy category $K(\mathcal{A})$ is defined by

$$Ob\, K(\mathcal{A}) = Ob\, C(\mathcal{A}),$$

$$Hom_{K(\mathcal{A})}(X^\bullet, Y^\bullet) = Hom_{C(\mathcal{A})}(X^\bullet, Y^\bullet)/\{morphisms\, homotopic\, to\, 0\}.$$

Proposition 1.3.6 *The class of quasi-isomoprhisms staisfies the Ore conditions in $K(\mathcal{A})$*

Proof. First note that any map of complexes $C \to D$ is homotopic to the embedding $C \to C \oplus Cone(C)$. Now suppose we have a diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "qis"', "\phi"'] & B \\ C & \end{tikzcd}$$

where a is an embedding and ϕ is a quasi-isomorphism, then we can apply the pushout construction to obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "a"] \arrow[d, "\phi"] & B \arrow[r] \arrow[d, "\psi"] & L \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C \arrow[r, "b"] & D \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

where ψ is also a quasi-isomorphism (by 5-lemma), and this is exactly what the Ore conditions require us to prove. □

Definition 1.3.7 The derived category $D(\mathcal{A})$ is defined to be the localization of the homotopy category $K(\mathcal{A})$ by the class Φ of all quasi-isomorphisms.

* * *

We will now give two criteria to verify Ore conditions for a multiplicative subset S of a ring A. First, given $x \in A$ write $adx(y) = xy - yx$.

Proposition 1.3.8 *If for any $s \in S$ the operator ad s on A is (locally) nilpotent then the Ore conditions hold for S.*

Proof. Let a, s be as in the first Ore condition. Then, there exists $n \in \mathbb{N}$ such that

$$0 = (ad\ s)^n(a) = \sum_{r=0}^{n} (-1)^r \binom{r}{n} s^r a s^{n-r}.$$

(The last equality can be easily proved by induction on n.) Hence if take $t = s^n$, we get $0 = as^n + sb$, where $b = -\sum_{r=1}^{n} (-1)^r \binom{r}{n} s^{r-1} a s^{n-r}$. $\square$

To state the second (less trivial) criterion, assume that we have an almost commutative algebra A such that gr$A$ is finitely generated over an algebraically closed field. Start with a multiplicative subset $\bar{S} \subset \mathrm{gr}A$ which contains no zero divisors (this assumption is not imprtant but that is what we will have in applications anyway). Define

$$S = \{s \in A \mid \sigma(s) \in \bar{S}\}.$$

Proposition 1.3.9 *The subset $S \subset A$ satisfies the Ore conditions.*

Proof. We need to show that $\forall a \in A, s \in S$, there is an element $t \in S$ such that $ta \in As$. Define

$$I = \{x \in A \mid xa \in As\}$$

(apriori I could be zero). Hence, if we denote by $r(a)$ the right multiplication map $r(a) : A \to A$, $x \mapsto xa$, then I is the kernel of the composition $\phi : A \xrightarrow{r(a)} A \to A/As$. Therefore $A/I \hookrightarrow A/As$ is an injective map of A-modules and hence $\mathrm{SS}(A/I) \subset \mathrm{SS}(A/As)$. The point here is that the inclusion is *not* compatible with filtrations (since $r(a)$ is not comatible with them) but we use the fact that the characteristic variety does not depend on the choice of filtration.

With respect to the natural filtrations we have $\mathrm{gr}(A/I) = \mathrm{gr}A/\mathrm{gr}(I)$ and $\mathrm{gr}(A/As) = \mathrm{gr}A/(\mathrm{gr}A) \cdot \sigma(s)$ (the latter equality uses that $\sigma(s)$ in not a zero-divisor). It follows now from the inclusion $\mathrm{SS}(A/I) \subset \mathrm{SS}(A/As)$ established above, that $\sigma(s)$ vanishes at $\mathrm{SS}(A/I) = \{$*the zero-variety of* gr$I\}$. Hence $(\sigma(s)^n \in \mathrm{gr}(I)$, by Hilbert Nullstellensatz. But then $\sigma(s^n) \in \mathrm{gr}(I)$ and therefore $\exists t \in I$ such that $\sigma(t) = \sigma(s^n)$. Since $\bar{S}$ is multiplicatively closed $\sigma(s^n) = (\sigma(s)^n \in \bar{S}$, hence $t \in S$ by definition of S $\square$

1.3.10 Remark In contrast with the first criterion the element t provided by (1.3) is not a power of s but one rather has $t \equiv s^n$ (modulo lower terms of the filtration). $\Diamond$

Suppose that gr$A$ has no zero divisors. Then by (1.3) for any multiplicatively closed subset $\bar{S} \subset \mathrm{gr}A$ we can construct a subset $S \subset A$ satisfying Ore conditions. Hence the localization $S^{-1}A$ exists.

We introduce the filtration on $S^{-1}A$ by requiring that $s^{-1}a \in A_{j-i}$ if $deg\,\sigma(s) = i$ and $deg\,\sigma(a) = j$. This is well-defined since we assume that two left fractions are equal iff they are equal to the same right fraction and $s^{-1}a = bt^{-1}$ implies $at = sb$ hence $\sigma(a)\sigma(t) = \sigma(s)\sigma(b)$. However this filtration on $S^{-1}A$ is in fact a $\mathbb{Z}$-filtration with non-trivial terms in all negative degrees.

Definition 1.3.11 The completion $A_{\bar{S}}$ of $S^{-1}A$ in the topology defined by the filtration above is called the *formal microlocalization* of A at S. We define the formal microlocalization of an A-module M to be $M_{\bar{S}} = A_{\bar{S}} \otimes_A M$ viewed as a left $A_{\bar{S}}$-module.

Definition 1.3.12 Let $f : M \to N$ be a morphism of filtered A-modules. We say that f is *compatible with filtrations* if $f(M_i) \subset N_i$. We say that f is *strictly compatible* if $f(M_i) = f(M) \cap N_i$ (i.e. the two possible filtrations on $f(M)$ arising from viewing $f(M)$ either as the surjective image of M or as a subobject of N, in fact coincide).

1.3.13 Properties of the formal microlocalization.

(1) The natural embedding $A \hookrightarrow A_{\bar{S}}$ is sitrictly compatible with filtrations.

(2) $A_{\bar{S}}$ is flat over A (since the localization and completion are both exact functors).

(3) $\mathrm{gr}(A_{\bar{S}}) = (\bar{S})^{-1}\mathrm{gr}A$ (this is true since taking associated graded "does not feel" the completion).

To formulate the fourth property, recall first that the scheme $Spec(\mathrm{gr}A)$ has a natural $\mathbf{G_m}$-action, and is *cone-scheme* over $Spec(A_0)$. Consider *Zariski cone-topology* on $Spec(\mathrm{gr}A)$, i.e. the topology generated by open *cone*-subsets $U \subset Spec(\mathrm{gr}A)$. For each open cone-subset U we denote by $\bar{S}(U)$ the set of all elements of gr$A$ that are invertible on U. This is clearly a multiplicative subset of gr$A$, and abusing the notation we write A_U (instead of $A_{\bar{S}(U)}$) for the microlocalization of A with respect to $\bar{S}(U)$.

Given a finitely generated A-module M, and an open cone-subset U as above, put $M_U := A_{\bar{S}(U)} \otimes_A M$, an A_U -module. Then we have

Proposition 1.3.14 *(i) The assignment $U \mapsto A_U$ defines a sheaf of algebras on the cone-scheme $Spec(\mathrm{gr}A)$.*

(ii) For any finitely generated A-module M, the assignment $U \mapsto M_U$ gives a sheaf of modules. $\square$

1.3.15 REMARK Notice that in the case when $Spec(A_0)$ is a point (= the vertex of the cone $Spec(\mathrm{gr}A)$), we have $A_U = A$ if U contains $Spec(A_0)$. One can invert much more elements on A if U does not contain $Spec(A_0)$. $\diamondsuit$

Digression: non-commutative determinants.

Recall that to any $(n \times n)$-matrix $\|a_{ij}\| \in M_n(A)$ with entries in a commutative ring A one can associate its determinant, $\det \|a_{ij}\| \in A$, given by the standard alternating sum over all permutations $s \in S_n$:

$$\det \|a_{ij}\| \;=\; \sum_{s \in S_n} (-1)^s \cdot a_{1\,,s(1)} \cdot a_{2\,,s(2)} \cdot \ldots \cdot a_{n\,,s(n)}\ . \tag{1.3.16}$$

This formula does not make sense, however, for a noncommutative ring since, in the non-commutative case, the order of factors in (1.3.16) becomes essential.

Now, let A be an almost commutative filtered ring. We will see that, under certain mild conditions, to any A-valued $(n \times n)$-matrix $P \in M_n(A)$, one can associate in a canonical way an element $Det(P) \in \mathrm{gr}A$ satisfying most of the expected properties of a determinant.

Theorem 1.3.17 *Assume that $\mathrm{gr}A$ is a unique factorization domain. Then, there is a natural map $Det : M_n(A) \to \mathrm{gr}A$ which satisfies the following properties:*

(i) $Det(P) \cdot Det(Q) = Det(P \cdot Q) \quad , \quad \forall P, Q \in M_n(A)$,

(ii) $Det \begin{pmatrix} P & Q \\ 0 & R \end{pmatrix} = Det(P) \cdot Det(R)$,

(iii) P is invertible iff $Det(P)$ is invertible,

(iv) Let $P = \|a_{ij}\|$. Assume that $\exists l \in \mathbb{Z}$ such that, for any permutation $s \in S_n$, $a_{1\,,s(1)} \cdot a_{2\,,s(2)} \cdot \ldots \cdot a_{n\,,s(n)} \in A_l$, and the following holds:

$$\sigma(a_{1\,,s(1)}) \cdot \sigma(a_{2\,,s(2)}) \cdot \ldots \cdot \sigma(a_{n\,,s(n)}) \neq 0 \quad \textit{and} \quad \det \|\sigma(a_{ij})\| \neq 0\,.$$

Then we have $Det(P) = det\|\sigma(a_{ij})\|$.

Proof. Put $\bar{S} := \mathrm{gr}A \setminus \{0\}$ and let $S \subset A$ be as in (1.3). Then the localization $K = S^{-1}A$ (without completion) is a skew-field. The filtration on A induces an increasing $\mathbb{Z}$-filtration on K, moreover, $\mathrm{gr}(K)$ is just the field of fractions of $\mathrm{gr}A$.

Given a matrix P over any skew-field, K, one can associate to it its *Dieudonne determinant*, $det(P) \in K^\times/[K^\times, K^\times]$, where $K^\times = K \setminus \{0\}$ denotes the multiplicative group of K, $[K^\times, K^\times]$ denotes the derived group, the multiplicative subgroup of $K^\times$ generated by the elements $ghg^{-1}h^{-1}$. Recall the construction of $det(P)$. First, one puts P into uppertriangular form by elementary transformations in $M_n(K)$. If one of the elements on the diagonal is zero, we set $det(P) = 0$. Otherwise we multiply the elements on the diagonal and denote the product by $det(P)$. However, since the diagonal entries do not commute in general, and there is no natural order in which we should multiply these elements, $det(P)$ is only well-defined as an element of $K^\times/[K^\times, K^\times]$.

Next, fixing some order of the diagonal elements defines a lift $\widetilde{det(P)}$ of $det(P)$ to $K^\times$. Consider the symbol $\sigma(\widetilde{det(P)}) \in \mathrm{gr}(K)^\times$. Any other lift of $det(P)$ to $K^\times$ will differ by a product of elements of the type $ghg^{-1}h^{-1}$. Since $\mathrm{gr}(K)$ is almost commutative, this other lift will have the same symbol in $\mathrm{gr}(K)$. Thus, we have associated to $P \in M_n(A)$ a well-defined element $Det(P) := \sigma(\widetilde{det(P)}) \in \mathrm{gr}K$.

All the properties of the theorem hold for $Det(\bullet)$: they follow readily from the corresponding properties of commutative determinants and multiplicativity of the symbol map. The only thing that remains is the following claim:

Lemma 1.3.18 *For any $P \in M_n(A)$ we have* $\sigma(\widetilde{det(P)}) \in \mathrm{gr}A$.

Sketch of proof. *Step 1.* Assume that $\mathrm{gr}A$ is a discrete valuation ring. In this case we can easily say when an element is integral, therefore to verify directly that $\sigma(\widetilde{det(P)})$ is integral.

Step 2. Take any prime element $p \in \mathrm{gr}A$ and localize at p. One can show that $\sigma(\widetilde{det(P)}) \in \mathrm{gr}A_{(p)}$ and the rest of the proof follows by the fact that $\mathrm{gr}A = \bigcap_p \mathrm{gr}A_{(p)}$. $\square$

1.4. Sato-Kashiwara filtration and Duality.

Let A be a positively filtered almost commutative algebra over a field k of characteristic zero. Put $B := \mathrm{gr}A$. Then $B = \oplus B_i$ is a graded k-algebra with augmentation ideal $B_+ = \oplus_{i>0} B_i$ and $A_0 = B_0 \simeq B/B_+$.

Assume further that $B = k[X]$ is the coordinate ring of a *smooth* affine algebraic variety. The grading on B gives an algebraic $\mathbf{G_m}$-action on X. The projection $B \to B/B_+ = B_0$ induces an imbedding

$i : X_0 = Spec(B_0) \to X = SpecB$, and X_0 is just the fixed point set of the $\mathbf{G_m}$-action. By assumption, X is smooth. Hence, X_0 is also smooth.

Consider the normal bundle $T_{X_0}X$ of X_0 in X. Since X_0 is the fixed point set, the group $\mathbf{G_m}$ acts along the fibers of $T_{X_0}X$. It follows that $T_{X_0}X$, viewed as the locally free sheaf on X_0, splits into a finite direct sum of weight components, each of them a locally free sheaf on X_0.

To interpret the normal bundle algebraically, recall that, for any graded B-module N the quotient N/B_+N is naturally a B_0-module and if N is finitely generated over B then N/B_+N is finitely generated over B_0. We see that the weight decomposition of $T_{X_0}X$ corresponds to the decomposition of the B_0-module B_+/B_+^2 into its graded components.

Lemma 1.4.1 *Assume that B_+/B_+^2 is free over B_0 and that N/B_+N is free and finitely generated over B_0. Then if N is projective graded (i.e. a graded direct summand of a free graded B-module) then N is free over B.*

Proof. Note that, for any graded B-module N, any set of B_0-generators in N/B_+N lifts (non-uniquely) to a set of B-generators in N. By our assumption $N \oplus N' \simeq B^{\oplus k}$. Then $N/B_+N \oplus N'/B_+N' \simeq B_0^{\oplus k}$. Choose a base of N/B_+N over B_0 and lift it to a set of generators of N over B. There can be no relations between these generators as can be seen by passing to the field of fractions of B_0. □

Lemma 1.4.2 *Let M be a filtered $A-$module such that* $\mathrm{gr}M$ *has a finite presentation*

$$0 \leftarrow \mathrm{gr}M \leftarrow B^{\oplus k} \leftarrow B^{\oplus l}.$$

If $\mathrm{gr}M/B_+\mathrm{gr}M$ *is free over B_0 then one can lift the presentation above to the non-filtered level*

$$0 \leftarrow M \leftarrow A^{\oplus k} \leftarrow A^{\oplus l}$$

so that all maps are strictly compatible with filtrations.

Sketch of proof. The first arrow $\mathrm{gr}M \leftarrow B^{\oplus k}$ of the presentation gives k generators $m_1, \ldots, m_k$ of $\mathrm{gr}M$ (of degrees $d_1, \ldots, d_k$ respectively) which we lift to elements $m_1', \ldots, m_k'$ of M. Consider the exact sequence

$$0 \leftarrow M \leftarrow A^{\oplus k} \leftarrow Ker \leftarrow 0$$

Notice that $A^{\oplus k}$ has a shifted filtration:

$$\left(A^{\oplus k}\right)_j = \bigoplus_{i=1}^{k} A_{j-d_i}.$$

First of all we need to show that $\left(A^{\oplus k}\right)_j$ maps surjectively onto M_j (that will imply that the first arrow is surjective and strictly compatible with filtrations). To that end, we induct on j. Take an element m of M_j and write $\sigma_j(m)$ as a linear combination of m_i's. If we try to lift this linear combination to M, it will not in general be equal to m, but will differ from m by an element of M_{j-1}. By inductive assumption any element of M_{j-1} is an image of some element in $\left(A^{\oplus k}\right)_{j-1}$. Hence, any element of M_j is an image of some element of $\left(A^{\oplus k}\right)_j$.

The arrow $B^{\oplus k} \leftarrow B^{\oplus l}$ gives us l relations between $m_1, \ldots, m_k$. We lift these relations to non-graded level as follows. Let $b_1 m_1 + \ldots b_k m_k = 0$ be one such relation (and $deg(b_1) + d_1 = deg(b_2) + d_2 = \ldots = deg(b_k) d_k =: d$). Suppose that $b_i = \sigma(a_i)$. Then $a_1 m_1' + \ldots a_k m_k$ is not necessarily zero but rather an element of M_{d-1}. By the strong compatibility of the first arrow we have $a_1 m_1' + \ldots a_k m_k = a_1'' m_1' + \ldots a_k'' m_k$ where $a_i'' \in M_{d-d_i-1}$. But then $\sigma(a_i - a_i'') = \sigma(a_i) = b_i$ hence $(a_1 - a_1'') m_1' + \ldots + (a_k - a_k'') m_k' = 0$ is the lift of the relation form gr$M$ to M.

The l lifts of relations to M define a map $A^{\oplus l} \to Ker$. We need to show that this map is surjective and strictly compatible with filtrations. The proof of this repeats the proof of correspoding facts for the lift of the first arrow of the resolution. $\Box$

Let A be a positively filtered almost commutative ring. Notice that for any $a_0 \in A_0$ the operator ad $a_0 : A \to A$ is nilpotent, since it sends A_i to A_{i-1} for all $i = 0, 1, \ldots$. Therefore, any multiplicative subset S of A_0 satifies the Ore conditions (cf. 1.3) and we can localize A with respect to S. The localization is positively filtered (hence, in particular, we do not have to take the completion).

Corollary 1.4.3 *Given a graded projective resolution of* gr$M$

$$\ldots \to N_2 \to N_1 \to N_0 \to \mathrm{gr}M \to 0$$

one can (localizing with respect to A_0 if necessary) lift this resolution to a free resolution of M itself

$$\ldots \to A^{k_2} \to A^{k_1} \to A^{k_0} \to M \to 0\,,$$

where all morphisms are strictly compatible with filtrations.

Proof. We can shrink $Spec(A_0)$ to make the resolution free (since both $Spec(A_0)$ and $Spec(\mathrm{gr}A)$ are smooth). Then we apply the lemma above step by step to lift the resolution. $\Box$

Corollary 1.4.4 *If $X = Spec(\mathrm{gr}A)$ is smooth, then the algebra A has the same homological dimension as* gr$A$, *that is $d_A = \dim X$.* $\Box$

Proposition 1.4.5 *Let M be finitely generated A-module. Then*
(i) codim $\mathrm{SS}(Ext^j_A(M,A)) \geq j$, $\forall j$.
(ii) $Ext^j_A(M,A) = 0$ *unless codim* $(\mathrm{SS}M) \leq j \leq \dim X$.

Proof. If we replace A by $\mathrm{gr}A$, M by $\mathrm{gr}M$, and "SS" by "*Supp*", the corresponding statement in commutative algebra is well-known, see e.g. [**Eis**].

Now, given M, we first choose a free graded resolution of $\mathrm{gr}M$, and then use Lemma 1.4 to lift it to a free filtered resolution $\ldots \to F_1 \to F_0 \to M$, with F_i finitely generated. Then $Ext_A(M,A)$ can be computed using the complex

$$0 \to Hom_A(F_0, A) \to Hom_A(F_1, A) \to \ldots \tag{1.4.6}$$

of free right A-modules. Observe that the *Hom*-group between any two filtered objects, X, Y, has a natural filtration: a morphism $\phi : X \to Y$ is contained in the lth term of the filtration, if $(X_i) \subset Y_{i+l}$, $\forall i$. Thus, all the groups $Hom_A(F_j, A)$ are filtered. Furthermore, it is easy to verify that $\mathrm{gr}Hom_A(F_j, A) = Hom_{\mathrm{gr}A}(\mathrm{gr}F_j, \mathrm{gr}A)$. Hence the spectral sequence of the filtered complex (1.4.6) implies that $\mathrm{gr}Ext^j_A(M,A)$ is a subquotient of $Ext^j_{\mathrm{gr}\,A}(\mathrm{gr}M, \mathrm{gr}A)$. Since passing to a subquotient does not increase the support, the proposition follows. $\square$

Duality. Given a left A-module M define its naive dual to be $Hom_A(M,A)$. The multiplication on A on the right induces a right A-module structure on $Hom_A(M,A)$. Similarly, given a right A-module M define its naive dual to be $Hom_A(M,A)$, a left A-module. The main defect of this naive construction is that the module $Hom_A(M,A)$ is often zero (e.g. when $M = A/Aa$ for some element $a \in A$).

The standard way to go around this difficulty is to use the language of derived categories. Let $D^b(A\text{-}mod)$ be the derived category of bounded complexes of *projective* left A-modules, and write $D^b(mod\text{-}A)$ for a similar derived category of right A-modules. Then, the naive duality above, applied term by term to a bounded complex $\ldots \to P_i \to P_{i+1} \to \ldots$ of projective left A-modules gives a bounded complex

$$\ldots \to Hom_A(P_{i+1}, A) \to Hom_A(P_i, A) \to \ldots$$

of projective right A-modules. This way one gets duality functors

$$\mathbf{D} : \; D^b(mod\text{-}A) \;\rightleftarrows\; D^b(mod\text{-}A)\,. \tag{1.4.7}$$

We say that the ring A has finite homological dimension d_A, provided any finitely-generated left A-module M has a projective resolution of length $\leq d_A$. This is equivalent to the requirement that

$Ext^i_A(M,N) = 0$ for any A-modules M, N, and all $i > d_A$. In such a case any finitely-generated left A-module M gives rise to a well-defined object of $D^b(A\text{-}mod)$ represented by its projective resolution. Thus, applying the above duality one gets a well-defined object of $\mathbf{D}M \in D^b(mod\text{-}A)$.

1.4.8 EXAMPLE

(i) We have $\mathbf{D}A = A$.

(ii) If $M = A/Aa$ then we have a free resolution of M:

$$0 \to A \xrightarrow{\cdot a} A \to M \to 0$$

which we can use to compute $RHom$. Since the dual of the complex $0 \to A \xrightarrow{\cdot a} A \to 0$ is $0 \to A \xrightarrow{a\cdot} A \to 0$ we have $\mathbf{D}(A/Aa) = A/(aA)[1]$ where $[1]$ stands for the shift of complexes in the derived category.

1.5. Sato-Kashiwara filtration.

Throughout this subsection A stands for a filtered algebra over a field of characteristic zero such that $Spec\,\mathrm{gr}A$ is a smooth algebraic variety. In particular, by Corollary 1.4, A has finite homological dimension $d = \dim Spec\,\mathrm{gr}A$.

Lemma 1.5.1 *There is a functorial quasi-isomorphism in $D^b(A-$ -mod): $\mathbf{DD}(M) \overset{qis}{\simeq} M$.*

Proof. The assertion is true for a free module, hence for a projective module M. The proof is now completed by induction on the length of finite projective resolution. $\square$

Definition 1.5.2 Consider a complex of abelian groups

$$K_\bullet = \{\ldots \to K_{i-1} \xrightarrow{d_{i-1}} K_i \xrightarrow{d_i} K_{i+1} \to \ldots\}.$$

The *i-th truncation*, $\tau_{i\leq}K_\bullet$, is defined as

$$\tau_{i\leq}K_\bullet = \{\ldots \to 0 \to Im(d_{i-1}) \hookrightarrow K_i \to K_{i+1} \to \ldots\} \quad .$$

We have a natural morphism of complexes $K_\bullet \to \tau_{i\leq}K_\bullet$, given by

$$\begin{tikzcd}
\cdots \arrow[r] & K_{i-2} \arrow[r] \arrow[d] & K_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "d_{i-1}"] & K_i \arrow[r, "d_i"] \arrow[d, equal] & K_{i+1} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & Im(d_{i-1}) \arrow[r, hook] & K_i \arrow[r, "d_i"] & K_{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Also, it follows from the definition that

$$H^j(\tau_{i\leq}K_\bullet) = \begin{cases} 0 & \text{for} \quad j < i, \\ H^j(K_\bullet) & \text{for} \quad j \leq i. \end{cases}$$

This cohomological property implies that the truncation functor gives rise to a well defined functor on the derived category.

For a finitely generated A-module M, and any i, we have the object $\tau_{i\leq}(\mathbf{D}M) \in D^b(A\text{-}mod)$, and the canonical morphism $\mathbf{D}M \to \tau_{i\leq}(\mathbf{D}M)$. Applying the functor $\mathbf{D}$ to this morphism we get a canonical morphism $\mathbf{D}(\tau_{i\leq}(\mathbf{D}M)) \to \mathbf{DD}(M) \simeq M$

Definition 1.5.3 Given an A-module M, and any $i \in \mathbb{Z}$, write $S_j(M)$ for the image of the induced map on zero cohomology

$$H^0(\mathbf{D}(\tau_{i\leq}(\mathbf{D}M))) \to H^0(\mathbf{D}M) = M\,.$$

The $\{S_jM\}_{j=0,1,\dots}$ form an increasing filtration of M by A-submodules (it is clear that $S_jM = 0$ if $j < 0$). This filtration is called *Sato-Kashiwara filtration.*

1.5.4 REMARK Since M has a finite projective resolution, the Sato-Kashiwara filtration is finite too. In fact, $S_j(M) = M$ if $j > d_A$ since the corresponding truncation complex vanishes. $\Diamond$

Recall that earlier we have defined the Gabber filtration $G_\bullet M$ of M by

$$G_iM = \{\textit{largest } A\textit{-submodule } \ M' \subset M \textit{ such that } \dim \mathrm{SS}(M') \leq i\}.$$

Gabber filtration is relatively easy to define but hard to handle while Sato-Kashiwara filtration, due to its abstract definition, has good functorial properties. Fortunately, the following theorem says that they coincide anyway.

Theorem 1.5.5 *The Gabber filtration is equal to the Sato-Kashiwara filtration, i.e.*

$$G_i(M) = S_i(M) \quad , \quad \forall i = 0, 1, \dots\,.$$

Before we prove (1.5) we need two lemmas.

Lemma 1.5.6 *Sato-Kashiwara filtration is functorial, i.e. for any morphism $f : M_1 \to M_2$ of A-modules one has*

$$f(S_jM_1) = S_jM_2.$$

Proof. Everything is functorial. $\square$

For an A-module M denote by $d(M)$ the dimension of $\mathrm{SS}(M)$.

Lemma 1.5.7 (i) $d(S_k(M)) \leq k$,
(ii) $S_{d(M)}(M) = M$.

Proof. To prove the lemma we reformulate (i) and (ii) above in the following way:

$$\begin{array}{ll} (i') & d(S_i(M)/S_{i-1}(M)) \leq k, \quad \forall i \leq k, \\ (ii') & S_i(M)/S_{i-1}(M) = 0 \quad \forall i > d(M). \end{array}$$

We now prove (i') and (ii') by using the spectral sequence for the composition $\mathrm{Id}_{D^b(A\text{-}mod)} = \mathbf{D} \circ \mathbf{D}$ of the duality functors, applied twice. The spectral sequence reads:

$$E_2^{i,j} = Ext_A^j(Ext_A^{d(M)-i}(M, A), A) \Rightarrow E_\infty^{i,j} = \begin{cases} S_i(M)/S_{i-1}(M) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

This spectral sequence arises as follows. Choose a finite projective resolution of M

$$0 \to M_{-d} \to \ldots \to M_{-2} \to M_{-1} \to M_0 \to M \to 0\,,$$

so that $\mathbf{D}M$ can be computed using the complex $\{Hom_A(M_i, A)\}_{-d \leq i \leq 0}$. To apply the duality functor again, we have to replace this complex by a quasi-isomorphic complex of projectives. It is more convenient however to use a different approach and to choose instead a (possibly infinite) resolution of the $(A\text{-}A)$-bimodule A by graded injective bimodules:

$$0 \to A \to I_0 \to I_1 \to \ldots\,.$$

Then $\mathbf{DD}M$ can be computed using the simple complex associated with the double complex $K^{i,j} = Hom_A(\mathbf{D}M_i, I_j)$. Note that since M_i is projective, $\mathbf{D}M_i$ is projective. But for a free module, hence for the projectives M_i, one has canonical isomorphisms:

$$K^{i,j} = Hom_A(\mathbf{D}M_i, I_j) \simeq Hom_A(Hom_A(M_i, A),\, I_j) \simeq I_j \otimes_A M_i\,.$$

Thus, the spectral sequence above becomes the standard spectral sequence of the following double complex:

$$\begin{tikzcd}
0 \arrow[r] & I_0 \otimes_A M_{-d} \arrow[r] \arrow[d] & I_0 \otimes_A M_{-d+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & I_0 \otimes_A M_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_1 \otimes_A M_{-d} \arrow[r] \arrow[d] & I_1 \otimes_A M_{-d+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & I_1 \otimes_A M_0 \arrow[r] \arrow[d] & 0 \\
 & \cdots \arrow[d] & \cdots \arrow[d] & & \cdots \arrow[d] & \\
0 \arrow[r] & I_d \otimes_A M_{-d} \arrow[r] \arrow[d] & I_d \otimes_A M_{-d+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & I_d \otimes_A M_0 \arrow[r] \arrow[d] & 0 \\
 & {} & {} & & {} &
\end{tikzcd}$$

Now we can apply (1.4) for the *Ext*-groups in this spectral sequence (twice). The assertion follows. □

Proof of (1.5). By definition of Gabber filtration and (1.5(i)) we have $S_k(M) \subset G_k(M)$. To show the opposite inclusion, consider the map $G_k(M) \hookrightarrow M$. By functoriality of the Sato-Kashiwara filtration (1.5) we get

$$S_i(G_kM) \subset S_i(M).$$

Take $i = k$. Since $d(G_k(M)) = k$ we can appply (1.5(ii)) to get $G_k(M) = S_k(G_k(M)) \hookrightarrow S_k(M)$. □

Lemma 1.5.8 *Taking Gabber filtration commutes with microlocalization*

Proof. Formal microlocalization is exact and hence commutes with truncation functors. □

Proof of Gabber's Equi-dimensionality Theorem. Recall that we need to prove that $G_iM/G_{i-1}M$ is of pure dimension i. Suppose that $\mathrm{SS}(G_iM/G_{i-1}M)$ is not of pure dimension i. Then it has a component Y of dimension $\dim Y \leq i-1$. Choose an open affine subset $U \subset X$ such that

$$U \cap \mathrm{SS}(G_iM) \subset Y \setminus Y^{sing},$$

Taking microlocalization at U we clearly have

$$(M/G_{i-1}(M))_U \neq 0, \qquad (1.5.9)$$

since $\mathrm{SS}((G_i(M)/G_{i-1}(M))_U)$ is a subset of $\mathrm{SS}(M/G_{i-1}(M))_U$ containing $Y \setminus Y^{sing}$.

On the other hand $d(M_U) \leq i - 1$ since by localizing at U we excluded the components of characteristic variety of dimension $\geq i$. Therefore $M_U = G_{i-1}(M_U) = (G_{i-1}(M))_U$. Hence localizing the short exact sequence

$$0 \to G_{i-1}(M) \to M \to M/G_{i-1}(M) \to 0$$

at U we obtain $(M/G_{i-1}(M))_U = 0$ which contradicts (1.5.9). $\square$

2. Algebraic differential operators on a manifold.

2.1. Sheaf of algebraic differential operators.

2.1.1 Quasi-coherent sheaves.

We will assume that all varieties are defined over the field of complex numbers $\mathbb{C}$.

Let X be an algebraic variety and U be its affine open subset. For any regular function $f \in \mathcal{O}(U)$ we denote by U_f the open subset $U \setminus f^{-1}(0)$. Given an $\mathcal{O}(U)$-module M, set $M_f = M[f^{-1}] = \mathcal{O}(U_f) \otimes_{\mathcal{O}(U)} M$.

Lemma 2.1.2 The following conditions on a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules are equivalent:

(i) $\mathcal{F}$ is a direct limits of its coherent subsheaves.

(ii) For any Zariski open affine subset $U \subset X$ and any $f \in \mathcal{O}(U)$ one has $\Gamma(U_f, \mathcal{F}) = \Gamma(U, \mathcal{F})_f$. $\square$

Definition 2.1.3 A sheaf of $\mathcal{O}_X$-modules satisfying either of the two equivalent conditions above is called *quasi-coherent*.

Now we present two different approaches to differential operators on X.

2.1.4 Coordinate approach. We define a notion of "local coordinates" on X as follows. Assume that X is a smooth affine variety (all the constructions will be local in Zariski topology). Fix a point $x \in X$. By Noether Normalization Lemma we can find an open neighbourhood $U \subset X$ of x and a finite unramified morphism $\phi : U \to V$ of U to an open affine neghbourhood V of $\{0\} \in \mathbb{C}^n$ (where $n = \dim X$) such that $\phi(x) = 0$. In fact, and n regular functions $\phi_1, \ldots, \phi_n$ with linearly independent differentials $d\phi_1, \ldots, d\phi_n$ in U will define such a map ϕ. While it is not true that ϕ gives local coordinates on U, it still induces an isomorphism of tangent spaces. Hence any vector field on V can be lifted in a unique way to U. We can see that algebraically it we recall that $B = \Gamma(U, \mathcal{O}_X)$ is a finite extension of $A = \Gamma(V, \mathcal{O}_{\mathbb{C}^n})$. Hence any element $b \in B$ satisfies some monic equation $b^k + a_1 b^{k-1} + \ldots + a_k = 0$ with $a_1, \ldots, a_k \in A$. If ∂ is a derivation of A then, differentiating the equation above, we obtain

$$k\, b^{k-1}\, \partial(b) + \big((k-1)a_1\, b^{k-2}\, \partial(b) + b^{k-1}\partial(a_1)\big) + \ldots + \partial(a_k) = 0$$

and this defines $\partial(b)$ uniquely.

Denote by $x_1, \ldots, x_n$ the standard coordinates on $\mathbb{C}^n$ and by ∂_i, $i = 1, \ldots, n$, the lift of the vector field $\partial/\partial x_i$ from V to U. Then the vector fields ∂_i on U commute with each other and $\Gamma(U, T_X) = \mathcal{O}(U) \cdot \partial_1 + \ldots + \mathcal{O}(U) \cdot \partial_n$ since $\partial_1, \ldots, \partial_n$ generate the tangent space at each point of U. We call the pullbacks of $x_1, \dots, x_n$ to U an *algebraic "local coordinate"* system. It must be kept in mind that these "local coordinates" do not separate points in the same fiber of $\phi : U \to V$.

Definition 2.1.5 We define the sheaf $\mathcal{D}_X$ of differential operators on X as the subsheaf $End_{\mathbb{C}}(\mathcal{O}_X)$ of all *complex-linear* endomorphisms of $\mathcal{O}_X$ generated locally at each point by $\mathcal{O}_X$ and T_X.

Lemma 2.1.6 In algebraic "local coordinate" system we have

$$\Gamma(U, \mathcal{D}_X) = \mathcal{O}(U)[\partial_1, \dots, \partial_n]$$

where the RHS stands for the polynomial algebra over $\mathcal{O}(U)$ generated by commuting vector fields $\partial_1, \dots, \partial_n$. *Proof.* An exercise left to reader (one checks that any section u of $\mathcal{D}_X$ can be uniquely expressed in terms of $\partial_1, \dots, \partial_n$ by applying u to pullbacks of functions from V). $\square$

In particular $\mathcal{D}_X$ is a quasi-coherent sheaf.

Corollary 2.1.7

(i) The filtration $\{\mathcal{D}_X^i\}_{i \in \mathbb{Z}_+}$ of $\mathcal{D}_X$ by the order of the operator is preserved under coordinate change, thus making $\mathcal{D}_X$ into a filtered sheaf of rings.

(ii) For any $i \in \mathbb{Z}_+$ the subsheaf $\mathcal{D}_X^i$ of differential operators of order $\leq i$ is coherent.

(iii) The associated graded sheaf gr $\mathcal{D}_X$ is isomorphic to the symmetric algebra $S\mathcal{T}_X$ of the tangent bundle T_X. In particular, $\mathcal{D}_X$ is almost commutattive. $\square$

Corollary 2.1.8 (Poincaré-Birkhoff-Witt for differential operators) Let U be any affine open subset of X such that the module of global sections $\Gamma(U, T_X)$ is free over $\Gamma(U, \mathcal{O}_X)$ with generators $v_1, \dots, v_n$. Then any differential operator on U of order $\leq k$ can be written uniquely in the form

$$\sum_{k_1 + \dots k_n \leq k} f_{k_1 \dots k_n} v_1^{k_1} \dots v_n^{k_n}.$$

Proof. The vector fields $v_1, \dots, v_n$ may no longer commute. From commutation relations one can deduce the existence of the representation above, while uniqueness can be proved by covering U with open subsets on which "local coordinates" exist. In fact, it suffices to prove that

$$\sum_{k_1 + \dots k_n \leq k} f_{k_1 \dots k_n} v_1^{k_1} \dots v_n^{k_n} \neq 0 \tag{2.1.9}$$

if at least one of the functions $f_{k_1 \dots k_n}$ is not zero. Without loss of generality we can assume that such function satisfies $k_1 + \dots k_n = k$. By shrinking U if necessary, we can find functions $\phi_1, \dots, \phi_n$ with linearly independent differentials that we deonte by $\partial_1, \dots, \partial_n$. Now rewrite (2.1.9) in terms of ∂_i instead of v_i and apply (2.1). $\square$

Since $S\mathcal{T}_X$ is an associated graded of $\mathcal{D}_X$, it carries a natural Poisson bracket $\{\cdot,\cdot\}_{\mathrm{gr}\,\mathcal{D}_X}$. One also has a Poisson structure on $S\mathcal{T}_X$ on it arising from symplectic topology as follows. Consider the cotangent bundle $\pi : T^*_X \to X$ with its natural symplectic form $\omega = \sum dp_i \wedge dq_i$. For any two functions f and g on T^*_X we have corresponding Hamiltonian vector fields ξ_f and ξ_g. Hence $\omega(\xi_f, \xi_g)$ is a function on T^*_X that we denote by $\{f, g\}_{symplect}$. Since $S\mathcal{T}_X \simeq \pi_*(\mathcal{O}_{T^*_X})$ one also has an induced Poisson bracket $\{\cdot,\cdot\}_{symplect}$ on $S\mathcal{T}_X$.

Proposition 2.1.10 The two brackets $\{\cdot,\cdot\}_{\mathrm{gr}\,\mathcal{D}_X}$ and $\{\cdot,\cdot\}_{symplect}$ on $S\mathcal{T}_X$ coincide. $\square$

2.1.11 Coordinate-free interpretation of $\mathcal{D}_X$ (Grothendieck).

Definition 2.1.12 Let A be a commutative ring. For any pair of A-modules M, N we define modules $\mathcal{D}\mathit{iff}^k_A(M,N)$ inductively by putting

(1) $\mathcal{D}\mathit{iff}^0_A(M,N) = Hom_A(M,N)$,

(2) $\mathcal{D}\mathit{iff}^{k+1}_A(M,N) = \{$additiive maps $M \to N$, s.t. $\forall a \in A$, $(au - ua) \in \mathcal{D}\mathit{iff}^k_A(M,N)\}$.

It follows from the definition that $\mathcal{D}\mathit{iff}^k_A(M,N) \subset \mathcal{D}\mathit{iff}^{k+1}_A(M,N)$. We put

$$\mathcal{D}\mathit{iff}_A(M,N) = \cup_k \mathcal{D}\mathit{iff}^k_A(M,N).$$

2.1.13 Exercise Show that if $M = N$ then $\mathcal{D}\mathit{iff}(M)$ is a filtered almost commutative ring.

2.1.14 Remarks.

(1) Note that for any $a \in A$ the operator $[\cdot, a]$ of commuting with a maps $\mathcal{D}\mathit{iff}^{k+1}(M,N)$ to $\mathcal{D}\mathit{iff}^k(M,N)$ hence it is locally nilpotent. Recall that this is precisely one of the properties that guarantees that the Ore conditions are satisfied (in the case $M = N$). This property is necessary is we want to show that $\mathcal{D}\mathit{iff}$ can be glued into a quasi-coherent sheaf of algebras (since we want to be able to localize $\mathcal{D}\mathit{iff}$. In a sense, $\mathcal{D}\mathit{iff} \subset Hom_{\mathbb{C}}(M,N)$ is the lagrest subspace on which the adjoint action of A is locally nilpotent.

(2) The abelian group $Hom_{\mathbb{C}}(M,N)$ has a natural structure of an $(A-A)$-bimodule. Since A is commutative, we can view $Hom_{\mathbb{C}}(M,N)$ as a sheaf on $Spec\,A \times Spec\,A$. Since the ideal of the diagonal $\Delta \subset$

$Spec\, A \times Spec\, A$ is generated by functions of the type $(a \otimes 1 - 1 \otimes a)$, we can say that $\mathcal{D}iff(M,N)$ is the largest submodule of $Hom_{\mathbb{C}}(M,N)$ that is supported on the diagonal.

(3) If X is not smooth it is hard to say anything about $\mathcal{D}iff(M,N)$ (for example, identify the associated graded of $\mathcal{D}iff(M,M)$). Moreover, for a singular X the sheaf of rings $\mathcal{D}iff(M,M)$ may not even be Noetherian.

Hence from now on we will assume that X is a smooth algebraic variety and $\mathcal{M}$, $\mathcal{N}$ are two coherent sheaves on X.

Definition 2.1.15
(i) We define the sheaf $\mathcal{D}(\mathcal{M},\mathcal{N})$ by gluing $\mathcal{D}iff(\mathcal{M},\mathcal{N})$ on open affine subsets, i.e. by requaring that for any affine open subset $U \subset X$

$$\Gamma(U, \mathcal{D}(\mathcal{M},\mathcal{N})) = \mathcal{D}iff_{\mathcal{O}(U)}(\mathcal{M}(U),\mathcal{N}(U)).$$

(ii) The sheaf $\mathcal{D}_X$ of differential operators is define by $\mathcal{D}_X := \mathcal{D}_X(\mathcal{O}_X, \mathcal{O}_X)$.

2.1.16 Exercise Let $\mathcal{M}$, $\mathcal{N}$ be locally free $\mathcal{O}_X$-sheaves. Show that

$$\mathcal{D}_X(\mathcal{M},\mathcal{N}) = \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{M}^*.$$

Show also that this isomorphism is compatible with filtrations and

$$\mathrm{gr}\, \mathcal{D}_X(\mathcal{M},\mathcal{N}) \simeq \mathcal{H}om(\mathcal{M},\mathcal{N}) \otimes S\mathcal{T}_X. \quad \square$$

2.2. Twisted differential operators (TDO).

Definition 2.2.1 A *sheaf $\mathcal{D}$ of twisted differential operators* (TDO for short) on X is a positively filtered sheaf of almost commutative algebras together with an ismorphism of Poisson algebras $\psi_{\mathcal{D}} : \mathrm{gr}\, \mathcal{D} \simeq S\mathcal{T}_X$.

2.2.2 Example $\mathcal{D}_X$ is a TDO on X.

It follows from the definition that for any TDO $\mathcal{D}$ on X one has $\mathcal{D}^0 \simeq \mathcal{O}_X$. Also one has a bracket $[\cdot,\cdot] : \mathcal{D}^1 \times \mathcal{D}^0 \to \mathcal{D}^0$ coming from commutator of elements in $\mathcal{D}$. This bracket by its definition satifies the Jacobi identity. Hence every $\partial \in \mathcal{D}^1$ induces a derivation of $\mathcal{D}^0 = \mathcal{O}_X$ which we denote by $\bar{\partial}$. In particular, one has a map of sheaves $\mathcal{D}^1 \to T_X$. One easily checks that this gives rise to a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{D}^1 \to T_X \to 0.$$

In fact, a TDO structure on a given filtered sheaf of algebras can be reconstructed from the following data:

(a) An isomorphism $\mathcal{D}^0 \simeq \mathcal{O}_X$ and almost commutativity on $\mathcal{D}$.

(b) The fact that the commutator of element induces a short exact sequence $0 \to \mathcal{O}_X \to \mathcal{D}^1 \to T_X \to 0$ wher $\mathcal{O}_X$ is identified with $\mathcal{D}^0$ and hence $\mathcal{D}^1/\mathcal{D}^0$ is isomorphic to T_X.

(c) The induced bijective map $S\mathcal{T}_X \to \operatorname{gr}\mathcal{D}$ is bijective (then its automatically a Poisson algebra isomorphism).

In fact, the three properties above may be taken as a definition of a TDO.

2.2.3 Exercise Prove that two TDO $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic if and only if

(a) $\mathcal{D}_1 \simeq \mathcal{D}_2$ as sheaves and

(b) the two embeddings $\mathcal{O}_X \hookrightarrow \mathcal{D}_1$, $\mathcal{O}_X \hookrightarrow \mathcal{D}_2$ agree with the isomorphism above.

(HINT: define $\mathcal{D}_i^1$ as $\{u \,|\, [u, \mathcal{O}_X] \subset \mathcal{O}_X\}$, etc.)

The previous exercise essentially says that the TDO structure is uniquely defined by the embedding $\mathcal{O}_X \to \mathcal{D}$.

2.2.4 Example If $\mathcal{L}$ is a line bundle on X then $\mathcal{D}_X(\mathcal{L}, \mathcal{L}$ is a TDO.

2.2.5 Remark For a vector bundle $\mathcal{E}$ of rank greater than 1, $\mathcal{D}_X(\mathcal{E}, \mathcal{E})$ is not a TDO since $\mathcal{D}_X^0(\mathcal{E}, \mathcal{E}) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$.

Definition 2.2.6 An *Atiyah algebra* on a smooth algebraic variety X is a sheaf $(\mathcal{A}, [\cdot,\cdot])$ of $\mathcal{O}_X$-modules with a Lie bracket $[\cdot,\cdot]$ such that

(a) There exists a short exact sequance of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X \to \mathcal{A} \to T_X \to 0$$

(we denote the map $\mathcal{A} \to T_X$ by $a \mapsto \bar{a}$),

(b) $[a_1, f \cdot a_2] = (\bar{a_1} f) a_2 + f[a_1, a_2]$ and

(c) $1_A \in \mathcal{O}_X \subset \mathcal{A}$ is a central element of $\mathcal{A}$.

2.2.7 Remark It is immediate from (b) and (c) that $\mathcal{O}_X$ is an abelian ideal in $\mathcal{A}$.

2.2.8 Example For any TDO $\mathcal{D}$ the first term $\mathcal{D}^1$ of the filtration on $\mathcal{D}$ is an Atiyah algebra.

Lemma 2.2.9 The set of isomorphism classes of Atiyah algebras has a natural structure of a vector space. Moreover, there is a linear map from this vector space to $Ext^1(T_X, \mathcal{O}_X)$ given by forgetting the Lie bracket. *Proof.* First of all, we let any $\lambda \in \mathbb{C}$ act on the set of classes of Atiyah algebras by multiplying the map $\mathcal{A} \to T_X$ (and leaving the embedding $\mathcal{O}_X \hookrightarrow \mathcal{A}$ unchanged. The sum of two Atiyah algebras is defined via the Bauer ??? sum of extensions (one checks that this standard explicit constructions actually gives a Lie bracket on the sum of two extensions). □

2.2.10 Construction of the Atiyah class. Any Atiyah algebra $\mathcal{A}$ by its definition gives a class in $Ext^1(T_X, \mathcal{O}_X)$. Since T_X and $\mathcal{O}_X$ are locally free, one has $Ext^1(T_X, \mathcal{O}_X) = H^1(X, \mathcal{H}om(T_X, \mathcal{O}_X)) = H^1(X, \Omega^1)$. The class in $H^1(X, \Omega^1)$ given by $\mathcal{A}$ is called *Atiyah class* of $\mathcal{A}$.

2.2.11 Example. If $\mathcal{L}$ is a line bundle then $\mathcal{D}_X(\mathcal{L}, \mathcal{L})$ is a TDO hence $\mathcal{D}^1_X(\mathcal{L}, \mathcal{L})$ is an Atiyah algebra defining a class in $H^1(X, \Omega^1)$. This class is nothing but the first Chern class of $\mathcal{L}$. If $\mathcal{L}$ is a vector bundle of rank ¿ 1, then $\mathcal{D}^1_X(\mathcal{L}, \mathcal{L})$ is *not* a TDO.

Definition 2.2.12 Given an Atiyah algebra $\mathcal{A}$, we define $\mathcal{U}_X(\mathcal{A})$ to be the quotient of the sheaf $\mathcal{U}_{\mathbb{C}}(\mathcal{A})$ of the universal enveloping algebras by the relations

$$1_{\mathcal{U}_{\mathbb{C}}(\mathcal{A})} = 1_{\mathcal{A}}; \quad f \otimes a = f \cdot a, \quad \forall f \in \mathcal{O}_X \subset \mathcal{A}, \forall a \in \mathcal{A}.$$

2.2.13 Remark This definition imitates an attempt to consider the universal enveloping algebra of $\mathcal{A}$ *over* $\mathcal{O}_X$. This is of course not possible, since $\mathcal{O}_X$ is not central in $\mathcal{A}$. However, the effect of the relations above is that $\mathcal{U}_X(\mathcal{A})$ has the "size" of $\mathcal{D}_X$. More precisely, one has the following

Proposition 2.2.14 For any Atiyah algebra $\mathcal{A}$, $\mathcal{U}_X(\mathcal{A})$ is a TDO. □

We can construct another object from $\mathcal{A}$ which is rather similar to $\mathcal{U}_X(\mathcal{A})$ and can be easily confused with it.

Definition 2.2.15 Define $S_X(\mathcal{A})$ to the the quotient of the sheaf $S^\bullet_{\mathbb{C}}(\mathcal{A})$ of symmetric algebras on X, by the following relations:

$$1_{S^\bullet_{\mathbb{C}}}(\mathcal{A}) = 1_{\mathcal{A}}; \quad f \otimes a = f \cdot a, \quad \forall f \in \mathcal{O}_X \subset \mathcal{A}, \forall a \in \mathcal{A}.$$

Proposition 2.2.16 For any Atiyah algebra $\mathcal{A}$, $S_X(\mathcal{A})$ is a sheaf of filtered commutative algebras on X such that

(i) $S^0_X(\mathcal{A}) = \mathcal{O}_X$,

(ii) $S_X(\mathcal{A})$ has a natural Poisson bracket $\{\cdot, \cdot\} : S^i_X(\mathcal{A}) \times S^j_X(\mathcal{A}) \to S^{i+j-1}_X(\mathcal{A})$,

(iii) $S^1_X(\mathcal{A})/S^0_X(\mathcal{A}) \simeq T_X$ and $\mathrm{gr}\, S^i_X(\mathcal{A}) \simeq S\mathcal{T}_X$. □.

Definition 2.2.17 A sheaf of filtered algebras satisfying the properties of (i) - (iii) of the proposition above is called *a twised symmetric algebra* of the tangent bundle T_X.

Theorem 2.2.17.1. *There exist natural bijections between the following classes of objects*

$$\{\mathit{TDO}\} \longleftrightarrow \{\mathit{Atiyah\ algebras}\} \longleftrightarrow \{\mathit{Twisted\ symmetric\ algebras}\}. \quad \square$$

2.2.18 Exercise Given an Atiyah algebra $\mathcal{A}$, construct a flat family $\mathbb{D}_{\approx}$ of sheaves of filtered associative algebras on X, parametrized by $t \in \mathbb{CP}^{\nVdash}$ such that

(1) For $t \neq \infty$ one has $\mathbb{D}_t = \mathcal{U}_X(t \cdot \mathcal{A})$ (in particular, $\mathbb{D}_1 = \mathcal{U}_X(\mathcal{A})$ and $\mathbb{D}_0 = \mathcal{D}_X$),

(2) $\mathbb{D}_\infty = S_X(\mathcal{A})$,

(3) $\mathrm{gr}_i(\mathbb{D}) \simeq S^i T_X(i)$ as sheaves on $X \times \mathbb{CP}^1$.

(HINT. Use two filtrations on $\mathcal{U}_X(\mathcal{A})$: the natural filtration of the enveloping algebra and the filtration induced by a two-step filtration on $\mathcal{A}$ in which $\mathcal{A}^0 = \mathcal{O}_X$ and $\mathcal{A}^1 = \mathcal{A}$. Then form an object of the type $\sum_{i,j\geq 0} x^i y^j M_{ij}$ where x and y are homogeneous coordinates on $\mathbb{CP}^1$.)

2.2.19 Remark The sheaf $\mathbb{D}_t$ is an example of a *mixed twistor structure* of Simpson.

2.3. Twisted cotangent bundles and Lagrangian fibrations.

Definition 2.3.1 A *twisted cotangent bundle* on X is an affine bundle $\pi : T^\# X \to X$ modeled on vector bundle $T^*_X \to X$ such that

(i) The total space of $T^\# X$ is endowed with a symplectice form,

(ii) The fibers of π are Lagrangian with respect to this symplectic form,

(iii) The direct image $\pi_*(\mathcal{O}_{T^\# X})$ is a sheaf of commutative algebras with the Poisson bracket $\{\cdot,\cdot\}$ sending $\pi_*(\mathcal{O}_{T^\# X})^{\leq 1} \times \mathcal{O}_X$ to $\mathcal{O}_X$, where $\pi_*(\mathcal{O}_{T^\# X})^{\leq 1}$ stands for polynomials of degree ≤ 1 along the fibers on π.

2.3.2 Remarks.

(1) Notice that on an *affine* space one only has a well-defined notion of a polynomial finction of degree $\leq i$. This is because we don't have a preferred zero point, hence the notion of a *homogeneous* polynomial of degree i is not well-define since it is not preserved by affine transformations.

(2) The property (iii) above is equivalent to

$$\{\pi_*(\mathcal{O}_{T^\# X})^{\leq i}, \pi_*(\mathcal{O}_{T^\# X})^{\leq j}\} \subset \pi_*(\mathcal{O}_{T^\# X})^{\leq i+j-1}.$$

2.3.3 Example Any Atiyah algebra $\mathcal{A}$ gives rise to a twisted cotangent bundle $T^{\mathcal{A}}X$ defined by

$$T^{\mathcal{A}}X = \{\phi \in \mathcal{A}^* \mid \langle \phi, 1_{\mathcal{A}} \rangle = 1\}.$$

Proposition 2.3.4 There exists a canonical isomorphism of Poisson algebras

$$S_X\mathcal{A} \simeq \pi_*(\mathcal{O}_{T^{\mathcal{A}}X})$$

which gives a bijection between twisted cotangent bundles and twisted symmetric algebras. $\square$

2.3.5 Exercise Show that in C^∞-category the all isomorphism classes of twisted cotangent bundles are of the form

$$\Big(T^*X, \omega + \pi^*(\beta)\Big)/\text{modulo exact forms } \beta,$$

where ω is the standard symplectic 2-form on T^*X and β is a closed 2-from on X which we pull back on T^*X. Hence in the C^∞-situation $T^{\#}X$ is always a vector bundle and symplectic forms on it giving rise to a twisted cotangent bundle structure, are parametrized by $H^2(X, \mathbb{R})$.

2.3.6 Twisted cotangent bundle associated with a linear bundle.

Let us construct explicitly the twisted cotangent bundle $T^{\mathcal{L}}X$ corresponding to the Atiyah algebra $\mathcal{D}^1_X(\mathcal{L}, \mathcal{L})$ of differential oprators of order ≤ 1 on a line bundle $\mathcal{L}$. To that end, take the total space L of the principal $\mathbb{C}^*$-bundle corresponding to $\mathcal{L}$. The natural $\mathbb{C}^*$-action on L lifts canonically to a Hamiltonian $\mathbb{C}^*$-action on T^*L. Moreover, there is a *canonical choice* of a moment map

$$\mu : T^*L \to \big(\mathit{Lie}\ \mathbb{C}^*\big)^* = \mathbb{C}$$

(this is because any element $x \in \mathit{Lie}\ \mathbb{C}^*$ gives rise to a vector field on X which can be viewed as a function on T^*L).

Claim **2.3.7** $T^{\mathcal{L}}X$ is naturally isomorphic to the symplectice reduction $\mu^{-1}(1)/\mathbb{C}^*$. Moreover, the quotient $\mu^{-1}(0)/\mathbb{C}^*$ is isomorphic to the usual cotangent bundle T^*_X. $\square$

2.3.8 Remark Denote by ϕ the projection from the $\mathbb{C}^*$ bundle L to X. Then the Claim above is nothing but a commutative version of the followig statement: the sheaf $\phi^*(\mathcal{D}\mathcal{L})$ is naturally isomorphic to the sheaf $(\mathcal{D}_L)^{\mathbb{C}^*}$ of $\mathbb{C}^*$-invariant differential operators on L.

Proposition 2.3.9 There exists a canonical connection on the pullback $\pi^*\mathcal{L}$ of $\mathcal{L}$ to $T^{\mathcal{L}}X$ with curvature equal to the standard symplectic form on $T^{\mathcal{L}}X$. Moreover, any connection on $\mathcal{L}$ is obtain from this canonical connection on $T^{\mathcal{L}}X$ via a uniquely defined section $X \to T^{\mathcal{L}}X$ of π. $\square$

2.3.10 Lagrangian fibrations and holomorphic coordinates.

Proposition 2.3.11 Let M be a symplectic manifold and assume that a map $\pi : M \to X$ to a manifold X is a Lagrangian fibration (i.e. its fibers are Lagrangian subvarieties). Then the fibers of π have canonical affine structure.

Proof. Of course, since π is not assumed to be proper, the affine structure means a local action of $\mathbb{R}^n$, i.e. n commuting vector fields. Let α be a local section of T^*X. Then $\pi^*\alpha$ is a 1-form on M and we can find a unique vector field ξ on M such that $\omega(\xi, \cdot) = \pi^*\omega$. It follows that ξ is tangent to the fiber. Hence we proved that locally M is a principal homogeneous space over the vector bundle T^*X. In particular, M has a local affine structure. $\square$

2.3.12 Remarks.

(1) There is another way to define the affine structure: define $\mathcal{O}^{\leq 0}$ to be the normalizer on $\pi^*\mathcal{O}$ with respect to the Poisson bracket on M. Then the Hamiltonian vector fields obtained from the sections of $\pi^*\mathcal{O}$ are tangent to the fibers and define an affine structure on them.

Proposition 2.3.13 If fibers of the Lagrangian fibration π above are *compact* then each fiber is a *torus*, i.e. a quotient of a vector space by a maximal rank lattice.

Proof. By compactness the local action of $\mathbb{R}^n$ on each fiber can be integrated to a global action, hence the fiber becomes a compact homogeneous space of $\mathbb{R}^n$ which is necessarily a torus. $\square$

We now give a very important application of the proposition above: a construction of canonical *holomorphic coordinates* in a neighbourhood of a point on a Kähler manifold. This construction is due to Bershadsky-Cecoti-Ooguri-Vafa.

2.3.14 Construction.

Let M be a Kähler manifold with a Kähler form ω. Denote by $\overline{M}$ the same manifold with conjugate complex structure. The underlying *real* manifold $M_{\mathbb{R}}$ can be embedded diagonally into $M \times \overline{M}$. Denote by $\mathcal{O}_{\mathbb{R}}$ the sheaf of complex-valued $\mathbb{R}$-analytic functions on $M_{\mathbb{R}}$. Any section f of this sheaf can be expanded locally in a series in z_i and $\bar{z}_j$, local holomorphic and antiholomorphic coordinates on M. If we view z_i as functions on M and $\bar{z}_j$ as functions on $\overline{M}$, then we can extend f locally to a function in the neighbourhood of $M_{\mathbb{R}}$ in $M \times \overline{M}$. We assume that ω is a real-analytic form (this assumption is not essential but it simplifies the construction). Consider the corresponding section $\omega_{\mathbb{C}}$ of the complexified tangent bundle $T_{\mathbb{C}}M_{\mathbb{R}}$. Note that $T_{\mathbb{C}}M_{\mathbb{R}} \simeq T_{M\times\overline{M}}|_{M_{\mathbb{R}}}$. Since ω is a closed $(1,1)$-form, it follows that the extension of ω to a small neightbourhood of $M_{\mathbb{R}}$ in $M \times \overline{M}$ is *holomorphic*. Moreover, each fiber of the projection $pr_2 : M \times \overline{M} \to \overline{M}$ is Lagrangian. By proposition above each fiber acquires an affine structure. Such an affine structure on the fiber over $x \in \overline{M}$ define a holomorphic exponential map $exp_x^{hol} : T_xM \to U$, where U is an open neighbourhood of x in M. This exponential map is defined only in some neighbourhood since $\omega_{\mathbb{C}}$ can be extended only to a neighbourhood of $M_{\mathbb{R}}$ in $M \times \overline{M}$. Moreover, this exponential map exp_x^{hol} depends on x in an *antiholomorphic way* (since we take the conjugate struture on $\overline{M}$.

2.3.15 Exercise
(1) Prove that for the Fubini-Studi metric on the Riemann sphere $\mathbb{CP}^1$, the exponential map coincides with the standard stereographic projection form the point opposite to $x \in \mathbb{CP}^1$.
(2) Generalize the result of (1) to Grassmanians.

2.3.16 Remark The existence of canonical holomorphic coordinates on the moduli space of complex structures on a Calabi-Yau manifold has been recently used in Mirror Symmetry (some equations look especially nice in these local coordiates).

2.4. Classification of TDO.

Definition 2.4.1 A TDO $\mathcal{D}$ is said to be *locally trivial* if the embedding of *algebras* $\mathcal{O}_X \hookrightarrow \mathcal{D}$ is locally isomorphic to the standard embedding of algebras $\mathcal{O}_X \hookrightarrow \mathcal{D}_X$.

2.4.2 Example For any line bundle $\mathcal{L}$, the TDO $\mathcal{D}(\mathcal{L})$ is locally trivial.

Theorem 2.4.2.2. *Locally trivial TDO's on a smooth variety X are classified by the first cohomology group $H^1_{Zar}(X, \Omega^1_{cl})$, where Ω^1_{cl} stands for the sheaf of closed algebraic differential forms and the cohomology is taken in the zariski toplogy.*

First we will prove the following "meta-lemma".

Lemma 2.4.3 The objects $\mathcal{P}$ of "sheaf nature" on X are classified by the first cohomology group $H^1(X, Aut\,\mathcal{P})$ of the sheaf $Aut\,\mathcal{P}$ of automorphisms of $\mathcal{P}$ if the latter is independent of $\mathcal{P}$.

Proof. A standard argument invloving choosing a Cech covering $\{U_i\}$ of X and considering transition functions $\phi_{ij} \in \Gamma(U_i \cap U_j, Aut\,\mathcal{P})$. $\square$

Proof of (2.4.2.2) We have seen bofore that the embedding $\mathcal{O}_X \hookrightarrow \mathcal{D}$ defines the TDO structure on $\mathcal{D}$ uniquely. Hence, by the meta-lemma above we have to establish the isomorphism of sheaves

$$\Omega^1_{cl} \simeq Aut(\mathcal{O}_X \hookrightarrow \mathcal{D}_X)$$

since locally $\mathcal{O}_X \hookrightarrow \mathcal{D}$ is isomorphic to $\mathcal{O}_X \hookrightarrow \mathcal{D}_X$. In other words we have to find a group of automorphisms of $\mathcal{D}_X$ as a TDO. Any such automorphism is uniquely defined by its restriction to $\mathcal{D}^1_X \simeq \mathcal{O}_X \oplus T_X$. Since the embedding $\mathcal{O}_X \hookrightarrow \mathcal{D}_X$ is fixed, any automorphism of $\mathcal{D}^1_X$ is given by sending a section ξ of T_X to a section with the same symbol, which is necessarily of the form $\xi + \phi(\xi)$ for some function $\phi(\xi)$. Since the correspondence $\xi \mapsto \xi + \phi(\xi)$ is to be $\mathcal{O}_X$-linear, it is given be a global 1-form ϕ. By Cartan's formula, the operation $\xi \mapsto \xi + \phi(\xi)$ preserves brackets if and only if ϕ is closed. Hence $\Omega^1_{cl} \simeq Aut(\mathcal{O}_X \hookrightarrow \mathcal{D}_X)$. $\square$

2.4.4 Example Let us give an example of a TDO which is *not* locally trivial. We will describe only the correspondong Atiyah algebra $\mathcal{A}$. We take $\mathcal{A}$ to be a direct sum $\mathcal{O}_X \oplus T_X$ as a sheaf of $\mathcal{O}_X$-modules and define the bracket $[\cdot,\cdot]$ by the rule

$$[f_1 \oplus \xi_1, f_2 \oplus \xi_2] = (\xi_1 \cdot f_2 - \xi_2 \cdot f_1 + \beta(\xi_1,\xi_2)) \oplus [\xi_1,\xi_2],$$

where β is a 2-form on X. The Jacobi identity for this bracket is satisfied if and only if β is closed. If we now suppose that β is not locally exact, then the corresponding TDO will not be locally trivial. Of course, this never happens in the holomorphic setup but it is quite possible in the algebraic situation.

To classify the TDO's in the general case we consider the *truncated De Rham complex*

$$\Omega^{\geq 1} := \big(\Omega^1 \xrightarrow{d} \Omega^2_{cl}\big).$$

Theorem 2.4.4.3. *The TDO's on a smooth variety X are classified by the first hypercohomology group $H^1(X, \Omega^{\geq 1})$ of the truncated De Rham complex $\Omega^{\geq 1}$. Moreover, the short exact sequence of complexes*

$$0 \to \begin{pmatrix} 0 \\ \downarrow \\ \Omega^2_{cl} \end{pmatrix} \to \begin{pmatrix} \Omega^1 \\ \downarrow \\ \Omega^2_{cl} \end{pmatrix} \to \begin{pmatrix} \Omega^1 \\ \downarrow \\ 0 \end{pmatrix} \to 0$$

induces the long exact sequence of (hyper)cohomology

$$\ldots \to H^0(\Omega^1) \to H^0(\Omega^2_{cl}) \to H^1(\Omega^{\geq 1}) \to H^1(\Omega^1) \to \ldots$$

and the last arrow is given by the Atiyah class of the corresponding Atiyah algebra.

Proof. Choose an affine covering $X = \cup U_i$ so that the restriction of any Atiyah algebra $\mathcal{A}$ to U_i is given as an $\mathcal{O}_{U_i}$-module by the direct sum $\mathcal{O}_{U_i} \oplus T_{U_i}$. Since the symbol of the bracket on $\mathcal{A}$ is fixed, any $\mathcal{A}$ is necessarily given by some closed 2-form β_i on U_i as in the example above. We reflect it by writing $\mathcal{A}|_{U_i} = \mathcal{A}_{\beta_i}$.

Moreover, on double intersection $U_i \cap U_j$ any map $\mathcal{A}_{\beta_i} \to \mathcal{A}_{\beta_i}$ is necessarily given by

$$f \mapsto f, \quad \xi \mapsto \xi + \alpha_{ij}(\xi),$$

where f is a function, ξ is a vector field and α_{ij} is a 1-form on $U_i \cap U_j$.

Hence, with respect to the covering $X = \cup U_i$, any Atiyah algebra $\mathcal{A}$ is given by the data

$$\{\alpha_{ij}, \beta_i\}, \quad \alpha_{ij} \in \Omega^1(U_i \cap U_j),\ \beta_i \in \Omega^2_{cl}(U_i).$$

These data a required to satisfy the following conditions

$$\beta_i - \beta_j = d\,\alpha_{ij} \quad \text{ on } U_i \cap U_j, \quad \alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0$$

which say exactly that $\{\alpha_{ij}, \beta_i\}$ represents a class in $H^1(\Omega^{\geq 1})$. $\square$

2.4.5 Remark To compare locally trivial TDO's with all TDO's note that there is an embedding of complexes

$$\begin{pmatrix} \Omega^1_{cl} \\ \downarrow \\ 0 \end{pmatrix} \to \begin{pmatrix} \Omega^1 \\ \downarrow \\ \Omega^2_{cl} \end{pmatrix}$$

In holomorphic situation this embedding is a quasi-isomorphism since $\Omega^1 \to \Omega^2_{cl}$ is surjective and Ω^1_{cl} is the kernel of it. In the lagebraic situation this map may not me a quasi-isomorphism and in general it only induces a map of the corresponding hypercohomology groups.

2.4.6 Vector bundle case. If $E \to X$ is a vector bundle of rank $r > 1$ then the ring of differential operators $\mathcal{D}(E, E)$ is not a TDO since $\mathcal{D}^0(E, E) \simeq \mathcal{E}nd(E)$. Hence the extension

$$0 \to \mathcal{E}nd(E) \to \mathcal{D}^1(E, E) \to T_X \otimes \mathcal{E}nd(E) \to 0$$

is not an Atiyah algebra because of its size.

We consider a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}nd(E) \arrow[r] & \mathcal{D}^1(E,E) \arrow[r] & T_X \otimes \mathcal{E}nd(E) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}nd(E) \arrow[r] \arrow[u, equal] \arrow[d] & \mathcal{A}_1 \arrow[r] \arrow[u] \arrow[d] & T_X \arrow[r] \arrow[u] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{A}_2 \arrow[r] & T_X \arrow[r] & 0
\end{tikzcd}$$

the second row of which is obtained from the first as a pullback with respect to $T_X = T \otimes \mathcal{O}_X \hookrightarrow T_X \otimes \mathcal{E}nd(E)$ and the third is obtained from the second as a pushout with respect to $\mathcal{E}nd(E) \xrightarrow{Tr} \mathcal{O}_X$. Notice that the standard construction of pullback and pushout endow $\mathcal{A}_1$ and $\mathcal{A}_2$ with a bracket. This barcket defines the structure of an Atiyah algebra on $\mathcal{A}_2$.

THEOREM 2.4.6.4. *Let E be a rank r vector bundle on X and $\mathcal{A}_2 = \mathcal{A}_2(E)$ be the Atiyah algebra obtained from the diagram above. Then there exists a natural isomorphism of Atiyah algebras $r \cdot \mathcal{A}_2(E) \simeq \mathcal{A}(\Lambda^r E)$.* $\square$

2.5. Sato's construction of differential operators on a curve.

To give an illustration, suppose we consider the ring of *smooth* differential operators on the real line $\mathbb{R}$. The basic example of such an operator is $\partial = \partial/\partial x$. If f is a smooth function on $\mathbb{R}$ then one can write

$$\partial \cdot f = -\int_{\mathbb{R}} \delta'(x-y) f(y) dy, \qquad \text{where } \delta' \text{ is the Dirac delta function.}$$

This formula suggests that differential operators on X should have something to do with "functions" on the diagonal in $X \times X$.

Let now X be a smooth algebraic curve. Denote by Δ the diagonal in the Cartesian product $X \times X$ which we sometimes identify with X itself. Let $\mathcal{O}_{X\times X}(\infty \cdot \Delta)$ be the sheaf of functions with pole of any finite order along Δ. We will also write $\Omega_X \boxtimes \mathcal{O}_X(\infty \cdot \Delta)$ for $\big(\Omega_X \boxtimes \mathcal{O}_X\big) \otimes \mathcal{O}_{X\times X}\mathcal{O}_{X\times X}(\infty \cdot \Delta)$.

THEOREM 2.5.0.5. *(Sato) One has a natural isomorphism*

$$\frac{\Omega_X \boxtimes \mathcal{O}_X(\infty \cdot \Delta)}{\Omega_X \boxtimes \mathcal{O}_X} \simeq \mathcal{D}_X$$

of sheaves on $X \times X$ supported on $\Delta \simeq X$.

Sketch of proof. For any section ϕ of $\Omega_X \boxtimes \mathcal{O}_X(\infty \cdot \Delta)$ we construct a differential operator u_ϕ on X which acts on functions on X as follows. Choose such a function f and denote the local coordinates on the factors of $X \times X$ by z and w respectively. We put

$$(u_\phi f)(w) = \int_{|z-w|=\varepsilon} f(z)\phi(z,w), \qquad \text{for } \varepsilon \text{ small enough }.$$

In other words, we pull f back to $X \times X$ via the first projection, multiply it by ϕ and take the fiberwise residues along the points on diagonal with respect to the second projection.

If ϕ is written locally as $\frac{d\,z}{(z-w)^{k+1}}$ then integrating by parts we obtain

$$(u_\phi f)(w) = \int_{|z-w|=\varepsilon} \frac{f(z)\,d\,z}{(z-w)^{k+1}} =$$

$$= \pm \int_{|z-w|=\varepsilon} \frac{\partial^k f(z)}{\partial\, z^k} \frac{d\,z}{(z-w)} = \pm \Big(\frac{\partial^k f(z)}{\partial\, z^k} \Big)(w)$$

and the rest of the proof follows by this local computation. □

More generally, let E and F be two vector bundles on X. We put $E^\vee = E^* \otimes_{\mathcal{O}_X} \Omega_X$.

Theorem 2.5.0.6.
(i) There exists a canonical isomorphism of sheaves

$$\frac{(E^\vee \boxtimes F)(\infty \cdot \Delta)}{E^\vee \boxtimes F} \simeq \mathcal{D}_X(E,F).$$

Moreover, the two $\mathcal{O}_X$ module structures on $\mathcal{D}_X(E,F)$ (left and right) correspond under this isomorphism to the two $\mathcal{O}_X$-actions on $E^\vee \boxtimes F$ (via pullback under the first and the second projections).
(ii) One the level of filtrations one has

$$\frac{(E^\vee \boxtimes F)((k+1)\Delta)}{E^\vee \boxtimes F} \simeq \mathcal{D}^k_X(E,F).$$

Proof. Take a section ϕ of $E^\vee \boxtimes F(\infty \cdot \Delta)$. For any section e of E, the product $e \cdot \phi$ is a section of $\Omega_X \boxtimes F(\infty \cdot \Delta)$ hence when we take the residue over the first factor of $X \times X$ we will be left with a section of F. The fact that $(E^\vee \boxtimes F)((k+1)\Delta)/E^\vee \boxtimes F$ indeed maps to $\mathcal{D}^k_X(E,F)$ is shown by a local computation as in the proof of the previous theorem. We will show that this map is an isomorphism by showing that it induces an isomorphism on graded objects. Indeed, since $(\mathcal{O}(\Delta)/\mathcal{O})|_\Delta$ is isomorphic to the tangent sheaf T_X on $\Delta \simeq X$, one has

$$\frac{(E^\vee \boxtimes F)((k+1)\Delta)}{(E^\vee \boxtimes F)(k\Delta)} \simeq S^{k+1}T_X \otimes (E^* \otimes \Omega) \simeq E^* \otimes F \otimes S^k T_X. \qquad \square$$

2.5.1 Exercise Understand how the ring structure on $\mathcal{D}_X(E,E)$ apperas from Sato construction (i.e. how to find the product in terms of sections with poles and residues).

2.5.2 Left and right $\mathcal{D}$-modules.

To establish the connection between left and right $\mathcal{D}$-modules, we first notice that any rigth $\mathcal{D}$-module may be viewed as a left $\mathcal{D}^{op}$-module. Since $\mathcal{D}$ is a TDO, $\mathcal{D}^{op}$ is also a TDO. We will compute $\mathcal{D}^{op}$ using Atiyah algebras. In fact, given any Atiyah algebra

$$0 \to \mathcal{O}_X \to \mathcal{A} \to T_X \to 0$$

we can define the second $\mathcal{O}_X$-module structure on $\mathcal{A}$ (which we identify by writing the fuctions of the right while for the old $\mathcal{O}_X$-module structure we write functions on the left):

$$a \cdot f := f \cdot a + \bar{a}(f).$$

where $\bar{a}$ stands for the image of a in T_X. Of course, this is not compatible with the old $\mathcal{O}_X$-module structure on $\mathcal{A}$, i.e. in general $(g \cdot a) \cdot f \neq g \cdot (a \cdot f)$. We also replace the old bracket $[\cdot,\cdot]$ on $\mathcal{A}$ by $-[\cdot,\cdot]$. This new $\mathcal{O}_X$-module structure together with the new bracket define a structure of an Atiyah algebra on the same sheaf $\mathcal{A}$ that we denote by $\mathcal{A}^{op}$.

Proposition 2.5.3

(i) For any $\mathcal{A}$, one has the following identity between isomorphism classes of Atiyah algebras:

$$[\mathcal{A}] + [\mathcal{A}^{op}] = [\mathcal{A}_{\Omega_X}].$$

(ii) $\mathcal{D}_X^{op} \simeq \mathcal{D}_X(\Omega) \simeq \Omega \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega^{-1}$. Hence if a sheaf $\mathcal{E}$ has a structure of a left (resp. right) $\mathcal{D}_X$-modules then the tensor product $\Omega \otimes_{\mathcal{O}_X} \mathcal{E}$ has a structure of a left (resp. right) $\mathcal{D}^{op}$-module.

(iii) For any left $\mathcal{D}_X$-module $\mathcal{M}$, the tensor product $\Omega_{\mathcal{O}_X}\mathcal{M}$ has a natural right $\mathcal{D}_X$-module structure. $\square$

2.5.4 Example Let us consider the case $\mathcal{M} \simeq \mathcal{O}_X$, i.e. describe the canonical right $\mathcal{D}_X$-module structure on Ω_X. If ξ is a vector field and ω is a top-degree differential form, the we can form a Lie derivative $L_\xi\omega$ but it does not define a left $\mathcal{D}_X$-module structure since

$$L_{f\cdot\xi}\omega = f \cdot L_\xi\omega + \xi(f)\cdot\omega \neq f \cdot L_\xi\omega.$$

However since we have an equality of differential operators

$$\xi \cdot f = f \cdot \xi + \xi(f)$$

hence $\xi : \omega \mapsto -L_\xi \omega$ defines a right $\mathcal{D}_X$-module structure on Ω.

In general, for any left $\mathcal{D}_X$-module $\mathcal{M}$ the formula

$$\xi : (\omega \otimes m) \mapsto -\omega \otimes \xi \cdot m - (L_\xi \omega) \otimes m$$

gives a right $\mathcal{D}_X$-module structure on $\Omega \otimes_{\mathcal{O}_X} \mathcal{M}$.

If E and F is a pair of vector bundles on X then for any differential operator $u \in \mathcal{D}_X(E, F)$ one can consider $u^t \in \mathcal{D}_X(F^\vee, E^v ee)$.

Proposition 2.5.5

(i) Given three vector bundles E, F and V on X and opertors $u \in \mathcal{D}_X(E, F)$, $w \in \mathcal{D}_X(F, V)$, one has

$$(u \circ w)^t = u^t \circ w.$$

(ii) For any vector bundle E one has a natural isomorphism of sheaves of rings

$$\mathcal{D}_X^{op}(E, E) \simeq \mathcal{D}_X(E^\vee, E^\vee). \qquad \square$$

2.5.6 Remark One can interpret the proposition above in terms of Sato's construction as follows. the isomorphism

$$\mathcal{D}_X(E, F) \simeq \frac{(E^\vee \boxtimes F)(\infty \cdot \Delta)}{E^\vee \boxtimes F}$$

allows us to write

$$\mathcal{D}_X(F^\vee, E^\vee) \simeq \frac{(F^{\vee\vee} \boxtimes E^\vee)(\infty \cdot \Delta)}{F^{\vee\vee} \boxtimes E^\vee} = \frac{(F \boxtimes E^\vee)(\infty \cdot \Delta)}{F \boxtimes E^\vee}$$

and the last term now coincides with $\mathcal{D}_X(E, F)$ with the order of the factors flipped.

2.6. Application: Riemann-Roch Theorem for curves.

Let $X \to S$ be a smooth mosphism of smooth algebraic varieties and E be a vector bundle on X. One can define a line bundle $Rdet(\pi_* E)$ on S by a formula

$$Rdet(\pi_* E) = \Big(\bigotimes_{i\ even} det(R^i \pi_* E) \Big) \otimes \Big(\bigotimes_{i\ odd} det^{-1}(R^i \pi_* E) \Big).$$

One can show that even though the individual sheaves $R^i \pi_* E$ are not locally free, the expression above defines a locally free sheaf of rank one on S.

Then the ordinary Grothendieck-Riemann-Roch theorem gives an answer to the following question:

> Compute the Chern class of $Rdet(\pi_* E)$ in terms of Chern classes of E.

The Riemann-Roch theorem in Arakelov geometry deals with the following situation:

> Given a hermitian metric h on E, compute the induced Quillen metric on $Rdet(\pi_* E)$.

The Atiyah algebra of Riemann answers the question which is somewhat half the way betweem ordinary Riemann-Roch and Arakelov Riemann-Roch

> Compute the Atiah algebra of $Rdet(\pi_* E)$ in terms of $\mathcal{D}^1_X(E, E)$.

2.6.1 Remark The problem of this type arises when one cosideres natural determinant bundles on various moduli spaces, i.e. moduli $\mathfrak{M}_{g,n}$ of genus g curves with n points or the moduli space of bundles on a fixed curve, etc.

the logical connection is weak here

We assume that X is a smooth projective curve. Let $\widetilde{\mathcal{D}}$ denote the quotient $(\Omega_X \boxtimes \mathcal{O}_X)(\infty \cdot \Delta)/(\Omega_X \boxtimes \mathcal{O}_X)(-\Delta)$. By Sato's construction one has a natural extension

$$0 \to \frac{\Omega_X \boxtimes \mathcal{O}_X}{(\Omega_X \boxtimes \mathcal{O}_X)(-\Delta)} \to \widetilde{\mathcal{D}} \to \mathcal{D}_X \to 0$$

Notice that $\Omega_X \boxtimes \mathcal{O}_X/(\Omega_X \boxtimes \mathcal{O}_X)(-\Delta)$ is nothing but Ω_X. Similarly, for any vector bundle E on X we can define the sheaf $\widetilde{\mathcal{D}}(E)$ and consider the diagram of sheaves on X

$$\begin{tikzcd}
0 \arrow[r] & \Omega \otimes_{\mathcal{O}_X} \mathcal{E}nd(E) \arrow[r] \arrow[d] & \widetilde{\mathcal{D}}(E) \arrow[r] \arrow[d] & \mathcal{D}_X(E, E) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega \arrow[r] & \widehat{\mathcal{D}}(E) \arrow[r] & \mathcal{D}_X(E, E) \arrow[r] & 0
\end{tikzcd}$$

where the second row is obtained as a pushout of the first under the trace map $\Omega \otimes_{\mathcal{O}_X} \mathcal{E}nd(E) \to \Omega$. Now the second row gives a long exact sequance of cohomology and in particualr the connecting homomorphism

$$H^0(X, \mathcal{D}_X(E, E)) \xrightarrow{\partial} H^1(X, \Omega_X).$$

Since we assumed that X is smooth and projective $H^1(X, \Omega_X)$ is canonically identified with $\mathbb{C}$ and for any $u \in H^0(X, \mathcal{D}_X(E, E))$, $\partial(u)$ is just a complex number. Notice also that any such differential operator u induces a $\mathbb{C}$-linear transformation of each cohomology group $H^i(X, E)$.

Proposition 2.6.2 For any operator $u \in H^0(X, \mathcal{D}_X(E, E))$ we have

$$\partial(u) = Tr(u, H^0(X, E)) - Tr(u, H^1(X, E)).$$

Corollary 2.6.3 In the situation above, $\partial(1)$ is equal to the Euler characteristic $\chi(E)$ of E. $\square$

Sketch of proof of (2.6).

Step 1. One has the following cahin of isomorphisms:

$$H^{\bullet+1}(X \times X, E \boxtimes E^\vee) \simeq \oplus_k H^k(X, E) \otimes_{\mathbb{C}} H^{\bullet+1-k}(X, E^\vee) \simeq$$

$$\oplus_k H^k(X, E) \otimes_{\mathbb{C}} \left(H^{k-\bullet}(X, E)\right)^* \simeq End^{\bullet}_{\mathbb{C}} H^*(X, E).$$

In particular, one has maps

$$End^0_{\mathbb{C}}(H^*(X, E)) \simeq H^1(X \times X, E \boxtimes E^\vee) \xrightarrow{\Delta^*}$$

$$\xrightarrow{\Delta^*} H^1(X, E \otimes E^* \otimes \Omega) \xrightarrow{Tr} H^1(X, \Omega) = \mathbb{C}.$$

It follows from the definitions that the composition $\phi : End^0_{\mathbb{C}}(H^*(X, E)) \to \mathbb{C}$ of maps above takes u to $Tr(u, H^0(X, E)) - Tr(u, H^1(X, E))$.

Step 2. The exact sequence

$$o \to E \boxtimes E^\vee \to E \boxtimes E^\vee(\infty \cdot \Delta) \to \mathcal{D}_X(E, E) \to 0$$

gives a connecting homomorphism $H^0(X, \mathcal{D}_X(E, E)) \to H^1(X \times X, E \boxtimes E^\vee) = End^0_{\mathbb{C}} H^*(X, E)$

Now the proof of the theorem follows from the following claim

Claim **2.6.4** The diagram

$$\begin{tikzcd}
H^0(\mathcal{D}_E) \arrow[r, "\partial"] \arrow[dr] & H^1(E \boxtimes E^\vee) \arrow[r, equal] \arrow[d, equal] & H^0(E) \otimes H^1(E^\vee) \oplus H^1(E) \otimes H^0(E^\vee) \arrow[d, equal] \\
 & End_0\, H^\bullet(E) \arrow[r, equal] & End\, H^0(E) \oplus End\, H^1(E)
\end{tikzcd}$$

commutes (where $\mathcal{D}_E$ stands for $\mathcal{D}(E, E)$, ∂ is the connecting homomorphism and the diagonal arrow is given by the action on cohomology).

Proof. Since we are essentially proving an equality of two operators, it suffices to check it on a particular vector. We will only consider the $End\, H^0(E)$ part (the other part is similar).

Therefore want to prove that the diagram

$$\begin{tikzcd}
H^0(\mathcal{D}_E)\otimes H^0(E) \arrow[r] \arrow[d] & H^1(E\boxtimes E^\vee)\otimes H^0(E) \arrow[d, "a"] \arrow[dl, equal] \\
End_0 H^\bullet(E)\otimes H^0(E) \arrow[r, "b"] & H^0(E)
\end{tikzcd}$$

commutes (where a is given by projecting $H^1(E\boxtimes E^\vee)\otimes H^0(E)$ to $H^0(E)\otimes H^1(E^\vee)\otimes H^0(E)\simeq End\,H^0(E)\otimes H^0(E)$ and applying the action map $End\,H^0(E)\otimes H^0(E)\to H^0(E)$, similarly for b).

Pick a point $x\in X$ and let E_x be the fiber of E over x. We need to show that the following diagram commutes

$$\begin{tikzcd}
H^0(\mathcal{D}_E)\otimes H^0(E) \arrow[r] \arrow[d, "\text{action}"'] & E_x\otimes H^1(E^\vee)\otimes H^0(E) \arrow[d] \\
H^0(E) \arrow[r, "\text{restriction to } x"] & E_x
\end{tikzcd}$$

In fact, take the element $\psi\otimes f\in H^0(\mathcal{D}_E)\otimes H^0(E)$. Mapping it to $H^0(E)$ and then the result to E_x we obtain $Res_{x_2}(\psi(x,x_2)f(x_2))$ (this follows from definitions). To compute the other composition, take a unit disk in some local coordinates centered at x and call it X_{in}. Also denote $X\setminus\{x\}$ by X_{out}. There is a theorem saying that both X_{in} and X_{out} are Stein manifiold, hence cohomologically trivial. Then by Meyer-Vietoris,

$$H^1(E^\vee)=\frac{\Gamma(X_{in}\cap X_{out},E^\vee)}{\Gamma(X_{in},E^\vee)+\Gamma(X_{out},E^\vee)}$$

In thhis represetation we see that an element $\psi(x,x_2)$ in fact gives an element of $H^1(E^\vee)$ and that the image of $\psi\otimes f$ is also given by $Res_{x_2}(\psi(x,x_2)f(x_2))$ since the residue is equal to zero for $\psi\in\Gamma(X_{in},E^\vee)+\Gamma(X_{out},E^\vee)$. (**This should be said better**) $\square$

2.7. Leray residue and cohomology with support.

Let us assume first that we deal with Hausdorf topology on $\mathbb{C}$-analytic sets. Choose an open disk U in $\mathbb{C}^n=\{x_1,\dots,x_n\}$ centered at 0 (so that $0\in U$ is given by equations $\{x_1=0,\dots,x_n=0\}$). Denote by U^* the punctured disk $U\setminus\{0\}$. Cover U^* by open subsets $U_i=U\setminus\{x_i=0\},\quad i=1,\dots,n$.

Consider a holomorphic differential form $\omega\in\Omega^n(U_1\cap\dots U_n)$. The residue $Res_{\,x=0}(\omega)$ of ω at $x=0$ is defined by restricting ω to $(S^1)^n=$

$\{x \in U^* \text{ such that } |x_i| = 1, \quad i = 1, \ldots, n\}$ and computing the integral

$$Res_{\,x=0}(\omega) = \underbrace{\int\int \ldots \int}_{n} \omega$$

over $(S^1)^n$.

We can also consider $U = \mathbb{C}^n$ with Zariski topology and put $U_i = \mathbb{C}^n \setminus \{x_i = 0\}, \quad i = 1, \ldots, n$. For any *algebraic* differential n-form $\omega \in \Omega^n_{alg}(U_1 \cap \ldots \cap U_n)$ the residue $Res_{\,x=0}(\omega)$ is defined via the same formula.

2.7.1 Cohomological interpretation.

Let X is any topological space and A is a sheaf of abelian groups on X. Choose an open covering $\{U_i\}_{i \in I}$ of X such that for any finite subset $J \subset I$ we have $H^j(\cap_{i \in J} U_i, A) = 0$ for all $j \geq 1$ (such a covering is called A-*acyclic*). It is well known that, for such a covering, the cohomology groups $H^j(X, A)$ can be computed via the Cech complex $C^\bullet(\{U_i\}, A)$.

In our situation above (holomorphic or algebraic) the open sets U_i form a covering of $U^* = U \setminus \{0\}$. One can prove that this covering is Ω^n-acyclic (in the algebraic situation this is true since all U_i and their intersection are affine, in the holomorphic setting one uses (??)). Therefore the Cech complex $C^\bullet(\{U_i\}, \Omega^n)$ computes $H^\bullet(U^*, \Omega^n)$.

Any holomorphic (or algebraic) n-form ω represents represents a Cech $n-1$ cocycle (since there are no $(n+1)$-multiple intersections). Coboundaries in the Cech complex of Ω^n are formed by all linear combinations of forms that extend to at least one of the U_i's. It is easy to see that the residue at $x = 0$ of any form that represents a coboundary, is zero. Hence we have a well-defined residue map:

$$Res_{x=0} : H^{n-1}(U^*, \Omega^n) \to \mathbb{C}.$$

Alternatively, in the holomorphic setup we could notice that one has

$$H^{n-1}(U^*, \Omega) \simeq H^{n,n-1}_{\bar{\partial}}(U^*),$$

since on (n, q)-forms $\bar{\partial}$ is equal to d. One has a natural map

$$H^{n,n-1}_{\bar{\partial}}(U^*) \to H^{2n-1}(U^*) = \mathbb{C}$$

that coincides with the residue map.

2.7.2 Cohomology with support.

Consider a closed subvariety Y of an algebraic variety X. We denote by $Y \overset{i}{\hookrightarrow} X$ the closed embedding of Y in X and by $U = X \setminus Y \overset{j}{\hookrightarrow} X$ the open embedding of the complement U of Y.

For any sheaf of abelian groups A on X consider the subsheaf $\Gamma_{[Y]}A$ of A formed by all section of A supproted on Y. One has an exact sequence

$$0 \to \Gamma_{[Y]}A \to A \to j_*j^*A.$$

Moreover, $\Gamma_{[Y]}$ is exact on injective (or flabby) sheaves. Hence there exist derived functors $\mathcal{H}^i_{[Y]}A$ which allow us to continue the exact sequence above. Namely, since $R^i(id) = 0$ for $i > 0$, we have an exact sequence

$$0 \to \mathcal{H}^0_{[Y]}A \to A \to j_*j^*A \to \mathcal{H}^1_{[Y]}A \to 0$$

and isomorphisms

$$0 \to R^ij_*j^*A \to \mathcal{H}^{i+1}_{[Y]}A \to 0, \qquad i \geq 1. \tag{2.7.3}$$

2.7.4 REMARK The functor j_* may not be exact. For example, if one removes a point from $\mathbb{C}^2$ the resulting variety is no longer affine, so the functor R^1j_* will in fact be non-trivial.

2.7.5 EXAMPLE In the algebraic situation described above one has a vector space isomorphism

$$H^n_{[0]}(U, \Omega^n_{alg}) \simeq \mathbb{C}\Big[\frac{1}{x_1}, \ldots, \frac{1}{x_n}\Big]\frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n}.$$

In fact, $\Omega^n(U_1 \cap \ldots \cap U_n) = \mathbb{C}\Big[x_1, \frac{1}{x_1}, \ldots, x_n, \frac{1}{x_n}\Big]dx_1 \wedge \ldots \wedge dx_n$. However, any form that does not have a pole along *all* the divisors $\{x_1 = 0\}, \ldots, \{x_n = 0\}$, represents a Cech coboundary. The quotient of $\Omega^n(U_1 \cap \ldots \cap U_n)$ by the linear subspace generated by all such for is exactly the RHS of the expression above.

Denote the defining ideal of Y by $I_Y \subset \mathcal{O}_X$. First, notice that

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I_Y, \mathcal{M}) \subset \Gamma_{[Y]}\mathcal{M},$$

(since any section killed by I_Y is supported on Y.) To go backwards, we notice that if a section of $\mathcal{M}$ is supported on Y then it is killed by some *power* of I_Y. In other words,

$$\Gamma_{[Y]}\mathcal{M} = \varinjlim_n \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I^n_Y, \mathcal{M}). \tag{2.7.6}$$

By uniqueness of derived functors, we deduce from (2.7.6) that

$$\mathcal{H}^i_{[Y]}\mathcal{M} = \varinjlim_n \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_X/I^n_Y, \mathcal{M}). \tag{2.7.7}$$

2.7.8 Special case: X is affine and Y is a smooth divisor in X given by the equation $\{f = 0\}$. In this case $U = X \setminus Y$ is affine hence j_* is exact and we just have an exact sequence

$$0 \to \Gamma_Y[M] \hookrightarrow M \to M[f^{-1}] \to \mathcal{H}^1_{[Y]}M \to 0.$$

We see now that $\Gamma_Y M = \{m \in M \mid f^{i(m)} \cdot m = 0 \text{ for some } i(m)\}$ and $\mathcal{H}^1_{[Y]}M = M[f^{-1}]/Image(M)$.

2.7.9 Example If $X = \mathbb{C}^1$, $Y = \{0\}$ and $M = \mathbb{C}[t] = \mathcal{O}_X$. Then $\mathbb{C}[t, t^{-1}]/\mathbb{C}[t] = \mathcal{D} \cdot \delta$ in the notations of (???).

2.7.10 Local duality.

Now we explain the relationship between differential operators and local cohomology. Let us assume first that $Y \hookrightarrow X$ is a point given locally by vanishing of n regular functions. Namely, we choose an affine open neighbourhood U of $x := Y$ and functions $f_i \in \Gamma(U, \mathcal{O}_X)$, $i = 1, \ldots, f_n$ such that I_x is generated by $f_1, \ldots, f_n$.

Then we can construct a pairing

$$RES : \mathcal{O}_x \times \mathcal{H}^n_{[x]}(\Omega_X) \to \mathbb{C} \tag{2.7.11}$$

as follows. Let $f \in \mathcal{O}_x$ and $\omega \in \mathcal{H}^n_{[x]}(\Omega_X)$. Consider the open covering of $U^* = U \setminus \{x\}$ by affine open subsets $U_i = U \setminus \{f_i = 0\}$. Using (2.7.3) we can represent ω by an algebraic differential n-form on $U_1 \cap \ldots \cap U_n$. Put $RES(f, \omega) = Res_{\,x}(f \cdot \omega)$.

2.7.12 Example Let $X = \mathbb{C}$ and $x = \{0\}$. Then $\mathcal{H}^1_{[x]}(\Omega^1) = \frac{dx}{x}\left[\frac{1}{x}\right]$ and

$$RES : \mathcal{O}_x \times \frac{dx}{x}\left[\frac{1}{x}\right] \to \mathbb{C}$$

is given by the usual one-dimensional residue of $f \cdot \omega$ (i.e. the coefficient of $\frac{dx}{x}$ in the Laurent expansion of $f \cdot \omega$).

Assume for simplicity that we are in the situation of the example above, i.e. $X = \mathbb{C}$ and $x = \{0\}$. Take $\omega = \frac{1}{x^k}\frac{dx}{x}$. Then integrating by parts we obtain

$$\int_{S^1} \frac{f(x)}{x^k}\frac{dx}{x} = \pm \int_{S^1} \frac{d^k f(x)}{dx^k}\frac{dx}{x}.$$

Similarly, when $X = \mathbb{C}^n$ and $x = \{0\}$ we can conclude that

(a) $RES(\,\cdot\,, \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n})$ acts on $\mathcal{O}_x$ as delta function supported at x, i.e. $f \in \mathcal{O}_x$ maps to $f(0)$.

(b) $RES(\,\cdot\,, \left(\frac{1}{x_1^{k_1}} \ldots \frac{1}{x_n^{k_n}}\right)\frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n})$ maps $f(x)$ to $(\frac{\partial^{k_1}}{\partial x_1^{k_1}} \ldots \frac{\partial^{k_n}}{\partial x_n^{k_n}} f)(0)$.

2.7.13 Cohomological residues. Using the direct limit presentation (2.7.7) of local cohomology we can view the residue map (2.7.11) as follows. By Serre duality, for any $m = 1, 2, \dots$ there exists a natural pairing

$$\mathcal{O}_X/I_x^m \times Ext_X^n(\mathcal{O}_X/I_x^m, \Omega_X^n) \to \mathbb{C}.$$

Notice that since X is smooth we have

$$Ext_X^n(\mathcal{O}_X/I_x^m, \Omega_X^n) = \Gamma(X, \mathcal{E}xt^n(\mathcal{O}_X/I_x^m, \Omega_X^n)).$$

For any $\omega \in Ext_X^n(\mathcal{O}_X/I_x^m, \Omega_X^n) \subset \mathcal{H}_{[x]}^n(\Omega_X)$ the image of the residue map $RES(\,\cdot\,, \omega)$ is equal to the composition of $\mathcal{O}_x \to \mathcal{O}_X/I_x^m$ with the Serre duality map.

Proposition 2.7.14 The pairing (2.7.11) is (topologically) perfect, i.e. it identifies $\mathcal{H}_{[x]}^n(\Omega_X^n)$ with a subspace of $Hom_{\mathbb{C}}(\mathcal{O}_x, \mathbb{C})$ formed by all linear functions that depend only on some finite jet of an element in $\mathcal{O}_x$. Moreover, under this identification, the subspace $Ext_X^n(\mathcal{O}_X/I_x^m, \Omega_X^n) \subset \mathcal{H}_{[x]}^n(\Omega_X^n)$ corresponds to $Hom_{\mathbb{C}}(\mathcal{O}_X/I_x^m, \mathbb{C}) \subset Hom_{\mathbb{C}}(\mathcal{O}_x, \mathbb{C})$. $\square$

2.7.15 The general case. Now suppose that $Y \hookrightarrow X$ is a smooth subvariety of codimension d. By passing to an open subset we can assume that Y is given by vanishing of d functions $t_1, \dots, t_d$.

Lemma 2.7.16 If Y, X and $t_1, \dots, t_d$ are as above, then

$$\mathcal{H}_{[Y]}^i = \begin{cases} 0 & \text{if } i \neq d \\ \mathbb{C}[\partial/\partial t_1, \dots, \partial/\partial t_d] \otimes_{\mathbb{C}} \mathcal{O}_Y & \text{if } i = d. \end{cases} \quad \square$$

To construct the analogue of (2.7.11), notice that $\mathcal{O}_Y \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X/I_Y)$. Hence one has a Yoneda pairing

$$\mathcal{O}_Y \times \mathcal{E}xt^d(\mathcal{O}_X/I_Y, \Omega_X) \to \mathcal{E}xt^d(\mathcal{O}_Y, \Omega_X) \simeq \Omega_Y$$

(**I don't know what to to with higher powers of I_Y !!**)

Proposition 2.7.17 There exists a (topologically) perfect pairing of sheaves on X

$$RES : \mathcal{O}_{X,Y} \times \mathcal{H}_{[Y]}^d(\Omega_X) \to \Omega_Y \tag{2.7.18}$$

where $\mathcal{O}_{X,Y}$ denotes the completion of $\mathcal{O}_X$ along Y.

2.7.19 Higher-dimensional Sato construction.

Denote by Δ the diagonal in $X \times X$ and by i_Δ its embedding in $X \times X$. We want to represent $\mathcal{D}_X$ as some sheaf on $X \times X$ supported at Δ. Let $\dim X = n$.

THEOREM 2.7.19.1. *There exists an isomorphism*

$$(i_\Delta)_*(\mathcal{D}_X) = \mathcal{H}^n_{[\Delta]}(\Omega_X \boxtimes \mathcal{O}_X) \qquad (2.7.20)$$

of sheaves on $X \times X$. □

Sketch of proof. Suppose one has an section ω of $\mathcal{H}^n_{[\Delta]}(\Omega_X \boxtimes \mathcal{O}_X)$ and a function $f(x)$ on X. Then $\omega \cdot f$ is a function on X which can be described as follows. Given a point $x_2 \in X$, the restriction of ω to $pr_2^{-1}(x_2) \simeq X_2$ is an element of $\mathcal{H}^n_{[x_2]}(\Omega_X \boxtimes \mathcal{O}_X)$. We put the value of the function $\omega \cdot f$ at x_2 to be equal to $RES(f, \omega)$. □

We can describe the isomorphism of the theorem above from the point of view of local duality. For any function f on X the pull-back $pr_1^*(f)$ is a section of $\mathcal{O}_{X\times X}$ which we can project to a section of $\mathcal{O}_{X\times X,\Delta}$. The proposition (??) above provides a pairing

$$RES : \mathcal{O}_{X\times X,\Delta} \times \mathcal{H}^n_{[\Delta]}(\Omega_{X\times X}) \to \Omega_\Delta.$$

However since ω is a section of $\mathcal{H}^n_{[\Delta]}(\Omega_X \boxtimes \mathcal{O}_X)$, not of $\mathcal{H}^n_{[\Delta]}(\Omega_{X\times X})$, the result of the pairing will be a section of $\mathcal{O}_\Delta$, which is a function on $\Delta \simeq X$ (i.e. we "untwist" Ω_X on the second factor of $X \times X$).

2.7.21 Local cohomology modules.

Let $\mathcal{M}$ be a $\mathcal{D}_X$-module and let $Y \overset{i}{\hookrightarrow} X$ is a submanifold.

CLAIM **2.7.22** For any $i \geq 0$, $\mathcal{H}^i_{[Y]}\mathcal{M}$ has a natural structure of a $\mathcal{D}_X$-module.

Proof. Since this claim is very important we will give two equivalent constructions of the $\mathcal{D}_X$-module structure.

2.7.23 First construction. We use the isomorphisms $\mathcal{H}^{i+1}_{[Y]}(X, A) \simeq \mathcal{H}^i(X \setminus Y, A)$. Choose a Cech covering for $X \setminus Y$. All the groups $C^\bullet(U_{i_1} \cap \ldots \cap U_{i_n}, \mathcal{M})$ have a natural $\mathcal{D}_X$-module structure which descends to the cohomology due to combinatorial nature of the Cech differential.

2.7.24 Second construction. For the second construcion of the $\mathcal{D}_X$-module structure we extend isomorphism (2.7.6):

$$\Gamma_{[Y]}\mathcal{M} \simeq \varinjlim_n \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I^n_Y, \mathcal{M}) \simeq \varinjlim_n \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/(\mathcal{D}_X \cdot I^n_Y), \mathcal{M})$$

(the second isomorphism follows from $\mathcal{D}_X/(\mathcal{D}_X \cdot I^n_Y) \simeq \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X/I^n_Y$. Notice also that the first term in the line above makes sense in the general sheaf-theoretic setup, the second term in the category of $\mathcal{O}_X$-modules while the third only in the category of $\mathcal{D}_X$-modules.

Take $u \in \mathcal{D}^n_X$. The right multiplication map $\mathcal{D}_X \xrightarrow{\cdot u} \mathcal{D}_X$ does not descend to an endomorphism of $\mathcal{D}_X/(\mathcal{D}_X \cdot I^k_Y)$ since $(\cdot u)$ does not preserve I_Y^k. However, using the Leibniz rule we can show that $(\mathcal{D}_X \cdot I_Y^k) \cdot u \subset \mathcal{D}_X \cdot I_Y^{k-n}$ hence there is an induced map $\mathcal{D}_X/(\mathcal{D}_X \cdot I_Y^k) \xrightarrow{\cdot u} \mathcal{D}_X/(\mathcal{D}_X \cdot I_Y^{k-n})$. Therefore the right action of u on $\varprojlim_k \mathcal{D}_X/(\mathcal{D}_X \cdot I_Y^k)$ is well-defined.

Hence $\varprojlim_k \mathcal{D}_X/(\mathcal{D}_X \cdot I_Y^k)$ has a $\mathcal{D}_X$-bimodule structure. Since

$$\Gamma_{[Y]}\mathcal{M} \simeq \varinjlim_n \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/(\mathcal{D}_X \cdot I_Y^n), \mathcal{M}) = \mathcal{H}om_{\mathcal{D}_X}(\varprojlim_n \mathcal{D}_X/(\mathcal{D}_X \cdot I_Y^n), \mathcal{M}),$$

$\Gamma_{[Y]}\mathcal{M}$ inherits a left $\mathcal{D}_X$-module structure.

2.7.25 Warning. There is *no* $\mathcal{D}_X$-module structure on any of the individual terms $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I_Y^n, \mathcal{M})$, only on the direct limit above.

To extend this construction of higher cohomology with support (i.e. to
$\varinjlim \mathcal{E}xt^i(\dots))$ we can do two *a priori* different things:
consider $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/(\mathcal{D}_X \cdot I_Y^n), \mathcal{M})$ or consider $R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I_Y^n, \mathcal{M})$.

However these two functors coincide since we can compute the latter using locally free $\mathcal{O}_X$-resolutions of $\mathcal{O}_X/I_Y^n$. Given such resolution, we can tensor it with $\mathcal{D}_X$ over $\mathcal{O}_{\mathcal{X}}$ and obtain a $\mathcal{D}_X$-resolution computing the former functor.

3. $\mathcal{D}$-modules: basic constructions.

3.1. $\mathcal{D}$-modules on a line.

Here we will consider the case of $\mathcal{D}$-modules on a one-dimensional affine space over the field of complex numbers $\mathbb{C}$ (any other algebraically closed field of characteristic zero will do). In other words, we consider the ring of all differential operators of the form $\sum a_i(x)\frac{d^n}{dx^n}$ where $a_i(x) \in \mathbb{C}[x]$ are polynomials in x.

3.1.1 Notation Everywhere below we will write ∂ instead of $\frac{d}{dx}$.

The ring $\mathcal{D}$ of differential operators on the line can be also represented as a quotient of the ring $\mathbb{C}\langle x, \partial\rangle$ of polynomials in non-commuting variables x and ∂, by the relation $x\partial - \partial x = 1$.

We define an anti-involution $(\cdot)^t : u \mapsto u^t$ by

$$x^t = x, \quad \partial^t = -\partial$$

(then, for example, $(x\partial)^t = -\partial x$). The meaning of this involution becomes clear if we think of differential operators on $\mathbb{C}$ as acting on the space of smooth functions on $\mathbb{C} = \mathbb{R}^2$ (say, the Schwartz space $\mathcal{S}(\mathbb{R}^2)$). Then for any differential operator P and any pair of functions f, g, one has

$$\int_{\mathbb{R}^2} (Pf)g = \pm \int_{\mathbb{R}^2} f(P^t g).$$

Let us give several examples of $\mathcal{D}$-modules.

3.1.2 Examples.

(i) Consider $\mathcal{O} := \mathbb{C}[x]$. Then $\mathcal{O}$ is a simple $\mathcal{D}$-module generated by $1 \in \mathcal{O}$ and

$$\mathcal{O} = \mathcal{D} \cdot 1 = \mathcal{D}/\mathcal{D}\partial$$

(ii) We can act by a differential operator on the ring $\mathbb{C}[x, x^{-1}]$ of Laurent polynomials in x. This ring is not finitely generated as a $\mathcal{O}$-module (and in general, many $\mathcal{D}$ modules that we will consider will not be coherent). However, over $\mathcal{D}$, $\mathbb{C}[x, x^{-1}]$ is generated by the element $\frac{1}{x}$ which is annihilated by (∂x):

$$\mathbb{C}[x, x^{-1}] = \mathcal{D} \cdot \frac{1}{x} = \mathcal{D}/\mathcal{D}(\partial x)$$

(iii) For any $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ introduce the formal symbol x^λ and put $\partial \cdot x^\lambda = \lambda x^{\lambda-1}$. Generalizing the previous example, we can write

$$\mathbb{C}[x, x^{-1}]x^\lambda = \mathcal{D} \cdot x^\lambda = \mathcal{D}/\mathcal{D}(x\partial - \lambda).$$

(iv) Note that $\mathcal{O}$ is a submodule of $\mathbb{C}[x, x^{-1}]$. Consider the short exact sequence

$$0 \to \mathcal{O} \to \mathbb{C}[x, x^{-1}] \to \frac{\mathbb{C}[x, x^{-1}]}{\mathbb{C}[x]} \to 0,$$

and denote the image of x^{-1} in $\frac{\mathbb{C}[x,x^{-1}]}{\mathbb{C}[x]}$ by δ. The reader can easily prove that

$$\frac{\mathbb{C}[x, x^{-1}]}{\mathbb{C}[x]} = \mathcal{D} \cdot \delta = \mathcal{D}/\mathcal{D}x$$

3.1.3 Remark What are the characteristic varieties of the modules above? If we notice that $\mathrm{gr}(\mathcal{D})$ is $\mathbb{C}[x, \xi]$ (where ξ is the symbol of ∂) then $Spec(\mathrm{gr}(\mathcal{D})) = \mathbb{C}^2$ and

(i) $SS(\mathcal{O}) = \{\xi = 0\}$.

(ii) $SS(\mathbb{C}[x, x^{-1}]) = \{x\xi = 0\}$.

(iii) $SS(\mathbb{C}[x, x^{-1}]x^{\lambda}) = \{x\xi = 0\}$. Notice that in this case the characteristic variety is not irreducible even though $\mathbb{C}[x, x^{-1}]x^{\lambda}$ is a simple $\mathcal{D}$-module.

(iv) $SS(\mathcal{D} \cdot \delta) = \{x = 0\}$.

Note that in the fourth example the element $\delta \in \frac{\mathbb{C}[x,x^{-1}]}{\mathbb{C}[x]}$ satisfies $x \cdot \delta = 0$: a property that is true for the delta function in calculus. What will happen when we try to find "$\int \delta$"? In calculus this integral is given by the Heaviside function $\theta(x)$. To see what happens for $\mathcal{D}$-modules we consider the following

3.1.4 Example

Introduce a formal symbol $log(x)$ with the property $\partial \cdot log(x) = \frac{1}{x}$ and consider the $\mathcal{D}$-module $\mathbb{C}[x]log(x) + \mathbb{C}[x, x^{-1}]$. This module is generated over $\mathcal{D}$ by $log(x)$ and one has

$$\mathbb{C}[x]log(x) + \mathbb{C}[x, x^{-1}] = \mathcal{D} \cdot log(x) = \mathcal{D}/\mathcal{D}(\partial x \partial).$$

3.1.5 Remark In general, let $P = a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \ldots + a_0(x) \in \mathcal{D}$. The idea due to Sato is that one should study the properties of $M = \mathcal{D}/\mathcal{D} \cdot P$ instead of looking for solutions of $P \cdot a(x) = 0$.

3.1.6 Notation When a module M has a filtration $M_1 \subset M_2 \subset M_3 = M$ we write

$$M \rightsquigarrow \begin{pmatrix} M_3/M_2 \\ M_2/M_1 \\ M_1 \end{pmatrix}$$

and say that M_3/M_2 is *on the top* of the filtration while M_1 is *on the bottom.* For example, $\mathbb{C}[x, x^{-1}] \rightsquigarrow \begin{pmatrix} \mathcal{D}\cdot\delta \\ \mathcal{O} \end{pmatrix}$.

Lemma 3.1.7 $\quad \mathcal{D}\cdot log(x) \rightsquigarrow \begin{pmatrix} \mathcal{O} \\ \mathcal{D}\cdot\delta \\ \mathcal{O} \end{pmatrix}$

Proof. One has an exact sequence

$$0 \to \mathbb{C}[x, x^{-1}] \to \mathcal{D}\cdot log(x) \to \mathbb{C}[x] \to 0. \quad \square$$

Now mod out the bottom of $\mathcal{D}\cdot log(x)$ and denote by $\int\delta$ the image of $log(x)$ in the quotient module (this image generates the quotient). It follows from the definition that $\mathcal{D}\cdot\int\delta \rightsquigarrow \begin{pmatrix} \mathcal{O} \\ \mathcal{D}\cdot\delta \end{pmatrix}$. Denote by θ the generator of the quotient $\mathcal{O}$ of $\mathcal{D}\cdot\int\delta$. Thus, θ is "log(x) modulo $\mathbb{C}[x, x^{-1}]$". Geometrically θ gives the monodromy around zero. If we compare that with $\int\delta$ in calculus, we will see that the situation is similar to the relationship between real Morse theory (which tells how the topology changes when we go *through* a critical level) and complex Morse theory (which describes Picard-Lefschetz transformation, i.e., the result of going *around* a critical level).

3.1.8 Exercise The Verdier dual of a module was defined in (**??**). By its definition, dual of a left $\mathcal{D}$-module is a right $\mathcal{D}$-module. However, the anti-involution $u \mapsto (u)^t$ allows us to view any right $\mathcal{D}$-module as a left $\mathcal{D}$-module, so we will assume that the dual is a left $\mathcal{D}$-module.

(i) Prove that $\mathcal{O}$ is a self-dual module

(ii) Prove that $(\mathbb{C}[x, x^{-1}])^\vee = \mathcal{D}\cdot\int\delta$ and $(\mathcal{D}\cdot\int\delta)^\vee = \mathbb{C}[x, x^{-1}]$ (i.e. taking the dual flips the order of subquotients).

(iii) Prove that $(\mathcal{D}\cdot x^\lambda)^\vee = \mathcal{D}\cdot x^{-(\lambda+1)}$.

3.1.9 Remark Finally, we give an example which illustrates that in general one should be careful when considering the charactersitic varieties. Let $P_1 = x\partial$ and $P_2 = \partial x$. Consider the ideal $J = \{\mathcal{D}\cdot P_1 + \mathcal{D}\cdot P_2\}$. Then $J = \mathcal{D}$, so $SS\,\mathcal{D}/J = \emptyset$. However, if we just look at the zero-set of the principal symbols of P_1 and P_2, we get $\{x\xi = 0\}$. The point is that the statement $\{P_i$ generate $J\}$ does not imply in general that $\{\sigma(P_i)$ generate gr $J\}$.

3.2. Direct image from a submanifold.

When $\mathcal{M} = \mathcal{O}_X$ we can give a third construction of the $\mathcal{D}_X$-module structure on the local cohomology.

To that end, introduce the following

Definition 3.2.1 $\mathcal{D}_X^Y = \{u \in \mathcal{D}_X | u(I_Y^i) \subset I_Y^i, \quad \forall i\}$.

Proposition 3.2.2 *$\mathcal{D}_X^Y$ has the following properties:*

(i) $\mathcal{D}_X^Y$ is a subalgebra of $\mathcal{D}_X$.

(ii) $\mathcal{O}_X \subset \mathcal{D}_X^Y$.

(iii) Denote by $T_X^Y \subset T_X$ the subsheaf of all vector fields on X that are tangent to Y at any point of Y. Then $T_X^Y \subset \mathcal{D}_X^Y$.

(iv) $\mathcal{D}_X^Y$ is generated by $\mathcal{O}_X$ and T_X^Y

Proof. First notice that (i) and (ii) are clear, and we can prove (iii) and (iv) locally.

Since Y is smooth, it is locally a complete intersection in X. Therefore, for any point $y \in Y \subset X$, we can choose an affine neighborhood U of y in X and regular functions $y_1, \dots, y_k$, $t_1, \dots, t_l$ such that

(i) $Y \cap U$ is given by vanishing of $t_1, \dots t_l$,

(ii) $y_1, \dots, y_k$ form a system of "local coordinates" on Y (in the sense explained in **??**).

Then one can check that

(1) $\mathcal{D}_X = \mathcal{O}_X[\partial/\partial t_i, \partial/\partial y_l]$

(2) T_X^Y is generated by $\partial/\partial y_i, t_j\partial/\partial t_k$.

(3) $\mathcal{D}_X^Y = \mathcal{O}_X[t_i, t_j\partial/\partial t_k, \partial/\partial y_l]$.

Now the assertions of the proposition follow from (1)-(3). □

By definition of $\mathcal{D}_X^Y$, there is a well-defined action of $\mathcal{D}_X^Y$ on $\mathcal{O}_X/I_Y = \mathcal{O}_Y$. Moreover, one has the following

Claim **3.2.3** *There exists a diagram*

$$\mathcal{D}_X \hookleftarrow \mathcal{D}_X^Y \twoheadrightarrow \mathcal{D}_Y \simeq \mathcal{D}_X^Y/(I_Y \cdot \mathcal{D}_X) \cap \mathcal{D}_X^Y \tag{3.2.4}$$

Proof. The intersection $(I_Y \cdot \mathcal{D}_X) \cap \mathcal{D}_X^Y$ is in the kernel of the map: $\mathcal{D}_X^Y \to \mathcal{D}_Y$ since it sends $\mathcal{O}_X$ to I_Y. By choosing locally a suitable coordinate system we can show that $\mathcal{D}_X^Y \to \mathcal{D}_Y$ is indeed surjective and $(I_Y \cdot \mathcal{D}_X) \cap \mathcal{D}_X^Y$ coincides with the whole kernel of it. □

Considering $\mathcal{D}_X \cdot I_Y$ instead of $I_Y \cdot \mathcal{D}_X$ leads to a different situation:

Proposition 3.2.5 *The quotient $\mathcal{D}_X^Y/(\mathcal{D}_X \cdot I_Y) \cap \mathcal{D}_X^Y$ is isomorphic to the ring of twisted differential operators on the top exterior power $det(T_Y X)$ of the normal bundle to Y.*

Proof. First we observe that by Proposition 3.2, T_X^Y acts on I_Y/I_Y^2, the sheaf of sections of the conormal bundle T_Y^*X. The sections of the normal bundle $T_Y X$ are given by $\mathcal{H}om_{\mathcal{O}_Y}(I_Y/I_Y^2, \mathcal{O}_Y)$. If v and λ are sections of $T_Y X$ and $T_Y^* X$ respectively, and a vector field $\xi \in T_X^Y$ acts on v by the formula

$$< \xi \cdot v, \lambda >= \xi \cdot < v, \lambda > - < v, \xi \cdot \lambda > .$$

Thus, ξ also acts on $det(T_Y X)$ and it is easy to check that $[\xi, f] = \xi(f)$ for $f \in \mathcal{O}_X$. Since $\mathcal{D}_X^Y$ is generated by T_X^Y and $\mathcal{O}_X$, it follows that $\mathcal{D}_X^Y$ acts on $det(T_Y X)$. It is routine to check that I_Y acts trivially, so that $\mathcal{D}_X \cdot I_Y \cap \mathcal{D}_X^Y$ is in the kernel of this action. Checking that $\mathcal{D}_X^Y$ gives the full algebra of differential operators on $det(T_Y X)$ is easy to do locally. □

We can now define direct image for $\mathcal{D}_Y$ modules for the closed embedding $f : Y \to X$ by

$$f_* M = f_.(f^{\cdot}\mathcal{D}_X \otimes_{f^{\cdot}\mathcal{D}_X^Y} (M \otimes det(T_Y X)), \tag{3.2.6}$$

where $f_.$ and $f^{\cdot}$ are sheaf theoretic direct and inverse images. Here $\mathcal{D}_X^Y$ acts on M through $\mathcal{D}_Y$, and vector fields in $\mathcal{D}_X^Y$ act on the tensor product by derivations. On affine open subsets U, then the global sections of $f_* M(U)$ are given by

$$\mathcal{D}_X(U) \otimes_{\mathcal{D}_X^Y(U)} (M \otimes det(T_Y X)(U)).$$

When we are verifying local properties of f_*, we will often omit the sheaf functors $f_.$ and $f^{\cdot}$ from the notation.

3.2.7 Remark The twist by $det(T_Y X)$ in the definition of direct image can be explained if we think in terms of distributions. We would like to think of $f_*(\mathcal{O}_Y)$ as the sheaf of distributions supported on Y. In the case where $Y = \{0\} \subset \mathbb{C} = X$, the affine line, this means $f_*(\mathcal{O}_Y)$ should be spanned by the δ function at the origin and its derivatives. It is routine to check that the Euler vector field $x\partial \cdot \delta = -\delta$, so that $x\partial + 1 = \partial x$ should act as zero. This is the case for the action of $\mathcal{D}_X^Y$ on $det(T_Y X)$ but is not the case for the action of $\mathcal{D}_X^Y$ on $\mathcal{O}_Y$. Ignoring left vs. right difficulties that will be explained later, the factor $det(T_Y X)$ can also be explained as follows. Usually, we think of $\mathcal{D}$-modules as being like functions, so that it is natural to pull them back. If we want to push forward a $\mathcal{D}_Y$-module $\mathcal{M}$ to get a $\mathcal{D}_X$ module, we should think of $\mathcal{M}$ as a distribution by tensoring it with Ω_Y. Then it can be pushed forward naturally, and then we should tensor the push-forward with ${\Omega_X}^{-1}$ in order to think of the push-forward as functions again. This produces the factor $det(T_Y X)$.

3.2.8 Remark This construction of direct image is reminiscent of parabolic induction in representation theory. Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{p}$ be its parabolic subalgebra. Then $\mathfrak{p}$ can be written as a direct sum $\mathfrak{l} + \mathfrak{n}$ where $\mathfrak{n}$ is the nil-radical of $\mathfrak{p}$ (which is defined canonically) while $\mathfrak{l}$ is intrinsically defined only as a quotient $\mathfrak{p}/\mathfrak{n}$ (and the direct sum above is a non-canonical splitting). The diagram of maps of Lie algebras

$$\mathfrak{g} \hookleftarrow \mathfrak{p} \twoheadrightarrow \mathfrak{l}$$

gives rise to a diagram of maps of associative algebras

$$\mathcal{U}\mathfrak{g} \hookleftarrow \mathcal{U}\mathfrak{p} \twoheadrightarrow \mathcal{U}\mathfrak{l}.$$

hence for any $\mathcal{U}\mathfrak{l}$-module N we can consider a $\mathcal{U}\mathfrak{g}$-module $\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{p}} N$.

However, it is often more natural first to shift the action of $\mathfrak{p}$. For $x \in \mathfrak{p}$, let $\rho(x) = \frac{1}{2}Tr(x|_{\mathfrak{g}/\mathfrak{p}})$. ρ defines a one dimensional representation $\mathbb{C}_\rho$ of $\mathfrak{p}$. Then if N is a $\mathcal{U}\mathfrak{l}$-module, its twisted induced module is $\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{p}} (N \otimes \mathbb{C}_\rho)$. This twisted induction is much more natural in representation theory. The twist by ρ in representation theory plays the same role as the twist by $det(T_Y X)$ in $\mathcal{D}$-modules. □

Any morphism $f : Y \to X$ of algebraic varieties by definition induces a map of sheaves $f^{\cdot}\mathcal{O}_X \to \mathcal{O}_Y$. However, there is no covariant (or contravariant) map between differential operators, since they are constructed from functions (contravariant objects) and vector fields (covariant objects).

To illustrate the way in which this difficulty is resolved replace all the $\mathcal{D}$-modules with the corresponding graded objects which are modules over $\mathcal{O}_{T^*X}$ or $\mathcal{O}_{T^*Y}$. A map $f : Y \to X$ does not induce a map between the tangent spaces. So, instead of taking the graph of a map $T^*Y \to T^*X$ we consider the graph $Graph(f) \subset X \times Y$, and take its conormal bundle $T^*_{Graph(f)}(Y \times X) \subset T^*Y \times T^*X$ which is the only natural object that arises in this situation. This conormal bundle is only a *correspondence* (i.e. a multivalued map) between T^*Y and T^*X. The complexity of $\mathcal{D}$-module behaviour is depicted by deviation of $T^*_{Graph(f)}(Y \times X)$ from being a graph of a map.

To see how a correspondence can define a functor on some geometric objects, we think of sections of a $\mathcal{D}$-module as "functions" or "distributions". For any such "function" $F(y)$ on Y we can define its direct image under f by a formula used in physics:

$$(f_*F)(x) = \int_Y \delta(x - f(y))F(y)dy.$$

When we try to adapt this formula to our situation we view $\delta(x - f(y))$ as a $\mathcal{D}$-module supported on $Graph(f) \subset Y \times X$.

$$f_*(\mathcal{M}) = f_{\cdot}(\mathcal{B}_{Graph(f)|Y\times X} \otimes_{pr_Y^*\mathcal{D}_Y} \mathcal{M})$$

where $f_{\cdot}$ stands for sheaf-theoretic direct image which mimics the integral in the formula above. Here, if Z is a closed smooth subvariety of codimension d in a smooth variety W,

$$\mathcal{B}_{Z|W} := \mathcal{H}^d_{[Z]}(\mathcal{O}_W).$$

Proposition 3.2.9 *If Y is a smooth subvariety of X of codimension d, then*

$$\mathcal{H}^i_{[Y]}(\mathcal{O}_X) = \begin{cases} 0 & \text{if } i \neq d \\ f_*(\mathcal{O}_Y) & \text{if } i = d. \end{cases}$$

Remarks.

(1) The vanishing part of the proposition can be proved by a local computation.

(2) We will give the proof in 3.3.

3.2.10 Notation Denote by $\mathcal{D}_{Y\to X}$ the quotient $\mathcal{D}_X/I_Y \cdot \mathcal{D}_X$ (this is a particular case of the object to be introduced later).

With this notation, we can write the following equalities (the first one uses that $I_Y \cdot \Omega_Y = 0$):

$$\begin{aligned}\big(\Omega_Y \otimes_{\mathcal{D}_X^Y} \mathcal{D}_X\big) \otimes_{\mathcal{O}_X} \Omega_X^{-1} &= \big(\Omega_Y \otimes_{\mathcal{D}_X^Y} \mathcal{D}_X/I_Y \cdot \mathcal{D}_X\big) \otimes_{\mathcal{O}_X} \Omega_X^{-1} = \\ \big(\Omega_Y \otimes_{\mathcal{D}_X^Y} \mathcal{D}_{Y\to X}\big) \otimes_{\mathcal{O}_X} \Omega_X^{-1} &= \big(\Omega_Y \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}\big) \otimes_{\mathcal{O}_X} \Omega_X^{-1}.\end{aligned} \tag{3.2.11}$$

The point is that, while $\mathcal{D}_X$ is *not* locally free over $\mathcal{D}_X^Y$, we have the following

Claim **3.2.12** $\mathcal{D}_{Y\to X}$ *is locally free over* $\mathcal{D}_Y$

Sketch of proof. Locally we can choose d regular functions $t_1, \dots, t_d$ on X so that Y is given by vanishing of these functions. We can also choose (again locally) $(n-d)$ regular functions $y_1, \dots, y_{n-d}$ that have linearly independent differentials on Y (i.e. form an "etale local coordinate system" on Y). Then we can write a non-canonical splitting (depending on the choices above):

$$\mathcal{D}_X = I_Y \cdot \mathcal{D}_X \oplus \langle \partial_t, y, \partial_y \rangle.$$

which implies that $\mathcal{D}_{Y\to X} = \mathcal{D}_X/I_Y \cdot \mathcal{D}_X$ is a free module over $\mathcal{D}_Y$ with generators ∂_t. $\square$

3.2.13 Other definitions of f_* for closed embeddings.

(1) For any right $\mathcal{D}_Y$ module $\mathcal{N}$ one can form a right $\mathcal{D}_X$-module $\mathcal{N} \otimes_{\mathcal{D}_X^Y} \mathcal{D}_X$.

(2) For any left $\mathcal{D}_Y$-module $\mathcal{M}$ one can consider the left D_X-module $\mathcal{D}_X \otimes_{\mathcal{D}_X^Y} \mathcal{M}$.

We will see later that the latter of the two functors defined above is "wrong" (it does not agree with other operations on $\mathcal{D}$-modules to be defined later). So we have to redefine the functor on left $\mathcal{D}_Y$-modules by sending

$$\mathcal{M} \mapsto \Big(\big(\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y\big) \otimes_{\mathcal{D}_X^Y} \mathcal{D}_X\Big) \otimes_{\mathcal{O}_X} \Omega_X^{-1} \tag{3.2.14}$$

(i.e. we use tensoring with top degree differential forms on Y to make right modules from left modules, and then we tensor with Ω_X^{-1} to go back from right modules to left modules.) This last expression coincides with

$$\mathcal{M} \mapsto \Big(\big(\mathcal{M} \otimes_{\mathcal{O}_Y} \Omega_Y\big) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}\Big) \otimes_{\mathcal{O}_X} \Omega_X^{-1} \tag{3.2.15}$$

using (3.2.11) above.

3.2.16 Exercise Verify that $f_*(\mathcal{M})$ coincides with the expression defined in (3.2.15).

The exercise allows us to construct a good filtration on $f_*(\mathcal{O}_Y) = \mathcal{H}^d_{[Y]}\mathcal{O}_X$. It follows from the proof of claim 3.2 and the above exercise that locally $f_*(\mathcal{O}_Y) \cong \mathbb{C}[\partial/\partial t_k]\omega$, polynomials with constant coefficients in the normal derivatives times a section of $det(T_Y X)$. We filter by requiring locally that $\partial/\partial t_k \in (\mathcal{H}^d_{[Y]}\mathcal{O}_X)_1$. One deduces immediately that $SS(\mathcal{H}^d_{[Y]}\mathcal{O}_X) = T^*_Y X \subset T^*X$.

Corollary 3.2.17 *f_* is exact.*
Proof. This follows from Claim 3.2 and Exercise 3.2. □

3.3. Restriction to a submanifold; Kashiwara's theorem.

3.3.1 Pull-back of $\mathcal{D}$-modules (definition of f^+).

Let $f : Y \to X$ be a morphism. In our definition of a pullback, we want the following property to be satisfied:

> Let ϕ be a function on X and $Ann\ \phi = \{u \in \mathcal{D}_X \mid u\phi = 0\}$ be the sheaf of all differential operators that annihilate ϕ. Then the pullback of a $\mathcal{D}_X$-module $\mathcal{D}_X/Ann\ \phi$ is isomorphic to $\mathcal{D}_Y/Ann\ f^*(\phi)$.

This means that we pullback a $\mathcal{D}_X$ module by pulling back its solutions.

Recall that we have the sheaf-theoretic pullback functor $f^\cdot$ and the $\mathcal{O}$-module pullback functor f^* defined by

$$f^*(\mathcal{M}) = \mathcal{O}_Y \otimes_{f^\cdot\mathcal{O}_X} f^\cdot\mathcal{M}.$$

Claim **3.3.2** *For any left $\mathcal{D}_X$-module $\mathcal{M}$, the pullback $f^*(\mathcal{M})$ in the sense of $\mathcal{O}$-modules, has a natural $\mathcal{D}_Y$-module structure.*

Proof. Recall that we have a natural morphism $df : T_Y \to f^*T_X = \mathcal{O}_Y \otimes_{f^\cdot \mathcal{O}_X} T_X$ (the image of a tangent vector at a particular point of Y is a tangent vector at its image, but when we take a vector field of Y its image is well-defined only as a section of f^*T_X since a point in X can have many points in Y mapping to it).

Now define a $\mathcal{D}_Y$-module structure on $\mathcal{O}_Y \otimes_{f^\cdot \mathcal{O}_X} f^\cdot \mathcal{M}$ as follows. The sheaf of functions $\mathcal{O}_Y$ acts as usual, while for a vector field ξ on Y we write $df(\xi) = \sum g_i \otimes \eta_i$ and define

$$\xi(g \otimes m) = (\xi g) \otimes m + g \cdot (\sum g_i \otimes \eta_i m).$$

One can write the same formula in a slightly different form by choosing local coordinates $\{x_i\}$ on X:

$$\xi(g \otimes m) = (\xi g) \otimes m + \sum_i g(\xi f^*(x_i)) \otimes \partial_i m.$$

One checks that either of this two formulas above defines a $\mathcal{D}_Y$-module structure. $\square$

Definition 3.3.3 *We denote by f^+ the restriction of the pullback functor f^* from the category of $\mathcal{O}$-modules to the category of $\mathcal{D}$-modules.*

3.3.4 Remark One can give another definition of f^+: consider the sheaf $f^*\mathcal{D}_X$ on Y. By the claim above it has a left $\mathcal{D}_Y$-module structure and it also has a right $f^\cdot\mathcal{D}_X$-module structure. Then the equality

$$\mathcal{O}_Y \otimes_{f^\cdot \mathcal{O}_X} f^\cdot \mathcal{M} \simeq f^*\mathcal{D}_X \otimes_{f^\cdot \mathcal{D}_X} f^\cdot \mathcal{M}$$

allows us to define $f^+(\mathcal{M})$ as $f^*\mathcal{D}_X \otimes_{f^\cdot \mathcal{D}_X} f^\cdot \mathcal{M}$.

3.3.5 Notation The sheaf $f^*\mathcal{D}_X$ above will be denoted by $\mathcal{D}_{Y\to X}$.

3.3.6 Another pullback for closed embeddings (definition of $f^!$).

Let $f : Y \hookrightarrow X$ be a closed embedding. We define a functor

$$f^! : \mathrm{mod} - \mathcal{D}_X \to \mathrm{mod} - \mathcal{D}_Y$$

by

$$f^!(\mathcal{M}) = det(T_Y^* X) \otimes_{\mathcal{O}_Y} {}^{I_Y}\mathcal{M}$$

for a (left) $\mathcal{D}_X$-module $\mathcal{M}$. To see that $f^!(\mathcal{M})$ is a $\mathcal{D}_Y$-module, note that ${}^{I_Y}\mathcal{M}$ has a natural action by $\mathcal{D}_X^Y/(\mathcal{D}_X \cdot I_Y) \cap \mathcal{D}_X^Y$, the differential operators on $det(T_Y X)$ by Proposition 3.2. After we twist by $det(T_Y^* X)$, we have action by $\mathcal{D}_Y$.

3.3.7 Properties of f^+ and $f^!$ for closed embeddings.

Proposition 3.3.8 *Let $f : Y \hookrightarrow X$ be a closed embedding of codimension d. Then*

(i) $f^!$ is left exact (hence we can speak of right derived functors $R^i f^!$),

(ii) f^+ is right exact (and we can consider the left derived functors $L^i f^+$), and

(iii) For any left $\mathcal{D}_X$-module $\mathcal{M}$ there exist natural isomorphisms

$$R^d f^!(\mathcal{M}) \simeq f^+(\mathcal{M}) \text{ and } L^{-d} f^+(\mathcal{M}) \simeq f^!(\mathcal{M})$$

(iv) The functor f_* is left adjoint to $f^!$:

$$Hom_{\mathcal{D}_X}(f_*\mathcal{N}, \mathcal{M}) = Hom_{\mathcal{D}_Y}(\mathcal{N}, f^!\mathcal{M})$$

Proof. The first asserion follows from the definition of $f^!$ and left exactness of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \cdot)$. The second assertion is a consequence of right exactness of $(\mathcal{O}_Y \otimes_{f^\cdot \mathcal{O}_X} \cdot\,)$. The third claim follows from the equalities

$$R^i f^!(\mathcal{M}) \simeq det(T_Y^* X) \otimes_{\mathcal{O}_Y} \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}), \quad L^{-i} f^*(\mathcal{M}) \simeq \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M})$$

(which can be proved using uniqueness of derived functors) and the isomorphism of sheaves

$$\mathcal{E}xt^{d-i}_{\mathcal{O}_X}\big(\mathcal{O}_Y, \mathcal{M}\big) \simeq \mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{M}) \otimes_{\mathcal{O}_Y} det(T_Y X).$$

The latter isomorphism can be proved in local coordinates by considering the Koszul complex and using its self-duality property.

By easy sheaf arguments, it suffices to prove (iv) on an affine open cover and to check that the identifications agree on intersections. On affine open sets, the adjunction property (iv) follows by Frobenius reciprocity. We have

$$Hom_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{D}_X^Y} (\mathcal{M} \otimes det(T_Y X), \mathcal{N})) = Hom_{\mathcal{D}_X^Y}(\mathcal{M} \otimes det(T_Y X), \mathcal{N})$$

$$= Hom_{\mathcal{D}_{det(T_Y X)}}(\mathcal{M} \otimes det(T_Y X), \ {}^{I_Y}\mathcal{N}) = Hom_{\mathcal{D}_Y}(\mathcal{M}, det(T_Y^* X) \otimes {}^{I_Y}\mathcal{N})$$

where the second equality follows since I_Y annihilates $\mathcal{M} \otimes det(T_Y X)$. Identifications agree on intersections of affine open sets since they are canonical. □

3.3.9 Remark The proposition above essentially says that for a closed embedding the functor f^+ is "unnecessary" since it can be expressed via $f^!$.

Proposition 3.3.10

(i) If $g : Z \to Y$ and $f : Y \to X$ are arbitrary morphisms then for any $\mathcal{D}_X$-module $\mathcal{M}$ $(fg)^+(\mathcal{M}) \simeq g^+(f^+(\mathcal{M}))$.

(ii) If g and f are closed embeddings then $(fg)^!(\mathcal{M}) = g^!(f^!(\mathcal{M}))$ and $(fg)_*(\mathcal{N}) \simeq f_*(g_*(\mathcal{N}))$.

Proof. The first claim follows directly fron the definition of f^+ and the corresponding statement from algebraic geometry. Recall that one can also write $f^+(\mathcal{M}) = \mathcal{D}_{Y\to X} \otimes_{f^\cdot \mathcal{D}_X} \mathcal{M}$. Hence (i) applied to $\mathcal{M} = \mathcal{D}_X$ gives

$$\mathcal{D}_{Z\to X} \simeq \mathcal{D}_{Z\to Y} \otimes_{g^\cdot \mathcal{D}_Y} g^\cdot \mathcal{D}_{Y\to X}. \tag{3.3.11}$$

3.3.12 Exercise Prove (3.3.11) form the point of view of Sato construction.

The first claim of (ii) is routine to check, and the second claim follows from uniqueness of the adjoint functor from the first.

3.3.13 Kashiwara's Theorem.

Theorem 3.3.13.1. *Let $f : Y \hookrightarrow X$ be a closed embedding and $\mathcal{M}$ be a $\mathcal{D}_Y$-module. Then*

(i) $f^! f_*(\mathcal{M}) \simeq \mathcal{M}$ (i.e. $f^!$ is left inverse of f_*).

(ii) If a $\mathcal{D}_X$-module $\mathcal{N}$ is supported on Y (in the sense on $\mathcal{O}$-modules) then $\mathcal{N} \simeq f_* f^! \mathcal{N}$.

(iii) In this way one has an equivalence of categories:

$$f_* : \big(\mathcal{D}_Y\text{-modules}\,\big) \to \big(\mathcal{D}_X\text{-modules supported on } Y\big).$$

3.3.14 Remark

The analogue of Kashiwara's theorem for the category of $\mathcal{O}$-modules fails for the obvious reason: $\mathcal{O}_X$-modules supported on Y have a non-trivial invariant: *multiplicity* along Y. However, for $\mathcal{D}$-modules, differentiating in directions transversal to Y destroys this invariant. For example, on the affine line, $\mathbb{C}[t]/(t^n)$ is not a $\mathcal{D}$-module.

Proof of Kashiwara's Theorem. We will proceed in several steps.

Step 1. There exists a canonical adjunction map

$$\mathcal{M} \to f^! f_*(\mathcal{M}) = det(T_Y^* X) \otimes\ ^{I_Y}(\mathcal{D}_X \otimes_{\mathcal{D}_X^Y} (\mathcal{M} \otimes det(T_Y X)))$$

that in local coordinates maps m to $\phi \otimes 1 \otimes m \otimes \phi^{-1}$, where ϕ is a local nonvanishing section of $det(T_Y^* X)$ and ϕ^{-1} is the inverse section. Using local coordinates as in the proof of Claim 3.2 one can see that this map is an isomorphism. This proves (i).

Step 2. There exists a canonical adjunction map $f_* f^!(\mathcal{N}) \to \mathcal{N}$. To prove that it is an isomorphism we can represent Y locally as intersection of hypersurfaces and by functoriality of f_* and $f^!$ (cf. Proposition 3.3) we may assume that $codim(Y, X) = 1$. Hence locally Y is given by one equation $Y = \{t = 0\}$. Put $\partial := \partial/\partial t$ and $\theta := t \cdot \partial$. For $\mathcal{N}$ as in (ii), let

$$\mathcal{N}(i) := \{n \in \mathcal{N} | \theta \cdot m = i\, m\}.$$

Since $\partial \cdot t - t \cdot \partial = 1$, we have maps:

$$t : \mathcal{N}(i) \to \mathcal{N}(i+1), \quad , \quad \partial : \mathcal{N}(i+1) \to \mathcal{N}(i).$$

For $i < -1$, $\partial \cdot t$ and $t \cdot \partial$ are invertible operators on $\mathcal{N}(i)$, $\mathcal{N}(i+1)$, hence the maps t, ∂ above are isomorphisms.

Step 3. Consider $\mathcal{N}^I = \{n \in \mathcal{N} \mid tn = 0\}$. We want to show that locally $\mathcal{N} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} \mathcal{N}^I$. In fact, for any $n \in \mathcal{N}^I$, $\theta \cdot n = -n$. Therefore $\mathcal{N}_I \subset \mathcal{N}(-1)$ and $\partial^i \cdot \mathcal{N}^I \subset \mathcal{N}(-i-1)$. Now, by the Nullstellensatz, since $\mathcal{N}$ is supported on Y, any element n of $\mathcal{N}$ is annihilated by some power of t, say t^N. Hence $t^{N-1} n \in \mathcal{N}^I$ and $n \in \partial^{N-1} \cdot \mathcal{N}^I$. In particular $\mathcal{N}^I = \mathcal{N}(-1)$. The rest of the proof follows easily. □

3.3.15 $\mathcal{D}$-modules on singular varieties.

We assumed everywhere that Y is a smooth subvariety of X. For a singular variety Y, the category of $\mathcal{D}_Y$-modules can (and should) be defined by choosing a closed embedding $Y \hookrightarrow X$ and considering the subcategory of all $\mathcal{D}_X$-modules supported on Y. By Kashiwara's theorem, the definition of $\mathcal{D}_Y$-modules is independent of X. Using the equivalence between left and right $\mathcal{D}$ modules, we can define right $\mathcal{D}_Y$-modules in the same way.

3.3.16 $\mathcal{D}$-crystals. Let $\mu : X \to Y$ is a finite morphism of schemes. Then there is the Grothendieck functor $\mu^! : \mathcal{O}_Y - mod \to \mathcal{O}_X - mod$ satisfying the adjunction formula

$$\mathcal{H}om_{\mathcal{O}_Y}(M, \mu^! N) = \mathcal{H}om_{\mathcal{O}_X}(\mu.M, N).$$

When $X = SpecA$ and $Y = SpecB$, by letting $M = B$, we see that $\mu^!(N) = Hom_{B-mod}(A, N)$. In other words, if $A = B/J$ then $\mu^!(N)$ is the maximal submodule of N annihilated by J. It follows immediately from the adjunction property that for any pair of morphisms π, μ as above one has $(\mu \circ \pi)^! = \pi^! \circ \mu^!$.

Let Y be an affine variety (possibly singular). We say that a morphism $Y \hookrightarrow X$ is a *nilpotent thickening* of Y, if Y is a closed subscheme of X defined by a nilpotent ideal. Morphisms between two thickenings are defined in an obvious way.

We will consider in particular nilpotent thickenings of Y given by the diagonal embeddings $Y \to Y^m$, $y \mapsto (y, y, \dots, y)$. Let I be the ideal defining the diagonal and let Y^m_n be the *nth* infinitesimal neighborhood of the diagonal, the scheme defined by I^{n+1}, i.e., $Y^m_n = Spec(\mathcal{O}_{Y^m})/I^{n+1}$. We have the projections to the *ith* factor, $p_{i,n} : Y^m_n \to$

Y, and the obvious closed embeddings $i_n : Y \hookrightarrow Y_n^m$. Consider also the formal completion $Y^{<m>}$ corresponding to the diagonal embedding and the category of discrete $\mathcal{O}$-modules on $Y^{<m>}$, i.e., modules whose sections are supported on some Y_n^m. Note that from the description $p_{i,n}^! \mathcal{F} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{Y_n^m}, \mathcal{F})$, we have inclusions $p_{i,n}^! \mathcal{F} \subset p_{i,n+1}^! \mathcal{F}$. There are projections onto the *ith* factor $p_i : Y^{<m>} \to Y$. We define the discrete module $p_i^! \mathcal{F}$ as the union of all $p_{i,n}^! \mathcal{F}$. The functors $p_i^!$ satisfy similar adjunction properties.

In the remainder of this section, we will focus on the completions $Y^{<2>}$ and $Y^{<3>}$, and denote the projection on the *ith* factor $Y^{<3>} \to Y$ by q_i. We will denote the projections to the *ith* and *jth* factor $Y^{<3>} \to Y^{<2>}$ by p_{ij}. Functors $p_{ij}^!$ are defined as above, and are right adjoint to the corresponding direct image functors.

Proposition 3.3.17 [Grothendieck] *Let $\mathcal{F}$ be a $\mathcal{O}_Y$ module and assume that Y is smooth.*

$$p_1^! \mathcal{F} \cong \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$$

where the tensor product is defined using the left $\mathcal{O}_Y$ module structure on $\mathcal{D}_Y$. Moreover,

$$q_1^! \mathcal{F} \cong \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y.$$

Proof. First let $\mathcal{F} = \mathcal{O}_Y$. A differential operator ∂ of order n defines a section of $p_{1,n}^! \mathcal{F} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_{Y^n}, \mathcal{O}_Y)$ by the formula, $\phi_\partial : f \otimes g \mapsto f\partial(g)$. One checks by induction that if ∂ is a nth order operator, then ϕ_∂ annihilates the ideal defining Y_n, so ϕ_∂ is well-defined. The inverse map is given by $\phi \to \partial_\phi$, where $\partial_\phi(h) = \phi(1 \otimes h)$. Hence, $\mathcal{D}_Y \cong p_1^! \mathcal{O}_Y$. The general case follows from the isomorphism $p_1^! \mathcal{F} \cong \mathcal{F} \otimes p_1^! \mathcal{O}_Y$, which can be proved by using the freeness of $\mathcal{O}_{Y^{<2>}}$ over $\mathcal{O}_Y$. The second claim follows as above by defining a map $\phi : \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to q_1^!(\mathcal{O}_Y)$ by $\phi(\partial_1 \otimes \partial_2)(f_1 \otimes f_2 \otimes f_3) = f_1 \partial_1(f_2 \partial_2(f_3))$. $\square$

Definition 3.3.18 A $\mathcal{D}$-*crystal* on Y is a collection of $\mathcal{O}_X$-modules $\mathcal{F}_X$ on all nilpotent thickenings $Y \hookrightarrow X$, such that for any morphism

$$\begin{tikzcd}
 & X \arrow[dd, "\mu"] \\
Y \arrow[ur] \arrow[dr] & \\
 & X'
\end{tikzcd}$$

an isomorphism $\mu^!(\mathcal{F}_{X'}) \simeq \mathcal{F}_X$ compatible with compositions is given. The category of all $\mathcal{D}$-crystals on Y is denoted by $M_{crys}(Y)$.

We recall the notion of a formally smooth algebra due to Grothendieck ([**EGA IV**] or [**Mt**]). A noetherian topological ring A is said to be formally smooth if for any discrete ring R with a nilpotent ideal I, any homomorphism $A \to R/I$ lifts to a homomorphism $A \to R$. It suffices to requires the above lifting property for any ideal I with $I^2 = 0$. If A is the ring of functions on a smooth variety (with the discrete topology on A), then A is formally smooth (see [**Ha1**], Exer. II.8.6, pp. 188-189). Moreover, if A is formally smooth and I is an ideal of A, then the completion $\hat{A}$ of A along I (with I-adic topology) is formally smooth.

We give an equivalent formulation of the notion of a $\mathcal{D}$-crystal. We choose a closed embedding of Y into some smooth variety Z. By taking the formal completion V of Z along Y, we may find an embedding $k : Y \to V$, a formally smooth variety. As for $Y^{<2>}$, a $\mathcal{D}$-crystal determines a discrete $\mathcal{O}$-module on $V^{<2>}$. Note that when $Y = Z$, we may take $V = Z$.

$$\begin{tikzcd} & V^{<2>} \arrow[dd, "p_i"] \\ Y \arrow[ur, "k\times k"] \arrow[dr, "k"'] & \\ & V \end{tikzcd}$$

we obtain an $\mathcal{O}_{V^{<2>}}$ module isomorphism $\mathcal{F}_{V^{<2>}} \cong p_i^! \mathcal{F}_V$. By composing two of these we obtain an isomorphism

$$\tau \in \mathcal{H}om_{\mathcal{O}_{V^{<2>}}}(p_1^! \mathcal{F}_V, p_2^! \mathcal{F}_V).$$

It is easy to check that

$$p_{23}^!(\tau) p_{12}^!(\tau) = p_{13}^!(\tau). \tag{3.3.19}$$

Conversely, given $\tau \in \mathcal{H}om_{\mathcal{O}_{V^{<2>}}}(p_1^! \mathcal{F}_V, p_2^! \mathcal{F}_V)$ satisfying (3.3.19), we can recover the $\mathcal{D}$-crystal $\mathcal{F}$. Since both notions of $\mathcal{D}$-crystal are defined locally, we may assume Y is affine. Consider a nilpotent thickening $i : Y \to X$. Since V is formally smooth, we may choose a finite morphism $j : X \to V$ such that $j \circ i = k$. Then $j^! \mathcal{F}_V$ is a $\mathcal{O}_X$ module, which we can take to be $\mathcal{F}_X$. If $l : X \to V$ is another finite morphism such that $l \circ i = k$, then we claim there is a natural isomorphism $j^! \mathcal{F}_V \cong l^! \mathcal{F}_V$. Indeed, we obtain a morphism $(j, l) : X \to V^{<2>}$ and $j^! \mathcal{F}_V = (j, l)^! p_1^! \mathcal{F}_V$, $l^! \mathcal{F}_V = (j, l)^! p_2^! \mathcal{F}_V$. Then $(j, l)^!(\tau) : j^! \mathcal{F}_V \cong l^! \mathcal{F}_V$ is our natural isomorphism. We can show that $\mathcal{F}_X$ defines a $\mathcal{D}$-crystal.

Indeed, the structure isomorphisms can be given using the above argument, and compatibility follows from (3.3.19).

Thus, we have a description of $M_{crys}(Y)$ in terms of $V^{<i>}$. Using this description, we see that if $i : Y \to Z$ is a closed embedding into a smooth variety, there are functors $i_* : M_{crys}(Y) \to M_{crys}(Z)$ and $i^! : M_{crys}(Z) \to M_{crys}(Y)$. Indeed, consider V as above, and the induced map $\tilde{i} : V \to Z$. For i_*, consider the sheaf $\tilde{i}.\mathcal{F}_V$ with morphisms $(\tilde{i} \times \tilde{i}).p_1^!\mathcal{F}_V \to (\tilde{i} \times \tilde{i}).p_2^!\mathcal{F}_V$. The reader can check that $(\tilde{i} \times \tilde{i}).p_j^! = p_j^!\tilde{i}.$, and that (3.3.19) is satisfied. This defines i_*. The functors $i^!$ are left to the reader to define.

Let $M_{crys}(Z)_Y$ be the $\mathcal{D}$-crystals $\mathcal{F}_Z$ such that $j^*\mathcal{F}_Z = 0$, where $j : Z - Y \to Z$ is the open embedding. These are $\mathcal{D}$-crystals supported on Y.

Lemma 3.3.20 *The functors i_* and $i^!$ induce an equivalence of categories between $M_{crys}(Y)$ and $M_{crys}(Z)_Y$.*

The proof is easy, and left to the reader.

Proposition 3.3.21 *There exists a natural equivalence between the category of right $\mathcal{D}_Y$-modules (in the sense of the last subsection) and $M_{crys}(Y)$.*

Proof. Suppose first that Y is non-singular and $\mathcal{F}$ is a $\mathcal{D}$-crystal on Y. Then we are given an isomorphism $\tau \in \mathcal{H}om_{\mathcal{O}_{Y^{<2>}}}(p_1^!\mathcal{F}_Y, p_2^!\mathcal{F}_Y)$ satisfying (3.3.19). Using the isomorphism $p_1^!\mathcal{F}_Y \cong \mathcal{F}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$ and adjunction, we may regard τ as $a(\tau) \in \mathcal{H}om_{p_2^{-1}\mathcal{O}_Y}(\mathcal{F}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, \mathcal{F}_Y)$. We claim that $a(\tau)$ makes $\mathcal{F}_Y$ into a right $\mathcal{D}_Y$-module. The property (3.3.19) implies that $a(\tau)$ is a ring action. Indeed,

$$p_{13}^!(\tau) \in \mathcal{H}om_{\mathcal{O}_{Y^{<3>}}}(p_{13}^!p_1^!\mathcal{F}_Y, p_{13}^!p_2^!\mathcal{F}_Y) = \mathcal{H}om_{\mathcal{O}_{Y^{<3>}}}(q_1^!\mathcal{F}_Y, q_3^!\mathcal{F}_Y).$$

Using Proposition 3.3 and adjunction, we may regard

$$p_{13}^!(\tau) \in \mathcal{H}om_{\mathcal{O}_{Y^{<3>}}}(\mathcal{F}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, q_3^!\mathcal{F}_Y) =$$

$$\mathcal{H}om_{q_3^{-1}\mathcal{O}_Y}(\mathcal{F}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, \mathcal{F}_Y).$$

By chasing through the identifications, one can check that $p_{13}^!(\tau)$ corresponds to the map $v_1 \otimes \partial_2 \otimes \partial_3 \to a(\tau)(v_1 \otimes \partial_2\partial_3)$. Similarly, $p_{23}^!(\tau)p_{12}^!(\tau)$ corresponds to the map $v_1 \otimes \partial_2 \otimes \partial_3 \to a(\tau)(a(\tau)(v_1 \otimes \partial_2) \otimes \partial_3)$. Thus, $a(\tau)$ gives a right action. By pulling back to Y, one sees that $a(\tau)(v \otimes f) = fv$, for $v \in \mathcal{F}_Y$, $f \in \mathcal{O}_Y$. Hence, $\mathcal{F}_Y$ is a right $\mathcal{D}_Y$-module.

Conversely, if $\mathcal{F}_Y$ is a right $\mathcal{D}_Y$-module, by adjunction we get an $\mathcal{O}_{Y^{<2>}}$ morphism $\tau : p_1^!\mathcal{F}_Y \to p_2^!\mathcal{F}_Y$ which satisfies (3.3.19) from the action hypothesis. Moreover, we can prove by induction that τ is an isomorphism. Recall that $p_{1,k}^!\mathcal{F}_Y$ consists of functions vanishing on

I^{k+1}. It is clear that τ is an isomorphism on functions living on the $0th$ infinitesimal neighborhood and that $\tau : p^!_{1,k}\mathcal{F}_Y \to p^!_{2,k}\mathcal{F}_Y$. Assume that τ is an isomorphism through the kth infinitesimal neighborhood and let $\phi \in p^!_{1,k+1}\mathcal{F}_Y$ be such that $\tau(\phi) = 0$. Let $dx = 1 \otimes x - x \otimes 1 \in I$. Then $\tau(dx \cdot \phi) = dx \cdot \tau(\phi) = 0$. The morphism $dx \cdot \phi \in p^!_{1,k}\mathcal{F}_Y$, so $dx \cdot \phi = 0$ for all $x \in \mathcal{O}_Y$. Hence, ϕ vanishes on $I \cdot I^k$, so $\phi = 0$. Surjectivity follows from an induction argument, which we leave to the reader.

Now suppose Y is arbitrary, and choose a closed embedding $Y \to Z$. By Lemma 3.3, we may regard a $\mathcal{D}$-crystal on Y as a D-crystal on Z supported on Y. It is routine to check that the corresponding right $\mathcal{D}_Z$ module $\mathcal{F}_Z$ is supported on Y, and hence defines a $\mathcal{D}$-module on Y. □

3.3.22 Remark A more direct way to associate a $\mathcal{D}$-crystal to a right $\mathcal{D}_Y$-module can be given as follows. Let $Y \hookrightarrow X$ be a nilpotent thickening. Choose a closed embedding $X \hookrightarrow Z$ of X into a smooth variety Z. Then any object $\mathcal{M}$ of $\mathcal{D}_Y - mod$ is represented by some $\mathcal{D}_Z$-module $\mathcal{M}_Z$ supported on Y. Let $\mathcal{F}_X = Hom_{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{M}_Z)$.

See [**BD**] for more on $\mathcal{D}$-crystals.

Now we give an interpretation of local cohomology in terms of the functors introduced above.

Proposition 3.3.23 *If $i : Y \hookrightarrow X$ is a closed embedding then*

$$\Gamma_{[Y]}\mathcal{M} \cong i_* i^! \mathcal{M}.$$

Moreover, the same property is true for higher derived functors.
Proof. First note one has a diagram

$$\begin{tikzcd} i_* i^! \Gamma_{[Y]}\mathcal{M} \arrow[r] \arrow[d] & i_* i^! \mathcal{M} \arrow[d] \\ \Gamma_{[Y]}\mathcal{M} \arrow[r] & \mathcal{M} \end{tikzcd}$$

where the vertical arrows are the augmentation morphisms. It follows from the definitions that the top arrow is in fact an equality. Moreover, by Kashiwara's theorem the left arrow is an isomorphism. Hence $\Gamma_{[Y]}\mathcal{M} = i_* i^! \mathcal{M}$ as submodules of $\mathcal{M}$. To prove the property for higher derived functors, note that since $i^!$ is left exact and i_* is exact, $i_* i^!$ is left exact. Since $\Gamma_{[Y]}$ is also left exact, the property follows from uniqueness of higher derived functors. □
Proof of Proposition 3.2. Use the above Proposition and Proposition 3.3 and Kashiwara's theorem. □.

3.3.24 Remark The functor $i^!$ takes just those sections of $\mathcal{M}$ that are annihilated by the ideal sheaf I_Y itself. By "differentiating in transversal directions" the functor i_* restores the sections of $\mathcal{M}$ that are annihilated by higher powers of I_Y.

3.3.25 Applications of Kashiwara's Theorem.

Let $f : Y \hookrightarrow X$ be a closed submanifold. Recall that one has an exact sequence of vector bundles on Y:

$$0 \to T^*_Y X \to T^*X|_Y \xrightarrow{\pi} T^*Y \to 0.$$

Corollary 3.3.26 *(Bernstein) If $\mathcal{N}$ is a coherent $\mathcal{D}_Y$-module then $f_*\mathcal{N}$ is a coherent $\mathcal{D}_X$-module and, moreover, $SS(f_*\mathcal{N}) = \pi^{-1}(SS\,\mathcal{N})$.*

Proof. Locally Y is given by vanishing of some set of funtions $t_1, \dots , t_d$. Then locally one has $f_*\mathcal{N} = \mathcal{D}_Y[\partial_t] \otimes_{\mathcal{D}_Y} \mathcal{N} = \mathcal{N}[\partial_t]$ and both claims of the corollary follow. $\square$

Corollary 3.3.27 *(Weak Gabber theorem) If $\mathcal{M}$ is a $\mathcal{D}_X$-module and* $\dim SS\,\mathcal{M} < \dim X$ *then $\mathcal{M} = 0$.*
Proof. The characteristic variety $SS\,\mathcal{M} \subset T^*X$ always projects surjectively onto *Supp* $\mathcal{M} \subset X$. Hence if *Supp*$\mathcal{M} = X$ we are done. Hence we can assume that $Y =$ *Supp* $\mathcal{M}$ is a proper subvariety of X. We can choose an affine open subset U such that the intersection $Y^0 = U \cap Y$ coincides with the smooth locus of a certain irreducible component of Y of maximal dimension (i.e. $\dim Y$). Replace X by U and Y by Y^0 and apply Kashiwara's theorem to $\mathcal{M}|_U$. It follows that $M = i_*\mathcal{N}$ where $N = i^!\mathcal{M}$. By the previous corollary, $SS\,\mathcal{M} = \pi^{-1}(SS\,\mathcal{N})$. Arguing by induction on dimension we can assume that $\dim SS\,\mathcal{N} \geq \dim Y$ hence $\dim SS\,\mathcal{M} \geq \dim X$. $\square$

3.3.28 Remark Comparing this weak form of Gabber theorem with the original statement one notices two differences: first, "strong" Gabber's theorem implies that *every* irreducible component of $SS\,\mathcal{M}$ has dimension at least $\dim X$; and second, the weak version says nothing about the coisotropicity property.

3.3.29 f_* for open embeddings.

If $j : U \hookrightarrow X$ is an open embedding, then the higher direct image sheaves $R^ij_*(M)$ have $\mathcal{D}_X$-module structure (constructed exactly as in the case of cohomology with support, i.e. either via action on the Cech complex or via the local cohomology construction using $R^ij_*(M) \simeq \mathcal{H}^{j+1}_{[X\setminus U]}(M)$).

3.4. Beilinson-Bernstein theorem for projective space. Let X be a d-dimensional projective space $\mathbb{CP}^d$.

THEOREM 3.4.0.1. *(Beilinson-Bernstein)*
(i) The functor of global sections

$$\Gamma : \mathcal{M} \to \Gamma(X, \mathcal{M})$$

is exact on the categry of $\mathcal{D}$-modules on X.
(ii) Every $\mathcal{D}_X$-module is generated by its global sections.

3.4.1 Remarks.
(1) Both claims of the theorem can be restated as a claim that $\mathcal{D}_{\mathbb{C}\,\mathbb{P}^n}$ is a projective generator in the category of $\mathcal{D}$-modules on $\mathbb{C}\,\mathbb{P}^n$.
(2) A well known analogue of this theorem for the category of $\mathcal{O}_X$-modules (due to Serre) claims that the functor of global sections is exact on the category of $\mathcal{O}$-modules on an algebraic variety X if and only if X is affine. For this reason Beilinson and Bernstein call the varieties satisfying the property above *$\mathcal{D}$-affine.* The only known examples of such varieties are products of affine varieties and projective homogeneous spaces of reductive Lie groups.
(3) From this theorem one deduces immediately that there exists an equivalence of categories:

$$\{\text{coherent } \mathcal{D}_{\mathbb{CP}^d} - \text{modules}\} \longleftrightarrow \{\text{f.g. modules over } \Gamma(\mathbb{CP}^d, \mathcal{D}_{\mathbb{CP}^d})\}$$

Moreover, if $\mathbb{CP}^d = \mathbb{P}(V)$ and $x_0, \dots, x_d$ are coordinates on V, then the algebra $\Gamma(\mathbb{CP}^d, \mathcal{D}_{\mathbb{CP}^d})$ is generated over $\Gamma(\mathbb{CP}^d, \mathcal{O}_{\mathbb{CP}^d}) = \mathbb{C}$ by the global vector fields $v_{ij} = x_j \partial_j$ satisfying the following relations:

$$(x_i\partial_j)(x_k\partial_l) = (x_i\partial_l)(x_k\partial_j) + \delta_{jk} \cdot (x_i\partial_l) - \delta_{kl}(x_i\partial_j).$$

Thus $\Gamma(\mathbb{CP}^d, \mathcal{D}_{\mathbb{CP}^d})$ is a quotient of the universal enveloping algebra $\mathcal{U}\,\mathfrak{gl}(V)$. The reason for this is that $\mathfrak{gl}(V)$ acts on $\mathbb{P}(V)$ by vector fields, hence there is a morphism $\mathfrak{gl}(V) \to \Gamma(\mathbb{CP}^d, \mathcal{T}_{\mathbb{CP}^d})$ which extends to a morphism $\mathcal{U}\,\mathfrak{gl}(V) \to \Gamma(\mathbb{CP}^d, \mathcal{D}_{\mathbb{CP}^d})$ by the universal property of $\mathcal{U}\,\mathfrak{gl}(V)$.

Lemma 3.4.2 *The morphism of sheaves $\mathcal{O}_{\mathbb{CP}^d} \otimes \mathcal{U}\,\mathfrak{gl}(V) \to \mathcal{D}_{\mathbb{CP}^d}$ is surjective. In particular, $\mathcal{D}_{\mathbb{CP}^d}$ is generated by its global sections.*
Proof. This follows from the surjectivity of the associated graded map $\mathcal{O}_{\mathbb{CP}^d} \otimes S\,\mathfrak{gl}(V) \to S\,T_{\mathbb{CP}^d}$. $\square$

One can show in a similar way that the relations above form a complete set.

Proof of the theorem. As in the remark above, we assume that $\mathbb{CP}^d = \mathbb{P}(V)$. We also denote by V_* the open subset $V \setminus \{0\}$ of V and consider the maps:

$$\mathbb{P}(V) \xleftarrow{p} V_* \overset{j}{\hookrightarrow} V.$$

We write $j.$ for the sheaf-theoretic direct image and compare $\Gamma(V, j.p^*\mathcal{M})$ with $\Gamma(\mathbb{P}(V), \mathcal{M})$. First of all, note that there is a natural embedding $\Gamma(\mathbb{P}(V), \mathcal{M}) \hookrightarrow \Gamma(V, j.p^*\mathcal{M})$. To identify $\Gamma(\mathbb{P}(V), \mathcal{M})$ as a subspace of $\Gamma(V, j.p^*\mathcal{M})$, recall that the Euler vector field $\theta = \sum_i x_i\partial_i$ measures the homogeneous degree of any polynomial (or tensor) on V (or V_*). One can easily see that the image of $\Gamma(\mathbb{P}(V), \mathcal{M})$ in $\Gamma(V, j.p^*\mathcal{M})$ coincides with the subspace annihilated by θ.

Hence the theorem reduces to showing that the functor

$$\mathcal{M} \to \Gamma(V, j.p^*\mathcal{M})^\theta$$

is exact. To do that we consider the exactness properties of the various functors involved:

(1) p^* is exact since p is a smooth morphism.

(2) $j.$ is left exact, but it is not right exact since $j.$ is not affine.

(3) Γ is exact on V since V is affine.

(4) $\big(\dots\big)^\theta$ is exact on modules of the type $j.p^*\mathcal{M}$ since the action of θ on them is semisimple.

Therefore, a short exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to M'' \to 0$ of $\mathcal{D}_{\mathbb{P}(V)}$-modules induces a short exact sequence $0 \to p^*\mathcal{M}' \to p^*\mathcal{M} \to p^*M'' \to 0$. Applying $j.$ we obtain the long exact sequence

$$0 \to j.p^*\mathcal{M}' \to j.p^*\mathcal{M} \to j.p^*M'' \to R^1j.p^*\mathcal{M}' \to \dots \quad . \tag{3.4.3}$$

Now notice that *the eigenvalues of θ on the global sections of $R^{\geq 1}j.p^*\mathcal{M}$ are negative.* In fact, since j is an isomorphism away from $0 \in V$, we have $Supp(R^{\geq 1}j.p^*\mathcal{M}) = \{0\}$. We also know that each sheaf $R^{\geq 1}j.p^*\mathcal{M}$ has a structure of a $\mathcal{D}_V$-module by 3.3. By Kashiwara's theorem any such module has to be of the type $i_*(W)$ where i is the embedding $\{0\} \hookrightarrow V$ and W is a coherent $\mathcal{D}$-module on $\{0\}$, i.e. just a finite-dimensional vector space. We can assume without loss of generality that $W = \mathbb{C}$. Then $i_*(\mathbb{C}) = \mathbb{C}[\partial_0, \dots, \partial_d] \cdot \delta$. But $\theta \cdot \delta = -(\dim V)\delta$ (since $x_i\partial_i \cdot \delta = -\delta + \partial_i x_i \cdot \delta = \delta$), and applying ∂_i to δ can only decrease the eigenvalue of θ. Hence the eigenvalues of θ on $\Gamma(V, R^1j.p^*\mathcal{M}')$ are negative and the long exact sequence (3.4.3) induces the short exact sequence

$$0 \to \Gamma(V, j.p^*\mathcal{M}')^\theta \to \Gamma(V, j.p^*\mathcal{M})^\theta \to \Gamma(V, j.p^*\mathcal{M}'')^\theta \to 0$$

hence the short exact sequence

$$0 \to \Gamma(\mathbb{P}(V), M') \to \Gamma(\mathbb{P}(V), M) \to \Gamma(\mathbb{P}(V), M'') \to 0.$$

This proves (i).

To prove (ii) let $\mathcal{M}' \subset \mathcal{M}$ be the submodule generated by the global sections of $\mathcal{M}$. One has an exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

and by exactness of $\Gamma(\mathbb{P}(V), \cdot)$ this gives $\Gamma(\mathbb{P}(V), \mathcal{M}'') = 0$. We will show that this implies $\mathcal{M}'' = 0$. In fact, consider the decomposition of $\Gamma(j.p^*\mathcal{M}'')$ into a sum of eigenspaces with respect to θ:

$$\Gamma(j.p^*\mathcal{M}'') = \sum_{n\in\mathbb{Z}} \Gamma(n).$$

We have shown before that $\Gamma(\mathbb{P}(V), \mathcal{M}'') = \Gamma(0)$. Similarly one can show that $\Gamma(\mathbb{P}(V), \mathcal{M}''(n)) = \Gamma(n)$. Hence if $\mathcal{M}'' \neq 0$ then $\Gamma(n) \neq 0$ for $n \gg 0$. Take a non-zero section $m \in \Gamma(n)$ for such n. If all ∂_i annihilate m, then $\theta \cdot m = 0$, which is a contradiction. Hence, by appplying some nth order constant coefficient operator to m, we will get a non-zero element of $\Gamma(0)$, giving a contradiction. □

3.5. Functors $f^!$ and f_* for a general map.

Let $f : Y \to X$ be a morphism of varieties. The functor f^+ from $\mathcal{D}_X$ modules to $\mathcal{D}_Y$ modules was defined in 3.3. We want to generalize the functor $f^!$, but we will do this only on the derived category. Recall that we denote by $D^b(\mathcal{D}_X)$ the derived category of bounded complexes of $\mathcal{D}_X$ modules. Denote also by $f^+ : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$ the derived extension of f^+. It is computed by replacing a complex $\mathcal{M}$ with a locally free resolution $\mathcal{P} \to \mathcal{M}$ and then applying f^+.

Definition 3.5.1 *$f^! : D^b(\mathcal{D}_X) \to D^b(\mathcal{D}_Y)$ is the functor given by $f^!(\mathcal{M}) := f^+(\mathcal{M}[d])$, where $d = \dim(Y) - \dim(X)$.*

Note that this definition is consistent with the earlier definition in 3.2, by Proposition 3.3 (iii).

We also want to define a direct image functor $f_* : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$. This functor is defined as the composition of a left exact functor with a right exact functor, and as such does not have a good definition at the level of modules, but is only defined in the derived category. We will partially circumvent this problem by first defining direct image for a projection, where one can give an explicit resolution.

Let $p : Y = Z \times X \to X$ be projection on the second factor and let $q : Y \to Z$ be projection on the first factor. There is an isomorphism $\mathcal{D}_Y \cong p^{-1}\mathcal{D}_X \otimes_{\mathbb{C}_Y} q^{-1}\mathcal{D}_Z$.

Definition 3.5.2 *For $\mathcal{M} \in D^b(\mathcal{D}_Y)$,*

$$p_*(\mathcal{M}) := Rp.(q^*\Omega_Z \otimes^L_{q^*\mathcal{D}_Z} \mathcal{M}) = Rp.(q^{-1}\Omega_Z \otimes^L_{q^{-1}\mathcal{D}_Z} \mathcal{M}).$$

Here the symbol $\otimes^L$ means we take the derived functor defined by $\otimes$, i.e., we resolve $\mathcal{M}$ and/or $q^{-1}\Omega_Z$ by projectives and work with the complex. $Rp_{\cdot}$ is the usual direct image of sheaves. Since $p^{-1}\mathcal{D}_X$ commutes with $q^{-1}\mathcal{D}_Z$, it follows that $\mathcal{D}_X$ acts on $p_*(\mathcal{M})$.

In this definition, $q^{-1}\Omega_Z \otimes^L_{q^{-1}\mathcal{D}_Z} \mathcal{M}$ is not obviously a complex of $\mathcal{O}_Y$-modules, so it is not clear that the direct image can be computed in the $\mathcal{O}$ category. Thus, it is not clear that $p_*(\mathcal{M})$ is a quasi-coherent $\mathcal{O}_X$ module, and moreover, it is not clear that when Z is affine, $R^i p_{\cdot} = 0$ for $i > 0$. We give an equivalent definition for $q^{-1}\Omega_Z \otimes^L_{q^{-1}\mathcal{D}_Z} \mathcal{M}$ which exhibits the $\mathcal{O}$ module structure.

Let

$$\mathcal{D}_{X \leftarrow Y} = \mathcal{D}_{Y \to X} \otimes_{\mathcal{O}_Y} \Omega_{Y/X} = p^*(\mathcal{D}_X \otimes_{\mathcal{O}_X} {\Omega_X}^{-1}) \otimes_{\mathcal{O}_Y} \Omega_Y.$$

Since $\mathcal{D}_X$ has left and right $\mathcal{D}_X$ module structures, it follows that $\mathcal{D}_X \otimes {\Omega_X}^{-1}$ has two commuting $\mathcal{D}_X$ module structures. In particular, the pullback $p^*(\mathcal{D}_X \otimes {\Omega_X}^{-1})$ has a left $\mathcal{D}_Y$ module structure given by pulling back the shift of the right module structure, so $p^*(\mathcal{D}_X \otimes {\Omega_X}^{-1}) \otimes_{\mathcal{O}_Y} \Omega_Y$ is a right $\mathcal{D}_Y$ module, as well as a left $p^{-1}\mathcal{D}_X$ module. Thus, for $\mathcal{M} \in D^b(\mathcal{D}_Y)$, $\mathcal{D}_{X \leftarrow Y} {\otimes^L}_{\mathcal{D}_Y} \mathcal{M}$ has a $p^{-1}\mathcal{D}_X$ module structure. We claim that

$$\mathcal{D}_{X \leftarrow Y} {\otimes^L}_{\mathcal{D}_Y} \mathcal{M} \cong q^{-1}\Omega_Z {\otimes^L}_{q^{-1}\mathcal{D}_Z} \mathcal{M}.$$

The claim follows from the definition and the identification $\mathcal{D}_Y = q^{-1}\mathcal{D}_Z \otimes p^{-1}\mathcal{D}_X$.

Now we can show that $q^{-1}\Omega_Z {\otimes^L}_{q^{-1}\mathcal{D}_Z} \mathcal{M}$ can be treated as an $\mathcal{O}_Y$ module. Since we are working locally on the base, we can assume that X is affine, and we assume also that Z is affine. Then we can replace $\mathcal{M}$ by a free resolution $\mathcal{F} \to \mathcal{M}$ be a complex of free $\mathcal{D}_Y$ modules. Then $\mathcal{D}_{X \leftarrow Y} {\otimes^L}_{\mathcal{D}_Y} \mathcal{M} \cong \mathcal{D}_{X \leftarrow Y} {\otimes^L}_{\mathcal{D}_Y} \mathcal{F}$ is a direct sum of copies of $\mathcal{D}_{X \to Y}$ and hence is a quasi-coherent $\mathcal{O}_Y$ module. It follows that $p_*(\mathcal{M})$ is a quasi-coherent $\mathcal{O}_X$ module and $Rp_{\cdot}$ can be computed in the category of $\mathcal{O}_Y$ modules. In general, we use a covering of Z by affine open subsets, and the Cech resolution.

To compute $p_*(\mathcal{M})$, we give an explicit resolution of Ω_Z as a right $\mathcal{D}_Z$ module. Consider the complex of regular differential forms

$$\mathcal{A}^0_Z \to \mathcal{A}^1_Z \to \ldots \to \mathcal{A}^n_Z$$

with de Rham differential d, and $n = \dim(Z)$. Choose local coordinates $z_1, z_2, \ldots$ on Z and regard them also as coordinates on Y by pullback. If $\mathcal{M}$ is a $\mathcal{D}_Y$ module, we can consider the relative de Rham complex

$$\mathcal{A}^{\cdot}_{Y/X}(\mathcal{M}) := q^{-1}\mathcal{A}^{\cdot}_Z(\mathcal{M}) =$$

$$q^{-1}\mathcal{A}_Z^0 \otimes_{q^{-1}\mathcal{O}_Z} \mathcal{M} \to q^{-1}\mathcal{A}_Z^1 \otimes_{q^{-1}\mathcal{O}_Z} \mathcal{M} \to \ldots \to q^{-1}\mathcal{A}_Z^n \otimes_{q^{-1}\mathcal{O}_Z} \mathcal{M}$$

with differential d given by

$$d(\omega \otimes m) = dw \otimes m + \sum_i dz_i \wedge \omega \otimes \partial/\partial z_i \cdot m.$$

It is easy to check that $d^2 = 0$.

Let $L^i(\mathcal{M}) = H^i(\mathcal{A}^{\cdot}_{Y/X}(\mathcal{M})[n])$. Then

$$L^0(\mathcal{M}) = q^{-1}\Omega_Z \otimes_{q^{-1}(\mathcal{O}_Z)} \mathcal{M}/T_Z\mathcal{M}.$$

If the constant coefficient operators $\mathbb{C}[\partial/\partial_z]$ act freely on $\mathcal{M}$, then $L^{-i}(\mathcal{M}) = 0$ for $i > 0$. This follows by realizing $\Omega_Z^{\cdot}$ as a Koszul complex.

We apply this in particular to the case $\mathcal{M} = q^*(\mathcal{D}_Z)$, which is both a left $\mathcal{D}_Y$ module and right $q^*\mathcal{D}_Z$ module. Then $\mathcal{A}^{\cdot}_{Y/X}(q^*\mathcal{D}_Z)[n]$ is a complex of free right $q^*\mathcal{D}_Z$ modules. By the above comments, it follows that $L^j(q^*\mathcal{D}_Z) = 0$ if $j \neq 0$, and $L^0(q^*\mathcal{D}_Z) \cong \Omega_Z$ since $\mathcal{D}_Z/T_Z\mathcal{D}_Z \cong \mathcal{O}_Z$. Thus, $\mathcal{A}^{\cdot}_{Y/X}(q^*\mathcal{D}_Z)[n] \to q^*\Omega_Z$ is a resolution of $q^*\Omega_Z$ by free right $q^*\mathcal{D}_Z$ modules. Hence, we can identify

$$q^*\Omega_Z \otimes^L_{q^*(\mathcal{D}_Z)} \mathcal{M} \cong \mathcal{A}^{\cdot}_{Y/X}(q^*\mathcal{D}_Z)[n] \otimes^L_{q^*\mathcal{D}_Z} \mathcal{M}.$$

But this last module is isomorphic to $\mathcal{A}^{\cdot}_{Y/X}(\mathcal{M})[n]$ with the differential d. We conclude that

$$p_*(\mathcal{M}) = Rp.\mathcal{A}^{\cdot}_{Y/X}(\mathcal{M})[n].$$

3.5.3 Remark If $\mathcal{M}$ is a $\mathcal{D}_Y$ module, then $L^i(\mathcal{M})$ can have cohomology in degrees $-n \leq i \leq 0$. If p is affine, $p_*(\mathcal{M})$ can have cohomology only in degrees $-n \leq i \leq 0$, but in general $p_*(\mathcal{M})$ can have cohomology in degrees $-n \leq i \leq z$.

3.5.4 Examples. Consider the projection $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$. Then

$$\mathcal{H}^i(p_*\mathcal{O}_{\mathbb{C}^n}) = \begin{cases} 0 & \text{if } i \neq -1 \\ \mathcal{O}_{\mathbb{C}n-1} & \text{if } i = -1. \end{cases}$$

$$\mathcal{H}^i(p_*\mathcal{B}_{0|\mathbb{C}^n}) = \begin{cases} 0 & \text{if } i \neq 0 \\ \mathcal{B}_{0|\mathbb{C}^{n-1}} & \text{if } i = 0. \end{cases}$$

$$\mathcal{H}^i(p_*\mathcal{D}_{\mathbb{C}^n} = \begin{cases} 0 & \text{if } i \neq 0 \\ \mathcal{D}_{\mathbb{C}^n}/\partial/\partial_{z_n}\mathcal{D}_{\mathbb{C}^n} \otimes \mathcal{O}_{\mathbb{C}^{n-1}} dz_n & \text{if } i = 0. \end{cases}$$

Note that in this last case, direct image of the coherent module $\mathcal{D}_{\mathbb{C}^n}$ is not coherent.

We can factor an arbitrary map $f : Y \to X$ as $Y \to Y \times X \to X$, where the first map is $i : y \mapsto (y, f(y))$ and the second map is p, projection on the second factor.

Definition 3.5.5 $\quad$ **f_* for a morphism**

$$f_*(\mathcal{M}) := p_* i_*(\mathcal{M}), \mathcal{M} \in D^b(\mathcal{D}_Y).$$

3.5.6 Remark if $f : Y \to X$ is a closed embedding, then f_* coincides with the definition given before, as follows from Lemma 3.5 below. This would not be the case if we had not inserted the shift by n in the definition of direct image for a projection.

We want to prove that direct image is a functor. First we need some easy lemmas which are special cases of the claim.

Lemma 3.5.7 *Let $p = p_1 \circ p_2 : Z \times Y \times X \to Y \times X \to X$ be the sequence of projections. Then $p_* = p_{1*} \circ p_{2*}$.*
Proof. When Z and Y are affine, this is a routine exercise. The general case follows using Cech coverings.□

Lemma 3.5.8 *Let $i : Y \to W$ be a closed embedding, and consider the diagram*

$$\begin{tikzcd} X \times Y \arrow[r, "p_Y"'] \arrow[d, "id \times i"'] & Y \arrow[d, "i"'] \\ X \times W \arrow[r, "p_W"'] & W \end{tikzcd}$$

where the horizontal arrows are the projections. Then $i_* \circ p_{Y*} = p_{W*} \circ (id \times i)_*$.

Lemma 3.5.9 *Let $p : U \times V \to V$ and $q : W \times V \to V$ be the projections, and let $i : U \to W$ be a closed embedding, and $i \times id_V : U \times V \to W \times V$ be the induced embedding. Then $p_* = q_*(i \times id_V)_*$.*

We leave the proofs of these last two lemmas to the reader, noting the idea that direct image for a closed embedding adjoins normal derivatives, while direct image for a projection applied to a module free in the projected variables kills the normal derivatives.

Proposition 3.5.10 *Let $f : Z \to Y$ and $g : Y \to X$ be morphisms of smooth varieties. Then $(g \circ f)_* = g_* \circ f_*$.*
Proof. $g_* f_* = p_{Y \times X, X*} i_{g*} p_{Z \times Y, Y*} i_{f*}$, where $p_{U \times W, W} : U \times W \to W$ is the projection, and for a map $h : U \to V$, $i_h : U \to U \times V$ is the

closed embedding to its graph. By Lemma 3.5 applied to the diagram of embeddings and projections

$$\begin{tikzcd} Z \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \times Y \times X \arrow[r] & Y \times X \end{tikzcd} ,$$

$$g_* f_* = p_{Y\times X,X}{}_* p_{Z\times Y\times X,Y\times X}{}_* (i_{id\times g})_* i_{f}{}_*.$$

By Lemma 3.5 and Proposition 3.3, it follows that $g_* f_* = p_{Z\times Y\times X,X}{}_* (i_{f\times g\circ f})_*$. On the other hand,

$$(g\circ f)_* = p_{Z\times X}{}_* \circ (i_{f\circ g})_* = p_{Z\times Y\times X,X}{}_* (i_{f\times id_X})_* (i_{g\circ f})_* = p_{Z\times Y\times X,X}{}_* (i_{f\times g\circ f})_*,$$

using successfully, Lemma 3.5, Lemma 3.5, and Proposition 3.3.□

3.6. $\mathcal{D}$-modules and local systems.

Note that any $\mathcal{D}$-module $\mathcal{M}$ can be viewed as a quasi coherent $\mathcal{O}$-module with flat connection. In fact, the $\mathcal{D}$-module structure on an $\mathcal{O}$-module $\mathcal{M}$ is completely determined by the action map $T \otimes \mathcal{M} \to \mathcal{M}$, or equivalently, as a connection map $\mathcal{M} \to \mathcal{M} \otimes \Omega^1$ which is flat by definition of a $\mathcal{D}$-module. In particular, any *local system*, i.e. a $\mathcal{O}$-coherent locally free sheaf with flat connection, has a structure of a $\mathcal{D}$-module.

Proposition 3.6.1 *Let $\mathcal{M}$ be a $\mathcal{D}$-coherent module. Then the following properties are equivalent:*

(i) $SS\,\mathcal{M}$ is a subset of the zero section subvariety in T^*X (by coisotropicity property it then has to be equal to the zero zection),

(ii) $\mathcal{M}$ is $\mathcal{O}$-coherent,

(iii) $\mathcal{M}$ is a local system.

Proof. First of all, (ii) follows from (iii) by definition of a local system. To show that (i) and (ii) are equivalent, note that $SS\,\mathcal{M} = Supp\,\mathrm{gr}\mathcal{M}$ is a subvariety of the zero section iff the images of $\partial/\partial x_i$ in $\mathrm{gr}\mathcal{D}_X$ act on $\mathcal{M}$ by nilpotent operators. This in turn implies that $\mathrm{gr}\,\mathcal{M}$ is coherent over $\mathcal{O}_X \subset \mathrm{gr}D_X$, hence $\mathcal{M}$ is coherent over $\mathcal{O}_X$. If $\mathcal{M}$ is $\mathcal{O}$-coherent, we can consider a one-term filtration on $\mathcal{M}$ consisting of $\mathcal{M}$ itself and reverse the argument.

To show that (ii) implies (iii), suppose that $\mathcal{M}$ is $\mathcal{O}_X$-coherent and let $\mathfrak{m}_x \subset \mathcal{O}_X$ be the maximal ideal of the point. Then the geometric fiber $\mathcal{M}/\mathfrak{m}_x\mathcal{M}$ over x is finite-dimensional. It suffices to show that the function $\dim_{\mathbb{C}} \mathcal{M}/\mathfrak{m}_x\mathcal{M}$ is constant on X. Indeed, if we choose a set of n local sections of $\mathcal{M}$ generating the geometric fiber over x, then Nakayama's lemma implies that the sections generate the local ring $\mathcal{M}_x$, and hence they generate the geometric fibers in a neighborhood of x. Since the dimension of the fibers is constant, it follows that the sections are linearly independent in each geometric fiber and hence functionally independent in a neighborhood. [ASK ABOUT A REFERENCE FOR THIS] To show that the dimension of the geometric fiber is constant, choose two point x and y in X and assume that they can be connected by a smooth irreducible curve $Y \subset X$ (since in general we can always connect x and y with a chain of such curves).

Denote by i_x, i_Y and j the inclusions $\{x\} \hookrightarrow X, \{Y\} \hookrightarrow X$ and $\{x\} \hookrightarrow Y$, respectively. Then one can write $\mathcal{M}/\mathfrak{m}_x\mathcal{M} = i_x^+\mathcal{M} = j^+(i_Y^+\mathcal{M})$. The objects obtained at each step are $\mathcal{O}$-coherent (recall that the pullback $i_Y^+\mathcal{M}$ etc., in the sense of $\mathcal{O}$-modules has a natural $\mathcal{D}$-module structure). So, it suffices to prove the statement assuming that $X = Y$. In this one dimensional case a sheaf is locally free iff it has no torsion and this is what we need to show for our module $\mathcal{M}$ on Y. In fact, suppose this module has a torsion submodule at some point $z \in Y$. Then $i_z^!\mathcal{M} \neq 0$ and by right exactness of $(i_z)_*$ we have a non-zero submodule $(i_z)_*i_z^!\mathcal{M} = \Gamma_{[z]}\mathcal{M} \hookrightarrow M$. But by a local computation we know that the direct image of any non-zero module from $\{z\}$ is not an $\mathcal{O}$-coherent module on Y. $\square$

3.7. Simple non-holonomic modules. In was believed for some time that the characteristic variety of every simple $\mathcal{D}$-module has the minimal possible dimension, i.e., that every simple $\mathcal{D}$-module is holonomic, see §4.1 of the next chapter. We will show below (following Bernstein and Lunts) that in some sense "most" simple modules are *not* holonomic. In what follows we will assume that $X = \mathbb{C}^n$.

Conjecture 3.7.1 Let $u \in \mathcal{D}(\mathbb{C}^n)$. If $\sigma(u)$ is generic among homogeneous polynomials of $deg\, k \geq 4$ then $\mathcal{M} = \mathcal{D}/\mathcal{D} \cdot u$ is simple. Since $SS\, \mathcal{M}$ is equal to the zero-variety of $\sigma(u)$, $\mathcal{M}$ is not holonomic for $n \geq 2$.

THEOREM 3.7.1.1. *(Bernstein-Lunts*[**BeLu**]*) The conjecture above is true for $n = 2$.*

The proof of this theorem will occupy the rest of this section.

As in the previous section, we will use the Bernstein filtration defined by $deg\, x = deg\, \partial = 1$. Of course, this does not exist on a general smooth algebraic variety, and also the notion of holonomicity changes completely. However, we will work with this filtration (and hence a different notion of holonomicity) since it makes the basic idea of the proof simpler and in any case it can be repeated on the original situation with slight modifications.?????

Observe that $\mathrm{gr}\mathcal{D}(\mathbb{C}^n)$ can still be identified with functions on the cotangent bundle, so that the characteristic variety still lives in $T^*(\mathbb{C}^n)$. By the theorem of Involutivity of the Characteristic Variety Theorem (see section 1.2 REFERENCE), $SS\,\mathcal{M}$ is a coisotropic subvariety of $T^*\mathbb{C}^n = \mathbb{C}^{2n}$. We also note that $SS\,\mathcal{M}$ is a cone-subvariety of C^{2n} with respect to the *natural* $\mathbb{C}^*$-action (this is the main effect of changing the filtration since for the usual filtration the group $\mathbb{C}^*$ acts along the fibers of $T^*\mathbb{C}^n \to \mathbb{C}^n$). We will call any coisotropic cone-subvariety a *cc-(sub)variety.*

Definition 3.7.2 A cc-subvariety $\Sigma \subset \mathbb{C}^4$ is *minimal* if is does not contain any proper cc-subvariety $\Sigma' \subsetneq \Sigma$.

In particular, any minimal cc-subvariety is irreducible. Also, any irreducible Lagrangian cone-subvariety (equivalently, the closure of the conormal bundle to a smooth subvariety) is minimal. The main observation of Bernstein and Lunts is that there are plenty of minimal cc-subvarieties that are not Lagrangian.

More precisely, we will show that

THEOREM 3.7.2.2. *If P is a generic homogeneous polynomial of $\deg k \geq 4$ on $\mathbb{C}^4$ then the zero variety $\{P = 0\}$ is a minimal cc-subvariety*

We will show below that any hypersurface in a symplectic variety is automatically coisotropic, and thus $\{P = 0\}$ is a cc-subvariety. Once the theorem is proved, we can take any differential operator u with $\sigma(u) = P$ (of course, the symbol is taken here in the sense of the Bernstein filtration), to obtain an example of a simple non-holonomic module:

Lemma 3.7.3 If u is as above then $\mathcal{D}/\mathcal{D}\cdot u$ is simple, i.e. $\mathcal{D}\cdot u$ is a maximal left ideal.

Proof. If the claim is not true then there is a non-trivial inclusion of left ideals $\mathcal{D}\cdot u \subsetneq J$, which induces a nontrivial inclusion of associated graded ideals, and hence a nontrivial quotient map $\mathcal{D}/\mathcal{D}\cdot u \twoheadrightarrow \mathcal{D}/J$. It is easy to check that $SS\,\mathcal{D}/J$ is the zero set of $\sigma(J)$, the ideal generated by symbols of J. Since P is generic, it may be assumed irreducible, so the characteristic variety $SS\,\mathcal{D}/J$ is a proper cc-subvariety of the minimal cc-subvariety $SS\big(\mathcal{D}/\mathcal{D}\cdot u\big)$, which cannot happen. $\square$

3.7.4 Digresssion on coisotropic subarieties.

To prove that the zero variety of a generic polynomial is minimal we need to study the geometry of coisotorpic subvarieties in $\mathbb{C}^4$ endowed with the natural syplectic form $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. We will denote $\mathbb{C}^4$ with this symplectic structure by M. Recall that a subvariety

$\Sigma \subset M$ is called coisotropic is its ideal sheaf I_Σ satisfies $\{I_\Sigma, I_\Sigma\} \subset I_\Sigma$ where $\{\cdot\,,\cdot\}$ is the Poisson bracket (algebraic definiton). It follows that a hypersurface is coisotropic. Alternatively, we can say that for any smooth point $x \in \Sigma^{reg}$ the tangent space $T_x\Sigma \subset T_xM$ is a coisotropic subspace, i.e. $T_x\Sigma^\perp \subset T_x\Sigma$ (geometric definition) . Therefore in the tangent bundle to the smooth part Σ^{reg} we have a subbundle formed by $T_x\Sigma^\perp \subset T_x\Sigma$. This subbundle is called the *nul-foliation* and we summarize its properties in the following

Lemma 3.7.5
(i) $T\Sigma^{reg\perp}$ coincides with the radical of the form $\omega|_{\Sigma^{reg}}$.
(ii) $T\Sigma^{reg\perp}$ is integrable, i.e. a bracket of two vector fields in $T\Sigma^{reg\perp}$ is again a vector field in $T\Sigma^{reg\perp}$. $\square$

By Frobenius integrability theorem $T\Sigma^{reg\perp}$ is tangent to a familiy of curves in Σ^{reg}. In a particular case when Σ is a hypersurface in M given by an equation $P = 0$, the nullfoliation $T\Sigma^{reg\perp}$ is generated by the Hamiltonian vector field ξ_P corresponding to P.

Idea of proof of (3.7.2.2). The main idea behind the proof of Bernstein-Lunts is that a solution of an *algebraic* differential equation is usually just *analytic*. Suppose now that $\Sigma' \subsetneq \Sigma$ is a proper cc-subvariety. Then at a point x which is smooth for both Σ' and Σ one has $(T_x\Sigma')^\perp \supset (T_x\Sigma)^\perp$. Therefore Σ' is formed by the leaves of the foliation, i.e. ξ_P is tangent to Σ' at its smooth points. When we apply Frobenius integrability in general we will not get an *algebraic* subvariety Σ'.

Suppose that Σ' is algebraic, and denote the projective variety corresponding to Σ' by C. The condition of genericity we will impose on P basically says that if C exists then it has only ordinary double points as singularities. We know that C is tangent to the direction field $\bar{\xi}$ on the projectivization $\overline{\Sigma}$ obtained from ξ_P. Suppose for a moment that $\bar{\xi}$ is actually a vector field. Then, at each singular point, $\bar{\xi}$ defines a linear operator on its tangent space to $\overline{\Sigma}$. Suppose this operator is diagonalizable. Then the *ratio* of its eigenvalues makes sense even if $\bar{xi}$ is just a direction field (not a vector field). Now we can state the genericity conditions imposed on P:

(a) dP does not vanish away from the origin of $\mathbb{C}^4$

(b) For any singular point of $\bar{\xi}$, the ratio of the corresponding eigenvalues is *irrational*.

To show that these two conditions imply that C has only ordinary double points one uses the following theorem from the theory of differential equations:

THEOREM 3.7.5.3. *([**Ar**]) Suppose that a vector field ξ on a two-dimen sional manifold defines a diagonalizable operator on the tangent space to its singular point. Suppose futher that the ratio of the eigenvalues λ/μ is irrational. Then there exists a system of (formal) coordinates which linearizes ξ:*

$$\xi = \lambda y_1 \frac{\partial}{\partial y_1} + \mu y_2 \frac{\partial}{\partial y_2}. \qquad \square$$

Proposition 3.7.6 Suppose ξ is as in the theorem and ξ is tangent to a curve C at a point m. Then at m, either C is smooth or has an ordinary double point.

Now suppose ξ is tangent to a curve C at a point m and let f generate the ideal J of C. If m is a regular point for ξ, then we can find local coordinates (y_1, y_2) on a formal neighborhood such that $\xi = \partial/\partial y_1$, and it follows that the ideal of C is generated by y_2 in the formal neighborhood.

If m is a singular point, choose coordinates in a formal neighborhod as in the theorem. Write $f = \sum a_{\beta_1,\beta_2} {y_1}^{\beta_1} {y_2}^{\beta_2}$. We show that each monomial $y^\beta := {y_1}^{\beta_1} {y_2}^{\beta_2}$ is in the ideal of C. Since the ratio of eigenvalues is irrational, it follows that ξ has distinct eigenvalues on its eigenvectors, the monomials y^β. Hence, for any integer $N > 0$, we can find a polynomial Q_β such that $Q_\beta(\xi)(y^\beta) = y^\beta$ and $Q_\beta(\xi)(y^\nu) = 0$ for $deg(\nu) < N$, $\nu \neq \beta$. Since ξ preserves the ideal J, it follows that $y^\beta \in J + \mathfrak{m}^N$, where $\mathfrak{m}$ is the maximal ideal of m. Since J is closed in the $\mathfrak{m}$-adic topology by teh Artin-Rees lemma, it follows that $y^\beta \in J$. But now recall that f generates J and let y^β be a monomial of minimal degree of f. Then $y^\beta = hf$ for some power series h and $h(m) \neq 0$. Hence, h is invertible in the ring of formal power series, so y^β generates J. Since J is radical, it follows that $y^\beta = y_1, y_2$, or $y_1 y_2$. $\square$

Proof of (3.7.2.2). Recall that in our case $M = \mathbb{C}^4$ and Σ is a three-dimensional subvariety given by the equation $\{P = 0\}$ where P is a homogeneous polynomial of degree ≥ 4. Since be have a natural $\mathbb{C}^*$-action, we can pass to the projective picture, i.e. consider a projective algebraic surface $\overline{\Sigma} \subset \mathbb{CP}^3$. Then any proper cc-subvariety Σ' will give us a curve C on $\overline{\Sigma}$, so we need to study the curves on $\overline{\Sigma}$. A classical theorem of Noether asserts that since $k \geq 4$, any such curve is given by vanishing of a homogeneous polynomial Q of degree l [**De**]. Therefore, the ideal sheaf of Σ' is generated by two functions P and Q.

Recall that the Euler vector field $Eu = \sum_i x_i \partial_i$ allows us to define a 1-form $\alpha = i_{Eu}\omega$ such that $d\alpha = \omega$. One easily sees that a cone-subvariety of M is isotropic iff the restriction of α to it vanishes.

Since Σ' in our case is 2-dimensional and ω is non-degenerate, we have $(T_x\Sigma')^\perp = T_x\Sigma'$, so Σ is also isotropic and $\alpha|_{\Sigma'} = 0$. Since P and Q generate the ideal of Σ', dP and dQ generate the conormal bundle, and hence there exist regular functions f, g on $\mathbb{C}^4$ such that the equality

$$\alpha|_{\Sigma'} = f\, dP + g\, dQ.$$

holds on the regular part $(\Sigma')^{reg}$ of Σ' (where f and g are regular functions on $(\Sigma')^{reg}$. Since α, P and Q are homogeneous, we deduce that f and g are also homogeneous of degrees $(2-k)$ and $(2-l)$ respectively. We denote by $\mathcal{O}_C(i)$ the restriction of the linear bundle $\mathcal{O}(i)$ from $\mathbb{CP}^3$ to C. Hence we can think of f (resp. g) as a section of $\mathcal{O}_C(2-k)$ (resp. $\mathcal{O}_C(2-l)$) over the regular part of C.

Case 1. Assume C is smooth. Then by positivity of $\mathcal{O}_C(1)$ the bundles $\mathcal{O}_C(2-k)$ and $\mathcal{O}_C(2-l)$ have no global sections if $2-k<0$ and $2-l<0$. Since we assumed that $k \geq 4$, we have $f = 0$ and $l = 1$ or 2. [[DIDN'T UNDERSTAND THIS. REWROTE AS IN BERNSTEIN-LUNTS, BUT MAYBE I AM MISSING SOMETHING OBVIOUS We will consider here the case $l = 2$ (the other case is easier since for $l = 1$ the curve C is just a hyperplane section of $\overline{\Sigma}$).

Let $S_Q = \{x \in \mathbb{C}^4 \,|\, \xi_g = c \cdot Eu$ for some non-zero constant $c\}$. We claim that S_Q is a finite union of linear subpaces. In fact, define an operator $A : \mathbb{C}^4 \to \mathbb{C}^4$ by viewing Q as a symmmetric bilinear form and putting $Q(\cdot,\cdot) = \omega(A\cdot,\cdot)$ then the equation defining S_g translates into condition that x is an iegenvector of A. Since Σ' is coisotropic, it has to be a two-dimensional linear subpace. Hence C has to be a projective line which is impossible since $deg\, C = kl > 1$.]]

Thus, α is proportional to dQ at each point of Σ'. We claim that Σ' lies in a hyperplane. If $l = 1$, this is obvious. If $l = 2$, let $S_Q = \{x \in \mathbb{C}^4 \,|\, \xi_Q = c \cdot Eu$ for some non-zero constant $c\}$. We claim that S_Q is a finite union of linear subpaces. In fact, define an operator $A : \mathbb{C}^4 \to \mathbb{C}^4$ by viewing Q as a symmmetric bilinear form and putting $Q(\cdot,\cdot) = \omega(A\cdot,\cdot)$ then the equation defining S_Q translates into condition that x is an eigenvector of A. Since Σ' is irreducible, it follows that Σ' lies on a hyperplane.

Let ϕ be a linear function defining the hyperplane with constant Hamiltonian vector field ξ_ϕ. Choose some vector $v \in \Sigma'$ not proportional to ξ_ϕ. Since Σ' is a cone, $\mathbb{C}v \in \Sigma'$. Since ξ_ϕ is a constant vector field tangent to Σ', $\mathbb{C}v + \mathbb{C}\xi_\phi \subset \Sigma'$. Since Σ' is two-dimensional, it is a linear plane. Hence C has to be a degree one projective line which is impossible since $deg\, C = kl > 1$.

Case 2. C is singular. Since the vector field ξ_P is homogeneous of degree $(k-1)$, it does not descend to a vector field on $\mathbb{C}\,\mathbb{P}^3$. However, is still gives a one-dimensional subbundle in the tangent bundle (or a direction field) over an open part of $\mathbb{C}\,\mathbb{P}^3$ and our curve C is tangent to it. The preimage of singular points of C in $\mathbb{C}^4 \setminus \{0\}$ belongs to the zero set of the field ξ_P (since this preimage is defined by conditions $d\,P = d\,Q = 0$ and the first of the describes the zeros of ξ_P).

Recall that C has only ordinary double points (that is in some formal neighborhood with coordinates $y_1, \ldots, y_4$ in $\mathbb{C}^4$ of any singular point, $\Sigma = \{y_3 = 0\}$ and $\Sigma' = \{y_3 = 0 = y_1 y_2\}$. Let $\widetilde{C} \to C$ be the normalization. We claim that regular functions f and g on $(\Sigma')^{reg}$ defined by $\alpha|_{\Sigma'}^{reg} = f\,dP + g\,dQ$, viewed as sections of some line bundles on C^{reg}, extend to $\widetilde{C}$. For this we can use the formal local coordinates $y_1, \ldots, y_4$ above. One has to show that f and g do not have any poles on the two branches $y_1 = 0$ and $y_2 = 0$ of C in the neighbourhood of a singular point. In fact, $P = y_3$ (since $\Sigma = \{y_3 = 0\}$) and $Q = \varepsilon\,P + \delta\,y_1 y_2$ where ε and δ are formal power series in y and $\delta(0) \neq 0$. Then we can write an equality of differential forms on Σ'

$$\alpha = \sum \alpha_j dy_j = f\,dP + g\,dQ = (f + \varepsilon g)dP + gd(\delta \cdot y_1 y_2) + gPd\varepsilon,$$

and we can ignore the last term because P vanishes on Σ'. Since $d(\delta \cdot y_1 y_2) = \delta d(y_1 y_2) + y_1 y_2 d\delta$ and $y_1 y_2$ vanishes on Σ', we can absorb δ into g, i.e. assume that $Q = y_1 y_2$.

Recall that we want to extend f and g to $\Sigma'_i = \{y_i = y_3 = 0\}$, $i = 1, 2$. We see that $\alpha_4|_{\Sigma'} = 0$ and $\alpha_3|_{\Sigma'} = f + \varepsilon g$. Then, $g|_{\Sigma'_2} = \frac{\alpha_2}{y_1}$ and since α_2 vanishes at the origin of Σ'_2, the ratio does not have a pole at the origin of Σ'_2 which means that g extends to Σ'_2. It follows that $f = \alpha_3 - \varepsilon g$ extends to Σ'_1. Similarly, f, g extend to Σ'_1.

Now our goal is to show that for under the genericity condition for P described above we will in fact have only ordinary double points. Recall that Σ is given by the equation $\{P = 0\}$. First of all we assume that df does not vanish on $\mathbb{C}^4 \setminus \{0\}$ (in particular P is irreducible and Σ is smooth away from the origin). Let $S_P = \{x \in \mathbb{C}^4 |\, \xi_P = a \cdot Eu$ for some non-zero constant $a\}$.

Lemma 3.7.7 Under the conditions described above one has $\dim S_f = 1$ *Proof.* Define $S_P = \{x \in \mathbb{C}^4 |\, \xi_P = Eu\}$. Then $S'_P \subset S_P = \mathbb{C}^* \cdot S'_P$ so it is enough to prove that S'_P is a finite set. To that end, write $\xi_P = \sum P_i \frac{\partial}{\partial x_i}$ and note that by our assumption P_i have no common zeros outside the origin. Since $deg\,P_i = k - 1$, there exists a constant $C > 0$ such that $\sum |P_i|^2 \geq C \cdot \left(\sum |x_i|^2\right)^{k-1}$ for all $x \in \mathbb{C}^4$. The set S'_P is given by the equations $\{P_1 = x_2, P_2 = x_1, P_3 = x_4, \ldots\}$ hence for any point $x \in S'_P$ one has $\sum |x_i|^2 \leq C^{\frac{1}{k-2}}$. If S'_P is not finite we obtain a contradiction since any algebraic set of positive dimension is necessarily unbounded. $\square$

Now the theorem on a canonical form of a vector field quoted above finishes the proof. $\square$

3.7.8 A generalization of $\mathcal{H}ol(\mathcal{D}_X)$

Recall that by $\mathcal{H}ol(\mathcal{D}_X)$ we denoted the abelian category of holonomic $\mathcal{D}_X$-modules. Now we can also define $\mathcal{M}in(\mathcal{D}_X)$ to be the category of all $\mathcal{D}_X$-modules $\mathcal{M}$ such that every component of $SS\,\mathcal{M}$ is a *minimal* coisotropic subvariety. Generalizing from the lagrangian case one easily proves the following

Proposition 3.7.9 Every object in $\mathcal{M}in(\mathcal{D}_X)$ has finite length. $\square$

In other respects the category $\mathcal{M}in(\mathcal{D}_X)$ is not studied at all.

4. Holonomic $\mathcal{D}$-modules.

4.1. Category of holonomic modules.

Definition 4.1.1 A coherent $\mathcal{D}$-module $\mathcal{M}$ is called *holonomic* if

$$\dim SS\,\mathcal{M} = \dim X.$$

Proposition 4.1.2
(i) Holonomic $\mathcal{D}$-modules form an abelian subcategory $\mathcal{H}ol_X$ of the category of all coherent $\mathcal{D}$-modules. In particular, every subquotient of a holonomic module is itself holonomic.
(ii) Every holonomic module has finite length as a $\mathcal{D}$-module.
Proof. To prove the first claim, note that for any subquotient $\mathcal{M}'$ of a holonomic module $\mathcal{M}$ we have $SS\,\mathcal{M}' \subset SS\,\mathcal{M}$. Hence $\dim SS\,\mathcal{M}' \leq \dim X$. But by the Involutivity of Characteristic Variety theorem for any $\mathcal{D}$-module $\mathcal{M}'$ one has $\dim SS\,M' \geq dimX$. Hence in our case we have $\dim SS\mathcal{M}' = \dim X$.
To prove the second claim we need to show that any decreasing chain of submodules of a holonomic module $\mathcal{M}$ necessarily terminates. In fact, let $S_1, \dots, S_n$ be the irreducible components of $SS\,\mathcal{M}$ which have multiplicities $m_1, \dots, m_n$ respectively. Since the characteristic variety of any subquotient of $\mathcal{M}$ is a union of several components S_i and the multiplicities are additive with respect to short exact sequences, any strictly decreasing chain of submodules of $\mathcal{M}$ has at most $m_1 + \ldots + m_n$ terms. □

Now we prove a small lemma which gives a partial description of the characteristic variety of a holonomic $\mathcal{D}$-module. Recall that T^*X is a symplectic manifold (REFERENCE TO CHAPTER TWO) with a $\mathbb{C}^*$ action along the fibers and corresponding Euler vector field Eu. A subvariety of T^*X is called a cone-subvariety if it is $\mathbb{C}^*$-invariant. Let $\lambda = i_{Eu}\omega$, the "tautological" 1-form on T^*X, i.e., if ξ is tangent to T^*X at a covector α, then by definition $\lambda(\xi) = \alpha(\pi_*\xi)$, where π is the projection $T^*X \to X$.

Lemma 4.1.3 Let X be a manifold and Λ an irreducible Lagrangian cone-subvariety in T^*X. Then there exists a locally closed smooth subvariety Y of X such that $\Lambda = \overline{T^*_Y X}$ and $Y = \pi(\Lambda)^{reg}$.
Proof. Since Λ is a cone-subvariety, Eu is tangent to Λ^{reg}. Since Λ is Lagrangian, $\omega(Eu, \xi) = 0$ if ξ is tangent to Λ, so λ vanishes on Λ. Let $Y = \pi(\Lambda)^{reg}$, and choose $y \in Y$ and generic $\alpha \in \Lambda^{reg}$ such that $\pi(\alpha) = y$ and the tangent map $T_\alpha(\Lambda) \to T_y Y$ is onto using the Bertini-Sard theorem. Then it follows from the definition of λ that $\alpha \in T^*_Y X$.

Since a generic vector $\alpha \in \Lambda$ lies in $T^*_Y X$, and Λ and $\overline{T^*_Y X}$ are both irreducible and of the same dimension, it follows that they coincide. □

Corollary 4.1.4 Let $\mathcal{M}$ be a holonomic module and *Supp* $\mathcal{M} = Y$. Then $SS\,\mathcal{M} = \overline{T^*_{Y^{\mathtt{reg}}} X} \cup \{\text{other irreducible components}\}$. □

Proof. Let $Z \subset T^*X$ be the union of the irreducible components of $\Lambda \subset SS\,\mathcal{M}$ such that the image of the generic point of Λ is not the generic point of Y. Thus, $\pi(Z) \cap Y$ is a proper closed subset in Y. Write $Y^{\mathtt{reg}}$ for the smooth locus of Y, and set $Y^\circ := Y^{\mathtt{reg}} \smallsetminus \pi(Z)$. Choose a Zariski open subset $U \subset X$ such that $Y \cap U = Y^\circ$. Then, $i : Y^\circ \hookrightarrow U$ is a closed imbedding of a smooth submanifold. Hence, by Kashiwara theorem, we have $\mathcal{M}|_U = i_* L$, where L is a finitely generated $\mathcal{D}$-module L on $Y^{\mathtt{reg}}$. Moreover, Kashiwara's theorem says that $SS\,L$ is contained in the zero section of T^*Y° The result follows. □

4.2. b-function.

Let f be a (global) regular function on X and $U \overset{j}{\hookrightarrow} X$ be the open subset defined by $X \setminus U = \{f = 0\}$. Let $\mathcal{M}$ be a $\mathcal{D}_U$-module. We introduce an auxilary variable s and consider X and U as varieties defined over the field of rational functions $K := \mathbb{C}(s)$ (even though the equations defining X and U do not contain s). Denote $\mathcal{D}_U \otimes K$ by $\mathcal{D}_{U(K)}$ and similarly for $D_{X(K)}$. The subrings with $\mathbb{C}[s]$ coefficients will be denoted $\mathcal{D}_U[s]$ and $\mathcal{D}_X[s]$. Let $\mathcal{M}f^s = \{a(s) \cdot m \cdot f^s\}$, be the space formed by all formal expressions $a(s) \cdot m \cdot f^s$, where $a(s)$ is a rational function in s, m is a section of $\mathcal{M}$ and f^s is a formal symbol. The $\mathcal{O}$-module structure on $\mathcal{M}f^s$ is defined by the action of $\mathcal{O}_X$ on $m \in \mathcal{M}$. We define the $\mathcal{D}_X$-module structure by letting a vector field ξ act via the formula

$$\xi : mf^s \to \xi(mf^s) = \big[(\xi m) + s\frac{(\xi f)}{f} m\big] f^s.$$

This way, $\mathcal{M}f^s$ becomes a $\mathcal{D}_{U(K)}$-module.

4.2.1 Remark One can define an automorphism τ of $\mathcal{D}_U[s]$ by

$$\xi \mapsto \xi + s\frac{\xi f}{f}, \qquad \phi \mapsto \phi,$$

where ξ is a vector field and ϕ is a function. Then it follows immediately from the definitions that $\mathcal{M}f^s = \mathcal{M}^\tau$, i.e. $\mathcal{M}^\tau$ is obtained from $\mathcal{M}$ by twisting the $\mathcal{D}_U[s]$-module structure with τ (this means that $P \in \mathcal{D}_U[s]$ now acts by $\tau(P)$).

Notice that if $\mathcal{M}$ is a holonomic $\mathcal{D}_U$ module, then $\mathcal{M}f^s$ is a holonomic $\mathcal{D}_{U(K)}$ module. This follows by extending filtrations by scalars.

Notice also that if κ is an automorphism of $\mathbb{C}(s)$, then it induces an automorphism of $\mathcal{D}_{X(K)}$ modules, which does not affect holonomicity. We will apply this in particular to the automorphism induced by $s \to s-1$.

Lemma 4.2.2 (Lemma on the b-function.) Suppose that $\mathcal{M}$ is a holonomic $\mathcal{D}_U$ module. Then for any section $m \in M$ there exist $u \in \mathcal{D}_X[s]$ and a polynomial $b(s) \in \mathbb{C}[s]$ satisfying

$$u(mf^s) = b(s-1)\frac{m}{f}f^s. \tag{4.2.3}$$

4.2.4 Remark

The meaning of the lemma is that if we view s as a complex number (not a variable) then $\frac{m}{f}f^s$ can be written as mf^{s-1} and hence the Lemma says that we can achieve one extra f in the denominator applying u and dividing by $b(s-1)$ (which is possible for all but finitely many values of s, the roots of the polynomial $b(s-1)$). Obviously, if a polynomial $b(s)$ satisfies the property (4.2.3) of the theorem, then any multiple of it satisfies the theorem too (since u can be also multiplied by a polynomial). This justifies the following

Definition 4.2.5 The generator of the (principal) ideal of all polynomials $b(s)$ with the property (4.2.3) is called the *b-function*.

Proof of the lemma on b-function.

We will prove a slightly more general statement. To that end, consider $\mathcal{M}$ as a $\mathcal{D}_X$-module and let $\mathcal{M}_0 \subset \mathcal{M}$ be an $\mathcal{O}_X$-coherent subsheaf (rather than just submodule generated by one section). We will show that the module $\mathcal{N} := \mathcal{D}_{X(K)} \cdot \mathcal{M}_0 f^s$ is a *holonomic* $\mathcal{D}_{X(K)}$ module *over the ground field* K.

The strategy here is to notice that $\mathcal{N}|_U$ is a holonomic $\mathcal{D}_{U(K)}$-module since it is a submodule of the holonomic module $\mathcal{M}f^s$, and then extend $\mathcal{N}|_U$ to some holonomic $\mathcal{D}_{X(K)}$-module and finally compare this extension with $\mathcal{N}$ itself.

To that end, denote by $\mathcal{N}' \subset \mathcal{N}$ the *lowest* term of Gabber's filtration, see REFERENCE. Of course, this filtration was defined earlier only for affine varieties and to make sense of it in the general situation one needs to prove that this filtration is compatible with localizations, i.e. $G_j(\mathcal{M})|_U = G_j(\mathcal{M}|_U)$. The latter property follows from the fact that Gabber's filtration coincides with Sato-Kashiwara filtration. Since the Sato-Kashiwara filtration is defined via truncations and other homological operations, it is compatible with localization. One deduces further that $\mathcal{N}' \neq 0$ since $\mathcal{N}'|_U = \mathcal{N}|_U$.

Thus, we've obtained a holonomic submodule $\mathcal{N}' \subset \mathcal{N}$ such that $\mathcal{N}'|_U = \mathcal{N}|_U$. Then $\mathcal{N}/\mathcal{N}'$ is a $\mathcal{D}_{X(K)}$-module supported on $X \setminus U$. Therefore any particular section of $\mathcal{N}/\mathcal{N}'$ is annihilated by a certain power of f. Hence, by the Nullstelenzats, the image of $\mathcal{M}_0$ in $\mathcal{N}/\mathcal{N}'$ is annihilated by a certain power of f, in particular for some k $f^k \cdot \mathcal{M}_0 f^s \subset \mathcal{N}'$. Hence, $\mathcal{D}_{X(K)}\mathcal{M}_0 f^{s+k} \subset \mathcal{N}'$, and therefore $\mathcal{D}_{X(K)}\mathcal{M}_0 f^{s+k}$ is holonomic, as a submodule of a holonomic $\mathcal{D}$-module. But $\mathcal{D}_{X(K)}\mathcal{M}_0 f^s \simeq \mathcal{D}_{X(K)}\mathcal{M}_0 f^{s+k}$ as $\mathcal{D}_{X(K)}$-modules after an automorphism of K. Indeed, the map is $P(s)mf^s \mapsto P(s+k)mf^{s+k}$ and it is obvious that it it one-to-one and onto, and the translation $s \to s+k$ makes the map a $\mathcal{D}_{X(K)}$ morphism. Hence, $\mathcal{D}_{X(K)}\mathcal{M}_0 f^s$ is also holonomic.

To deduce the original statement from the holonomicity note that any holonomic module has finite length. Hence the chain of submodules

$$\mathcal{D}_{X(K)}(\mathcal{M}_0 f^1) \supset \mathcal{D}_{X(K)}(f \cdot \mathcal{M}_0 f^1) \supset \mathcal{D}_{X(K)}(f^2 \cdot \mathcal{M}_0 f^1) \supset \dots$$

necessarily stabilizes. Hence $\mathcal{D}_{X(K)}(f^r \cdot \mathcal{M}_0 f^1) = \mathcal{D}_{X(K)}(f^{r+1} \cdot \mathcal{M}_0 f^1)$ for some r. If $\mathcal{M}_0$ is now generated by one section m, we have $\mathcal{D}_{X(K)}(f^r \cdot mf^s) = \mathcal{D}_{X(K)}(f^{r+1} mf^s)$. But the LHS of this equality is isomorphic to $\mathcal{D}_{X(K)}(mf^s)$ after a field automorphism. Hence $\mathcal{D}_{X(K)}(mf^s) = \mathcal{D}_{X(K)}(f \cdot mf^s)$ and therefore there exists a section u' of $\mathcal{D}_{X(K)}$ such that $u'(mf^{s+1}) = mf^s$. But this u' may have denominators involving s, i.e. $u' = \frac{u}{b(s)}$ where u and b depend on s polynomially. Then $u \cdot mf^{s+1} = b \cdot mf^s$. □

In the previous chapter we have constructed a functor

$$j_* : \mathcal{D}_U - \mathrm{mod} \to \mathcal{D}_X - \mathrm{mod}.$$

In general this functor does not map finitely generated modules to finitely generated modules: for example $j_*(\mathcal{D}_U)$ is not finitely generated over $\mathcal{D}_X$. However we have seen that sometimes the finite generation property is preserved: for example we can take $X = \mathbb{C}$, $U = \{z \neq 0\}$ and consider $j_*(\mathcal{O}_U) = \mathbb{C}[z, z^{-1}]$. The theorem that we state below says that it is the size of the characteristic variety that makes the difference.

Theorem 4.2.5.1. *Let X and U be as above. Then j_* maps holonomic $\mathcal{D}_U$-modules to holonomic $\mathcal{D}_X$-modules. In particular, the image under j_* of any holonomic D_U-module is finitely generated.*

Proof. $\mathcal{M}$ is a holonomic $\mathcal{D}_U$-module generated by an $\mathcal{O}_X$ coherent module $\mathcal{M}_0$. Then

$$\mathcal{D}_{X(K)}\mathcal{M}_0 f^s = \cup_{k\in\mathbb{Z}} \mathcal{D}_{X(K)}\mathcal{M}_0 f^{s-k} = \mathcal{D}_{U(K)}\mathcal{M}_0 f^s = j_*\mathcal{M} f^s,$$

where the first equality follows from the lemma on the b-function. Thus, $j_*\mathcal{M}f^s$ is a holonomic $\mathcal{D}_{X(K)}$ module, again by the lemma on the b-function. We can assume that $\mathcal{M}_0$ is generated by a single section m (othewise induct on the number of generators). Then $\mathcal{D}_{X(K)} \cdot mf^s = \mathcal{D}_{X(K)}/I$ where I stands for the annihilator ideal of (mf^s). Then the zero variety of $\operatorname{gr} I$ is a Lagrangian variety in T^*X (again viewed over K). The ideal $\operatorname{gr} I$ is generated by finitely many (K-valued) functions on T^*X. If we specialize s to an integer n, for all but finitely many values of n the complex dimension of the characteristic variety of $\mathcal{D}_X \cdot mf^n$ is the same as the K dimension of the characteristic variety of $\mathcal{D}_{X(K)} \cdot mf^s = \dim X$, and in addition the multiplicities of components are preserved. It follows immediately that if $n \ll 0$ then $\mathcal{D}_X \cdot mf^n$ is holonomic. In addition, if $n \ll 0$ then $\mathcal{D}_X \cdot mf^n = \mathcal{D}_X \cdot mf^{n-1} = \mathcal{D}_X \cdot mf^{n-2} = \dots$. Indeed, if not $\mathcal{D}_X \cdot mf^n$ is a proper holonomic submodule of $\mathcal{D}_X \cdot f^m$ for some $m < n$, and the sum of the multiplicities of $\mathcal{D}_X \cdot f^m$ exceeds the sum of the multiplicities of $\mathcal{D}_{X(K)} \cdot mf^s$. It follows that $j_*\mathcal{M} = \mathcal{D}_X \cdot mf^n$ for $n \ll 0$, so $j_*\mathcal{M}$ is holonomic. □

4.2.6 Another proof for $X = \mathbb{C}^n$.

Now we will run through a proof of Theorem **??** in the case where our variety X is $\mathbb{C}^n$. Let $A = \mathcal{D}(\mathbb{C}^n)$. This ring is generated by elements $x_1, \dots, x_n, \partial_1, \dots, \partial_n$. Filter A by giving all the generators degree 1 (earlier we used a filtration where x_i had degree 0 and ∂_i degree 1). We still have

$$\operatorname{gr} A = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$$

but now all generators have degree 1. The advantage of this choice of degrees is that each homogeneous piece of $\operatorname{gr} A$ is a finite dimensional vector space. This new filtration is called the *Bernstein filtration.*

Given an A-module M, consider M with a filtration such that

$$A_i \cdot M_j \subset M_{i+j}.$$

As before (REFER TO CHAPTER 1.1) we have a notion of a good filtration. However, with such a filtration, each piece of $\operatorname{gr} M$ is a finite dimensional vector space. Let $h(\operatorname{gr} M)$ be the Hilbert polynomial of $\operatorname{gr} M$; so that

$$h(\operatorname{gr} M)(i) = dim(M_i/M_{i-1}) \quad \text{for } i \gg 0.$$

This is indepentdent of particular choice of a good filtration; so we just write $h(M)$. Now

$$h(M)(i) = \frac{c \cdot i^d}{d!} + \text{ terms of degree } \leq d-1$$

where $d = \dim SS\,M$.

Note that this characteristic variety $SS\,M$ is defined using our particular filtration on A. It does not coincide with the characteristic variety defined using the other filtration on A. Nevertheless, *the dimensions of these characteristic varieties do coincide.* This follows from the result in Chapter 1 relating dim SS M to vanishing of Ext groups, which is independent of filtration. This is done in 1.4. If so, we should probably EXPLAIN this a little bit here ??). We use our filtration of A where $deg\, x_i = 1$, and all notions like $SS\,M$ will be with regard to this filtration.

Proposition 4.2.7 For any A-module M we have $\dim SS\,M \geq n$.
Proof. The Hilbert poynomial $h(A)$ of A looks like

$$h(A)(i) = \frac{i^{2n}}{(2n)!} + \dots$$

Let $0 \neq M_0 \subset M_1 \subset \dots, \cup M_i = M$, be a good filtration of M.

Lemma 4.2.8 The natural map $A_i \hookrightarrow Hom(M_i, M_{2i})$ is injective for all $i \geq 0$.
Proof. Use induction on i. The case $i = 0$ just means $M \neq 0$. Suppose we proved the statement for values les then i. Proceed by contradiction. Suppose $P \in A_i$ and $P \cdot M_i = 0$. Note that it follows that each $[P, x_j]$ and $[P, \partial_j]$ act by zero on M_{i-1}. We may assume that P is not a scalar. Then for some j either $[P, x_j] \neq 0$ or $[P, \partial_j] \neq 0$. If $[P, x_j] \neq 0$ then $[P, x_j]$ is a non-trivial element of A_{i-1} that acts by zero on M_{i-1}, which cannot happen by the inductive assumption. Similarly, $[P, \partial_j] \neq 0$ also leads to a contradiction. $\Box$

End of proof of the Proposition. The lemma above implies that $h(A)(i) \leq H(M)(i)h(M)(2i)$. Since $h(A)$ has degree 2n, we conclude that $h(M)$ has degree $\geq n$. $\Box$

Corollary 4.2.9 Any holonomic module has finite length
Proof. Since $h(M) = c \cdot \frac{i^n}{n!} + \dots$ where c is a positive integer, and since the Hilbert polynomial is additive on short exact sequences, we immediately get $length\, M \leq c$. $\Box$

THEOREM 4.2.9.2. *If M is a holonomic A-module and $f \in \mathbb{C}[x_1, \dots, x_n]$ then $M[f^{-1}]$ is a holonomic A-module.*

Proof. Let $M_0 \subset M_1 \subset \dots$ be a good filtration on M. Write $N = M[f^{-1}]$. Let $d = deg\, f$. Define

$$N_j = \{mf^{-j} \mid m \in M_{(d+1)j}\}.$$

We will not try to prove that this filtration is good. However, one can check by direct computation that $h(\operatorname{gr} N)$ has degree n.

To prove that N is holonomic, take a finitely generated A-submodule N' of N and choose a good filtration $\{N'_j\}$ on N. There exists m_0 large enough such that $N'_j \subset N_{j+m_0}$ for all j. Hence $deg\, h(N') = n$, so N' is holonomic. Moreover, the leading coefficient is bounded by that of $h(\operatorname{gr} N)$. Hence any increasing sequence of submodules finitely generated over A terminates. Since we proved that any such submodule is holonomic, N itself is holonomic. $\square$

4.3. Functoriality for holonomic complexes.

Definition 4.3.1

For any smooth algebraic variety X of dimension n we define the *Verdier duality functor* by

$$\mathbb{D} : \mathcal{M} \mapsto R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, D_X)(n)$$

(i.e. we extend the algebraic definiton we used before to the setup of coherend $\mathcal{D}_X$-modules).

4.3.2 Remark By definition, for any left $\mathcal{D}_X$ module $\mathcal{M}$ the Verdier dual $\mathbb{D}\mathcal{M}$ is a complex of right $\mathcal{D}_X$-modules. However, by tensoring with Ω_X^{-1} as usual we can think of $\mathbb{D}\mathcal{M}$ as a complex of left $\mathcal{D}_X$-modules.

Apriori the complex $\mathbb{D}\mathcal{M}$ can be infinite. This would make using the duality functor very difficult. We will show now that this is not the case.

Recall (??) that we denote by $D^b_{coh}(\mathcal{D}_X)$ the derived category of all bounded complexes of $\mathcal{D}_X$-modules with $\mathcal{D}_X$-coherent cohomology.

Proposition 4.3.3 The Verdier duality functor $\mathbb{D}$ maps $D^b_{coh}(\mathcal{D}_X)$ to itself. In particular, for any $\mathcal{D}_X$-module $\mathcal{M}$ the Verideir dual $\mathbb{D}\,\mathcal{M}$ is a bounded complex.

Proof. It suffices to prove the statement for a signle $\mathcal{D}_X$-module $\mathcal{M}$ (the proof will extend to the general case easily). If $\mathcal{M} = \mathcal{D}_X$ we have $\mathbb{D}(\mathcal{M}) = \Omega^{-1} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ and the statement is true. For a general $\mathcal{D}_X$-module $\mathcal{M}$ we consider a resolution (possibly infinite) with free $\mathcal{D}_X$-modules:

$$\ldots \to \mathcal{M}_2 \to \mathcal{M}_1 \to \mathcal{M}_0 \to \mathcal{M} \to 0.$$

As in the commutative case we prove that the kernel of the map $\mathcal{M}^{2n-1} \to \mathcal{M}^{2n-2}$ is a projective $\mathcal{D}_X$-module (this follows from the vanishing of non-commutative Ext-groups which can be deduced from the vanishing of commutative Ext-groups for the corresponding graded objects). Note that the statement of the proposition is local in nature.

Hence, once we know that it's true for free modules it has to be true for projective modules. If we know it is true for projective modules, we can deduce it for any module using the resolution above and long exact sequences of $R\mathcal{H}om$'s. $\square$

Note that by our computations (reference !) of Ext-groups we know that $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ vanishes if $i > n$ or $i < \operatorname{codim} SS\,\mathcal{M}$. Therefore if $\mathcal{M}$ is holonomic, only one Ext group can survive, namely $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$. This can be expressed in the following

Corollary 4.3.4 If $\mathcal{M}$ is holonomic then

(i) the complex $R\mathcal{H}om(\mathcal{M}, \mathcal{D}_X)$ is quasi-isomorphic to the single object $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$,

(ii) $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ is a holonomic $\mathcal{D}_X$-module,

(iii) $\mathbb{D}$ defines a contravariant exact functor $\mathbb{D} : \mathcal{H}ol(\mathcal{D}_X) \to \mathcal{H}ol(\mathcal{D}_X)$ satisfying $\mathbb{D} \circ \mathbb{D} = Id$. *Proof.* The fact that $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ is holonomic can be proved using the estimates on $\dim\Big(SS\,\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)\Big)$ proved in Chapter 1 (REFERENCE !). $\square$

4.3.5 Example: duality and local systmes. For any pair of $\mathcal{D}_X$-modules $\mathcal{M}$ nad $\mathcal{N}$ one can defined the $\mathcal{O}$-tensor product $\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{N}$ with the $\mathcal{D}_X$-module structure defined by the Leibniz rule $\xi(m \otimes n) = \xi(m) \otimes n + m \otimes \xi(n)$. In general, this product is not considered, since it may not even be coherent even for $\mathcal{N} = \mathcal{D}_X$. However, it is immediate that if $\mathcal{N} = \mathcal{L}$ is a local system then the $\mathcal{O}$-tensor product is indeed a coherent $\mathcal{D}_X$-module.

Proposition 4.3.6 Let $\mathcal{L}$ is a local system and $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module then

(i) The tensor product $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is a holonomic $\mathcal{D}_X$-module,

(ii) One has $\mathbb{D}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}) \simeq \mathcal{L}^* \otimes_{\mathcal{O}_X} \mathbb{D}(\mathcal{M})$ where $\mathcal{L}^*$ is the dual in the sense of local systems. In particular, $\mathbb{D}(\mathcal{L}) \simeq \mathcal{L}^*$. $\square$

4.3.7 Digression: on derived categories of holonomic modules.

Recall that $D^b_{coh}(\mathcal{D}_X)$ denotes the derived category of bounded complexes of $\mathcal{D}_X$-modules with $\mathcal{D}_X$-coherent cohomology. Inside it one has an abelian subcategory $Coh(\mathcal{D}_X)$ formed by all $\mathcal{D}_X$-coherent modules. One might ask the following question: is it true that $D^b_{coh}(\mathcal{D}_X)$ is equivalent to the bounded derived category of $Coh(\mathcal{D}_X)$? The answer to this question is not known (and it might be "no").

However, if we consider the holonomic version of this problem, then the answer is known to be "yes". More precisely, consider the bounded derived category $D^b_{hol}(\mathcal{D}_X)$ of complexes of $\mathcal{D}_X$-modules with holonomic cohomology and its abelian subcategory $\mathcal{H}ol(\mathcal{D}_X)$ formed by all holonomic $\mathcal{D}_X$-modules. Then one has the following

Theorem 4.3.7.3. *There exists a natural equivalence of categories*

$$D^b(\mathcal{H}ol(\mathcal{D}_X)) \to \mathcal{D}^b_{hol}(\mathcal{D}_X).$$

This theorem will be proved later and now we will explain how its proof can be reduced to a statement about objects of $\mathcal{H}ol(\mathcal{D}_X)$.

4.3.8 Devissage.

Recall that for any complex A (of sheaves, $\mathcal{D}_X$-modules, etc.) and any integer k we defined the truncated complex $\tau_{k\geq}A$ (GIVE ITS CONSTRUCTION) together with the morphism $A \to \tau_{k\geq}A$. The crucial feature of $\tau_{k\leq}$ is that it defines an operation of the *derived category* of compexes.

Lemma 4.3.9 Suppose that $H^iA = 0$ for $i > k$. Then the natural morphism $A \to \tau_{k\geq}A$ induces a quasi-isomorphism $H^kA \xrightarrow{qis} \tau_{k\geq}A$. □

Now the devissage procedure says that if we want to prove some property for the derived category of complexes, it suffice to show that

(a) The property holds for indvidual objects(complexes concentrated in degree zero)

(b) The property respects the distingiushed triangles, i.e. if any two complexes in a distinguished triangle

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, "{[1]}"] & \end{tikzcd}$$

satisfy the property then the third complex also satisfies it.

In fact, in (b) we can take $B = \tau_{k\geq}A$, $C = Cone(A \to \tau_{k\geq}A)$ and this allows us to prove the statement by induction on the degree of the top non-zero cohomology of A.

One useful application of devissage is the following theorem

Theorem 4.3.9.4. *Let U be an open subset of X. If $\mathcal{M} \in D^b_{hol}(\mathcal{D}_U)$ then $j_*\mathcal{M} \in D^b_{hol}(\mathcal{D}_X)$.*

Proof. By the first part of devissage it suffices to prove the statement for $\mathcal{D}_U$-modules. We will prove later that if $j : U \hookrightarrow X$ is affine, then the statement follows from the theorem on b-function. If j is not affine then we consider an open affine cover $U = \cup U_i$ where each open subset U_i is given by non-vanishing of a regular function f_i. We assume, for simplicity, that $U = U_1 \cup U_2$ (the general case follows from this easily). Then one has a long exact sequence

$$0 \to j_* \mathcal{M} \to (j_1)_* \mathcal{M}|_{U_1} \oplus (j_2)_* \mathcal{M}|_{U_2} \to (j_{12})_* \mathcal{M}|_{U_1 \cap U_2} \to \dots$$

the second part of the devissage is obvious. □

4.3.10 A criterion of holonomicity.

Suppose that $Y = X \setminus U \overset{i}{\hookrightarrow} X$ is a smooth subvariety. Then for any $\mathcal{D}_X$-module $\mathcal{M}$ one has the following distinguished triangle:

$$\begin{tikzcd} i_* i^! M \arrow[rr] & & \mathcal{M} \arrow[dl] \\ & j_* \mathcal{M}|_U \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Since by the theorem above we know that $j_* \mathcal{M}|_U$ is holonomic, the module $i_* i^!, M$ is also holonomic. But holonimicity of $i_* i^! M$ is equivalent to the holonomicity of $i^! M$. In fact, one has exact sequence $0 \to T^*_Y X \to T^* X|_Y \xrightarrow{\pi} T^* Y \to 0$ and equality $\mathrm{SS}\, i_* i^! M = \pi^{-1}(SS\, i^! M)$ and the assertion now follows by counting dimensions. This proves the following

Corollary 4.3.11 For any closed embedding $i : Y \hookrightarrow X$ and any holonomic module $\mathcal{M}$ the $\mathcal{D}_Y$-module $i^! \mathcal{M}$ is also holonomic.

These results allow us to prove the following criterion of holonomicity:

THEOREM 4.3.11.5. *Let $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$. Then the following are equivalent:*
(i) $\mathcal{M} \in D^b_{hol}(\mathcal{D}_X)$,
(ii) For any point $i_x : x \hookrightarrow X$, the complex $i^!_x \mathcal{M}$ has finite dimensional cohomology.

Proof. The fact that (i) implies (ii) is an easy consequence of the previous corollary. We prove that (ii) implies (i) by induction on $\dim \mathit{Supp}\mathcal{M}$. Let $\mathit{Supp}\mathcal{M} = Y \hookrightarrow X$. Let $Y^{reg} \subset Y$ be the subset of regular points of Y. We can choose an open subset $U \subset X$ such that $U \cap Y = Y^{reg}$. We can replace X by U to get to the situation in which Kashiwara's theorem applies. By this theorem one has $\mathcal{M}|_U = i_* \mathcal{N}$ for some $\mathcal{D}$-module $\mathcal{N}$ on Y^{reg}. Since for a closed embedding the functors i^+ and $i^!$ coincide up to shift in derived category, we deducde that $i^!_x \mathcal{N}$ is finite-dimensional for any $x \in Y^{reg}$.

Choose a goood filtration on $\mathcal{N}$. The sheaf $\mathrm{gr}\, \mathcal{N}$ is a coherent sheaf on $T^* Y^{reg}$. By a basic result from algebraic geometry (**reference**) we can choose a smaller open set $U' \subset U$ such that $\operatorname{gr} \mathcal{N}$ becomes flat when restricted to $Y' = Y \cap U'$. In fact, first by shrinking U we acheive that $\operatorname{gr} \mathcal{N}$ is flat over T^*Y'. This will imply that each graded component of it is finitely generated and flat (hence projective) over Y'. Hence the first term $\mathcal{N}^0$ of the filtration on $\mathcal{N}$ is projective. Since any extension of two projective modules is again projective, we deduce that $\mathcal{N}^1$ is projective. Continuing in this manner we obtain that all $\mathcal{N}^i$ are flat. Hence $\mathcal{N}$, being a direct limit of projective modules, is flat over Y'. Then for any $x \in Y'$ the complex $i_x^!(\mathcal{N})$ is in fact one object $i_x^+\mathcal{N}$ concentrated in some degree (since $i_x^!$ is equal up to shift to i_x^+ and higher derived functors of i_x^+ all vanish by flatness). Hence $\mathcal{N}|_{Y'}$ is a flat sheave with finite-dimensional geometric fibers. This implies that $\mathcal{N}|_{Y'}$ is a local system, hence holonomic.

We have obtained a locally closed embedding $i' : Y' \hookrightarrow X$ such that $(i')^!\mathcal{M}$ is holonomic. Now we deduce that $(i')_*(i')^!\mathcal{M}$ is holonomic. This follows from factoring i' into a composition of an open embedding and a closed embedding and applying the holonomicity theorems above. Let $V' = X \setminus Y' \overset{j'}{\hookrightarrow} X$. Then the cone of the morphism $(i')_*(i')^!\mathcal{M} \to \mathcal{M}$ is isomorphic to $j'_*\mathcal{M}|_{V'}$ which has support of strictly smaller dimension. So, by the inductive assumption $j'_*\mathcal{M}|_{V'}$ is holonomic, hence $\mathcal{M}$ itself is holonomic. $\square$

4.3.12 Remark In terms of Corollary (4.1), one can describe the locally closed subvariety Y' as follows. Decompose $SS\,\mathcal{M}$ as $T^*_{Y^{reg}}X \cup \{\text{other components}\}$. Then $Y' = Y^{reg} \setminus \{\text{images of the other components}\}$.

4.3.13 The functors f^* and $f_!$.

Recall that earlier we have defined for any map $f : Y \to X$ of smooth algebraic varieties the functors f_* and $f^!$ on $\mathcal{D}$-modules.

In the special case of *holonomic* modules we can define another pair of functors, f^*, $f_!$, using Verdier duality functor:

$$f^! := \mathbb{D} \circ f^! \circ \mathbb{D}; \qquad f_! = \mathbb{D} \circ f_* \circ \mathbb{D}.$$

We want to emphasize that we assume that the domain of $f^!$ is $D^b_{hol}(\mathcal{D}_X)$ and the domain of $f_!$ is $D^b_{hol}(\mathcal{D}_Y)$, since in general the Verdier duality functor is ill-behaved.

There seems to be no nicer formula for f^* and $f_!$ and all the properties of these functors are established using Verdier duality.

4.3.14 Example Let $j : U \hookrightarrow X$ be an affine open embedding (i.e. $Y = X \setminus U$ is of codimension 1. Then j_* is exact. We will prove in the next section that for any $\mathcal{D}_U$-module $\mathcal{M}$, the module $j_*\mathcal{M}$ has no submodules supported on Y. By duality one deduces that $j_!\mathcal{M}$ has no quotient modules supported on Y. Similarly, if $i : Y \hookrightarrow X$ is the closed embedding and $\mathcal{M}$ is a $\mathcal{D}_X$-module, by Kashiwara's theorem we deduce that $i_* i^* \mathcal{M}$ is the maximal quotient module of $\mathcal{M}$ which is supported on Y. Notice that the notion of "maximal quotient module" is well defined only for holonomic modules due to their artinian properties (while, of course, the notion of "maximal submodule" makes sense for any Noetherian module).

4.4. Diagram $Y \hookrightarrow X \hookleftarrow U$ and the DGM-extension.

4.4.1 Direct image from U.

Let $Y \overset{i}{\hookrightarrow} X$ be a smooth closed submanifold and $U = X \setminus Y \overset{j}{\hookrightarrow} X$ be its complement.

Proposition 4.4.2 For any $\mathcal{D}_U$ module $\mathcal{N}$ one has $i^! j_* \mathcal{N} = 0$.
Proof. This is a consequence or (**??**). In fact, since i_* is fully faithful, it suffices to prove that $i_* i^! j_* \mathcal{N} = 0$. By (**??**) this amounts to showing that $\Gamma_{[Y]} j_* \mathcal{N} = 0$ which follows from definitions. □

For any holonomic $\mathcal{D}_X$-module $\mathcal{M}$ one has a canonical map $\mathcal{M} \to j_*(\mathcal{M}|_U)$. Hence by duality and definition of $j_!$ one obtains a map $j_!(\mathcal{M}|_U) \to \mathcal{M}$. If we take $\mathcal{M}$ of the form $j_* \mathcal{N}$ for some holonomic $\mathcal{D}_U$-module $\mathcal{N}$, then one gets a canonical map $j_! \mathcal{N} \to j_* \mathcal{N}$. It follows immediately that the kernel and cokernel of it are supported on Y.

Definition 4.4.3 For any holonomic module $\mathcal{N}$ on a locally closed subvariety $Y \overset{j}{\hookrightarrow} X$ we define the Deligne-Goresky-MacPerson extension (or *minimal* extension) $j_{!*} \mathcal{N}$ of $\mathcal{N}$ by

$$j_{!*} \mathcal{N} = \text{ Image} \, (\mathcal{H}^0 j_! \mathcal{N} \to \mathcal{H}^0 j_* \mathcal{N}).$$

The basic properties of the DGM-extension are summarized in the following

Proposition 4.4.4 Let $\mathcal{N}$ be a holonomic module on Y and $j_{!*} \mathcal{N}$ its minimal extension. Then

(i) $j^!(j_{!*} \mathcal{N}) = \mathcal{N}$,

(ii) $j_{!*} \mathcal{N}$ has neither submodules nor quotient modules supported on
$\overline{Y} \setminus Y$,

(iii) Taking the minimal extension commutes with Verdier duality:

$$j_{!*}(\mathbb{D}\mathcal{N}) = \mathbb{D}(j_{!*} \mathcal{N}).$$

Proof. To show (i) take an open set $U \subset Y$ such that $U \cap \overline{Y} = Y$ and apply Kashiwara's theorem to U and Y. Part (iii) follows by self-duality of the map $j_! \mathcal{N} \to j_* \mathcal{N}$. To prove (ii) note that $j_{!*} \mathcal{N} \subset j_* \mathcal{N}$ has no submodules supported on $\overline{Y} \setminus Y$. The statement about quotient modules follows by duality. □

4.4.5 Warning. It might well be the case that $j_{!*}\mathcal{N}$ has a filtration with submodules such that one of the *intermediate* terms is supported on the boundary.

THEOREM 4.4.5.6. *Let* $Y \overset{j}{\hookrightarrow} X$ *be a locally closed embedding of smooth varieties and* $\mathcal{L}$ *be an irreducible local system on* Y*. Then* $j_{!*} \mathcal{L}$ *is a simple holonomic* $\mathcal{D}_X$*-module. Moreover, any simple holonomic* $\mathcal{D}_X$*-module* $\mathcal{M}$ *arises from an appropriate pair* $(Y, \mathcal{L})$*.*

Proof. Let $\mathcal{N} \subset \mathcal{M} = j_{!*} \mathcal{L}$ be a submodule. By the proposition above $\mathcal{N}$ cannot be supported on $\overline{Y} \setminus Y$. Hence $0 \neq j^! \mathcal{N} \subset j^! \mathcal{M} = \mathcal{L}$ and this implies $j^! \mathcal{N} = j^! \mathcal{M}$. Therefore $\mathcal{N} = \mathcal{M}$.

To prove the second part notice that in fact the argument for the first works backwards too: as in the previous section we can find a locally closed subvariety $Y \overset{j}{\hookrightarrow} X$ such that $j^! \mathcal{M}$ is a local system. By a similar argument one showd that the minimal extensionof this system coincides with $\mathcal{M}$. Finally, the irreducibility of this local system follows from simplicity on $\mathcal{M}$. □

4.4.6 REMARK Suppose that Y is a locally closed submanifold of X and Y^0 is an open subset of Y. Since all the varieties are *complex*, the fundamental group of Y^0 maps surjectively. Hence the restriction any irreducible local system $\mathcal{L}$ on Y to Y^0 is still irreducible. Moreover, one can show that the minimal extension of $\mathcal{L}|_{Y^0}$ coincides with the minimal extension of $\mathcal{L}$. Hence we can always assume that $\overline{Y} \setminus Y$ is of codimension 1 in $\overline{Y}$ (if not, we can just shrink Y).

From now on we will often consider the diagram

$$U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Y = f^{-1}(0)$$

where Y is a divisor given (locally) by an equation $\{f = 0\}$. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_U$-module. Choose an $\mathcal{O}_X$-coherent subsheaf $\mathcal{M}_0 \subset j.\mathcal{M}$ such that $\mathcal{D}_U \cdot \mathcal{M}_0 = \mathcal{M}$.

Proposition 4.4.7 For any $k \gg 0$ one has $j_{!*} \mathcal{M} = \mathcal{D}_X \cdot (f^k \mathcal{M}_0)$.

Proof. The module $j_* \mathcal{M}/j_{!*} \mathcal{M}$ is supported on the divisor Y hence all sections of this quotient are annihilated by f^k if $k \gg 0$. This implies that $\mathcal{D}_X(f^k \mathcal{M}_0) \subset j_{!*}\mathcal{M}$. Recall that $j_{!*} \mathcal{M}$ is holonomic, in particular, artinian. We deduce that the descending chain of submodules $\mathcal{D}_X(f^k\mathcal{M}_0) \supseteq \mathcal{D}_X(f^{k+1}\mathcal{M}_0) \supseteq \dots$, stabilizes as k is increasing. The result follows. $\square$

Recall that for a singular Y we defined $D^b_{hol}(\mathcal{D}_Y)$ to be the full subcategory of $D^b_{hol}(\mathcal{D}_X)$ generated by holonomic modules supported on Y.

Lemma 4.4.8 For any $\mathcal{M} \in \mathcal{H}ol(\mathcal{D}_U)$ the following objects are quasi-isomorphic in $D^b_{hol}(\mathcal{D}_Y)$:

$$Cone(j_!\,\mathcal{M} \to j_*\,\mathcal{M}) \overset{qis}{\sim} i^! j_!\,\mathcal{M} \overset{qis}{\sim} i^* j_*\,\mathcal{M}[-1??]$$

Moreover, $\mathcal{H}^{-1}i^!j_!\,\mathcal{M} = \mathcal{H}^0 i^* j_*\,\mathcal{M}$ and $\mathcal{H}^0 i^! j_!\,\mathcal{M} = \mathcal{H}^1 i^* j_*\,\mathcal{M}$. *Proof.* One has the following self-dual diagram

$$0 \to i^! j_!\,\mathcal{M} \to j_!\,\mathcal{M} \to j_*\,\mathcal{M} \to i^* j_*\,\mathcal{M} \to 0$$

(notice that since Y is a divisor, $j_!\,\mathcal{M}$, etc. is in fact just one object rather than a complex). Here $i^*j_*\,\mathcal{M}$ is the maximal quotient module supported on the boundary, and $i^!j_!\,\mathcal{M}$ is obtained by duality. Furthermore, for any $\mathcal{D}_X$-module $\mathcal{N}$ one has the following exact sequence:

$$0 \to \mathcal{H}^0 i^!\,\mathcal{N} \to \mathcal{N} \to j_*\mathcal{N} \to \mathcal{H}^1 i^!\,\mathcal{N} \to 0$$

If we take now $\mathcal{N} = j_!\,\mathcal{M}$ and look at the cokernel, we will get $\mathcal{H}^0 i^* j_*\,\mathcal{M} \simeq \mathcal{H}^1 i^! j_!\,\mathcal{M}$. The other equality is proved similarly. $\square$

If Y is a divisor we can summarize all the information about the objects involved in the following diagram (**I'll straighten it later**)

$$
\begin{tikzcd}
0 \arrow[d] & & & & & & \\
\mathcal{H}^1 i^* \mathcal{M}_X \arrow[ddrr] \arrow[dddr] & & & & 0 & & \mathcal{H}^{-1} i^! \mathcal{M}_X \\
& & & j_{!*}\mathcal{M} \arrow[ur] \arrow[dr] & & & \\
& & j_!\mathcal{M} \arrow[ur] \arrow[dr] & & j_*\mathcal{M} \arrow[uurr] \arrow[dr] & & \\
& \mathcal{H}^0 i^! j_! \mathcal{M} \arrow[ur] \arrow[dr] & & \mathcal{M}_X \arrow[ur] \arrow[dr] & & i^* j_* \mathcal{M} \arrow[uuur] & \\
0 \arrow[ur] & & i^! \mathcal{M}_X \arrow[ur] \arrow[dr] & & \mathcal{H}^0 i^* \mathcal{M}_X \arrow[ur] & & \\
& 0 \arrow[ur] & & \mathcal{H}^0 i^{!*} \mathcal{M} \arrow[ur] & & &
\end{tikzcd}
$$

where $i^{!*}\mathcal{M} = \{\text{Image of } i^!\mathcal{M} \text{ in } i^*\mathcal{M}\}$, $\mathcal{M}_X$ stands for a D_X-module and $\mathcal{M}$ stands for its restriction to U.

In general (i.e. when j is not affine) we just have a three-dimensional octahedral diagram in the derived category

$$
\begin{tikzcd}
 & & \mathcal{M}_X \arrow[dl, dotted] \arrow[drr] \arrow[ddr] & & \\
 & i_*i^*\mathcal{M}_X \arrow[rrr, dotted] \arrow[dl, dotted] & & & i_*i^!\mathcal{M}_X \\
j_!j^*\mathcal{M}_X \arrow[uurr] \arrow[rrr] & & & j_*j^*\mathcal{M}_X \arrow[ur] \arrow[dl] & \\
 & & i_*i^*j_*j^*\mathcal{M}_X \arrow[ull] \arrow[uurr] \arrow[uul, dotted] & &
\end{tikzcd}
$$

(**To check the directions of the arrows**) The equivalence of the two diagrams in the case when j is affine is a non-trivial fact and we express it as a theorem

THEOREM 4.4.8.7. *If Y is a divisor (i.e. if j is affine) then the octahedral diagram is equivalent to the plane diagram above.*

4.4.9 REMARK The octahedral diagram above has a close relation to the octahedral diagram involved in the definition of a derived category. The latter is usually represented as two "hats" (upper and lower):

$$
\begin{tikzcd}
A' \arrow[dd] \arrow[dr, "{[1]}"] \arrow[rr, phantom, "dist" near end, shift right=3ex] & & C \arrow[ll] \\
\oplus & B \arrow[ur] \arrow[dl] & \oplus \\
C' \arrow[rr] \arrow[rr, phantom, "dist" near start, shift left=3ex] & & A \arrow[uu] \arrow[ul, "{[1]}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A' \arrow[dd] \arrow[rr, phantom, "\oplus", shift right=3ex] & & C \arrow[ll] \arrow[dl] \\
dist & B' \arrow[ul, "{[1]}"'] \arrow[dr, "{[1]}"] & dist \\
C' \arrow[rr] \arrow[ur] \arrow[rr, phantom, "\oplus", shift left=3ex] & & A \arrow[uu]
\end{tikzcd}
$$

The condition imposed on this diagram is that all triangles marked by *dist* are distinguished triangles, all triangles marked by $\oplus$ commute and the two maps from B to B' (via C and C') coincide (these three conditions imply **??** that the two possible maps from B' to B also coincide). The octahedron axiom of the derived category says that any upper hat can be completed by a lower hat, and any lower hat can be completed by an upper hat. This axiom originates from the *Ore conditions* which have to be satisfied by quasi-isomorphisms when the derived category of complexes (of abelian groups, say) is obtained as a localization of the homotopy category of complexes.

4.6. Vanishing cycles and f^s.

Let $\mathcal{C}^*$ be the category of holonomic $\mathcal{D}$-modules on $\mathbb{C}^*$ such that all simple subquotients are isomorphic to $\Gamma(\mathcal{O}_{\mathbb{C}^*}) = \mathbb{C}[t, t^{-1}] = R$.

Lemma 4.6.1 The category $\mathcal{C}^*$ is naturally equivalent to the category of all finite dimensional vectors spaces V endowed with a nilpotent operator $s : V \to V$.

Proof. Any object $\mathcal{M}$ of $\mathcal{C}^*$ is necessarily a local system on $\mathbb{C}^*$. Given such $\mathcal{M}$, put $V = \mathcal{M}|_{\{1\}}$ and let u be the monodromy operator. Since all the subquotients have trivial monodromy, u has to be unipotent. Hence the operator $s = log\, u$ is nilpotent. $\square$

The lemma above allows us to identify the indecomposable objects of $\mathcal{C}^*$. In fact, by Jordan normal form any indecomposable object of the category of vector spaces with a nilpotent operator, is given by a nilpotent Jordan block of the size n. One can easily check that this block corresponds to the $\mathcal{D}$-module $\mathcal{M}_n = R \cdot \log^{n-1} t + R \cdot log^{n-2} t + \ldots + R$ (with the monodromy operator u given by $exp(t\partial_t)$).

We can also see from the lemma above that $\mathcal{C}^*$ has no projective modules. We can consider a pro-object

$$\mathcal{E}^{proj} = \lim_{\leftarrow} \mathcal{M}_n \rightsquigarrow \begin{pmatrix} \mathcal{O} \\ \mathcal{O} \\ \cdot \\ \cdot \end{pmatrix}$$

and an ind-object

$$\mathcal{E}^{ind} = \lim_{\rightarrow} \mathcal{M}_n \rightsquigarrow \begin{pmatrix} \cdot \\ \cdot \\ \mathcal{O} \\ \mathcal{O} \end{pmatrix}$$

Now let $\mathcal{C}$ be the category of all $\mathcal{D}$-modules $\mathcal{M}$ on $\mathbb{C}$ such that $\mathcal{M}|_{\mathbb{C}^*} \in \mathcal{C}^*$. In particular, one has a simple object $\delta = i_*\mathbb{C}$ where $i : \{0\} \hookrightarrow \mathbb{C}$. Another simple object of $\mathbb{C}$ can be obtained as a minimal extension of a $\mathcal{D}_{\mathbb{C}^*}$-module R to $\mathbb{C}$. In fact, one can see that

$$j_!R \rightsquigarrow \begin{pmatrix} \mathcal{O} \\ \delta \end{pmatrix} \qquad j_*R \rightsquigarrow \begin{pmatrix} \delta \\ \mathcal{O} \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{O} \\ \delta \end{pmatrix} \searrow \begin{pmatrix} \delta \\ \mathcal{O} \end{pmatrix},$$

and the map $j_!R \to j_*R$ mods out the bottom of $j_!R$ and maps the quotient $\mathcal{O}$ isomprhically onto the bottom of j_*R. Hence $j_{!*}R = \mathcal{O}$.

From now on we will denote $\mathcal{M}_{n-1}$ by Log^{n-1}. Then one has $j_!Log^{n-1} = \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}} \cdot (t\partial)^n$, $j_*Log^{n-1} = \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}} \cdot (\partial t)^n$ hence

$$j_*Log^{n-1} \rightsquigarrow \begin{pmatrix} \delta \\ \mathcal{O} \\ \cdot \\ \cdot \\ \delta \\ \mathcal{O} \end{pmatrix}, \qquad j_!Log^{n-1} \rightsquigarrow \begin{pmatrix} \mathcal{O} \\ \delta \\ \cdot \\ \cdot \\ \mathcal{O} \\ \delta \end{pmatrix}$$

where in both cases on has n pairs in the columns. Notice that $\mathbb{D}j_!Log^{n-1} = j_*Log^{n-1}$. We introduce the following pro- and ind-objects:

$$\nabla^{proj} = j_!\mathcal{E}^{proj}, \quad \Delta^{proj} = j_*\mathcal{E}^{proj}, \quad \Delta^{ind} = j_*\mathcal{E}^{ind}, \quad \nabla^{ind} = j_!\mathcal{E}^{ind}.$$

Lemma 4.6.2 ∇^{proj} and Δ^{proj} are indecomposable and projective. □

Note that there exist two surjective maps $\nabla^{proj} \to \mathcal{O} \to 0$ and $\delta^{proj} \to \delta \to 0$.

Lemma 4.6.3 The category $\mathcal{C}$ is equivalent to the category of pairs of vector spaces V_0, V_1 together with two linear maps $u : V_0 \to V_1$, $v : V_1 \to V_0$ satisfying $(uv)^n = 0$ for some n.

Proof. We define the equivalence by putting

$$\mathcal{M} \mapsto (V_0 = Hom_{\mathcal{C}}(\Delta^{proj}, M), V_1 = Hom_{\mathcal{C}}(\nabla^{proj}, M)),$$

where the maps u and v are induced by the natural maps $\delta^{proj} \leftrightarrows \nabla^{proj}$. Note that V_1 is nothing but the fiber of $\mathcal{M}$ at 1. □

4.6.4 Notation For any object $\mathcal{M}$ of $\mathcal{C}$ we denote $Hom_{\mathcal{C}}(\Delta^{proj}, M)$ by $\Phi(\mathcal{M})$ (the functor of *vanishing cycles*) and $Hom_{\mathcal{C}}(\nabla^{proj}, M))$ by $\Psi(\mathcal{M})$ (the functor of *nearby cycles*).

We have a canonical **residue pairing** $\mathbb{C}[[s]] \times \mathbb{C}((s))/\mathbb{C}[[s]] \longrightarrow \mathbb{C}$ given by taking the residue $f \times g \longmapsto Res_{s=0}(f \cdot g)$. It is a perfect pairing making $\mathbb{C}((s))/\mathbb{C}[[s]]$ the *continuous* dual of $\mathbb{C}[[s]]$, viewed as a topological $\mathbb{C}$-vector space equipped with s-adic topology.

Lemma 4.6.5

$$Log^{n-1} = \frac{R\,t^s[s]}{s^n R\,t^s[s]}\,.$$

Proof. Define a map $R\,t^s[s] \to Log^{n-1}$ as follows. Let $\sum_{i=0}^{n} a_i s^i t^s$ be an element of $R\,t^s[s]$. Put formally $t^s = e^{s\,log\,t} = \sum_{k=0}^{\infty} \frac{s^k \cdot log(kt)}{k!}$. So, we send

$$\sum_{i=0}^{n} a_i s^i t^s \mapsto Res_{s=0}\Big(\frac{1}{s}\,(\sum_{i=0}^{n} a_i s^i)(\sum_{k=0}^{\infty} \frac{s^k\,log(kt)}{k!})\Big).$$

This induces a bijection

$$Log^{n-1} = \frac{R\,t^s[s]}{s^n R\,t^s[s]}. \quad \square$$

Using this lemma we can make the following key observation:

$$\mathcal{E}^{proj} = Rt^s[[s]] = \lim_{\leftarrow} \frac{Rt^s[s]}{s^n Rt^s[s]}.$$

Moreover, under this isomorphism the logarithm of monodromy corresponds to s. When we take the direct image with respect to j we obtain the following diagram of objects on $\mathbb{C}$

$$\Delta^{proj} = j_*\mathcal{E}^{proj} = Rt^s[[s]] \hookrightarrow Rt^s((s)) \twoheadrightarrow \frac{Rt^s(ss)}{Rt^s[[s]]} = \Delta^{ind},$$

where the middle term is a pro-ind-object (self-dual with respect to $s \mapsto s^{-1}$) and the last equality is proved by using the correspondence

$$\sum_{i\in\mathbb{Z}} a_j s^i t^s \mapsto Res_{s=0}\Big(\frac{1}{s}\,(\sum a_i s^i)(\sum_{k=0}^{\infty} \frac{s^k\,log(kt)}{k!})\Big)$$

and noting that $\frac{Rt^s((s))}{Rt^s[[s]]} \simeq R[log\,t] = \Delta^{ind}$.

Corollary 4.6.6 The following sequences are exact:

$$0 \to j_!\mathcal{E}^{proj} \to j_*\mathcal{E}^{proj} \to \delta \to 0,$$

$$0 \to \delta \to j_!\mathcal{E}^{ind} \to j_*\mathcal{E}^{ind} \to 0 \quad \square$$

Let $f : X \to \mathbb{C}$ be a regular function such that df never vanishes outside $Y = f^{-1}(0)$:

$$\begin{array}{ccc} U = X \setminus Y & \stackrel{j}{\hookrightarrow} & X \\ \downarrow & & \downarrow f \\ \mathbb{C}^* & \hookrightarrow & \mathbb{C} \end{array}$$

Define $\mathcal{E}_f^{proj} = f^+\mathcal{E}^{proj}$. (So, formally one has $t^s \mapsto f^s, log\, t \mapsto log\, f$.)

For any holonomic $\mathcal{D}_U$-module $\mathcal{M}$, we have

$$\mathbb{D}(\mathcal{E}_f^{proj} \otimes_{\mathcal{O}_U} \mathcal{M}) = \mathcal{E}_f^{ind} \otimes_{\mathcal{O}_U} \mathbb{D}\mathcal{M}.$$

Proposition 4.6.7 In our pro-ind category

$$j_!\big(\mathcal{M}f^s((s))\big) \simeq j_*\big(\mathcal{M}f^s((s))\big).$$

Proof. Recall that

$$j_{!*}\big(\mathcal{M}f^s((s))\big) = Im\big[j_!\, \mathcal{M}f^s((s)) \to j_*\, \mathcal{M}f^s((s))\big]$$

and this module can be computed as

$$j_{!*}\mathcal{N} = \mathcal{D}_X(f^k\mathcal{N}_0) \subset j_*\mathcal{N},$$

where $\mathcal{N}_0$ is an $\mathcal{O}$-coherent submodule of $\mathcal{N}$ and $k \gg 0$. Assume for simplicity that $\mathcal{M} = \mathcal{D}_X \cdot m$ where m is a section of $\mathcal{M}$ (the general case follows from this by induction on the number of generators). Then

$$j_{!*}\big(\mathcal{M}f^s((s))\big) = \mathcal{D}_X\big(mf^kf^s\big)((s)) = \mathcal{D}_X\big(mf^{k+s}((s))\big) = \mathcal{M}f^s((s))$$

by the lemma on b-function. Hence our map is surjective. By duality it is also injective, therefore it is an isomorphism. □

Corollary 4.6.8 The canonical map

$$j_!\, \mathcal{M}f^s[[s]] \to j_*\, \mathcal{M}f^s[[s]]$$

is injective. □

Corollary 4.6.9 If $\mathcal{M}$ is generated by an $\mathcal{O}_X$-coherent subsheaf $\mathcal{M}_0$, then

$$\begin{array}{ll} (i) & j_!\, \mathcal{M}f^s[[s]] = \mathcal{D}_X[[s]] \cdot (f^{k+s}\mathcal{M}_0), \quad k \gg 0 \\ (ii) & j_*\, \mathcal{M}f^s[[s]] = \mathcal{D}_X[[s]] \cdot (f^{ks}\mathcal{M}_0), \quad k \gg 0. \end{array}$$

Proof. $j_!\mathcal{M}f^s[[s]] = j_{!*}\mathcal{M}f^s[[s]] = \mathcal{D}_X[[s]] \cdot (f^{k+s}\mathcal{M}_0)$. The second assertion is proved similarly. □

Corollary 4.6.10 For $k \gg 0$ one has the follwoing ismorphisms

$$j_!\, \mathcal{M} = \frac{\mathcal{D}_X[s](f^{s+k}\mathcal{M}_0)}{s \cdot \mathcal{D}_X[s](f^{s+k}\mathcal{M}_0)}, \quad j_*\, \mathcal{M} = \frac{\mathcal{D}_X[s](f^{s-k}\mathcal{M}_0)}{s \cdot \mathcal{D}_X[s](f^{s-k}\mathcal{M}_0)}.$$

Proof. We have the following short exact sequance on U

$$0 \to \mathcal{M}f^s[[s]] \xrightarrow{s\cdot} f^s\mathcal{M}[[s]] \to \mathcal{M} \to 0.$$

By exactness of $j_!$ this gives

$$0 \to j_!\,\mathcal{M}f^s[[s]] \xrightarrow{s\cdot} j_!\,f^s\mathcal{M}[[s]] \to j_!\,\mathcal{M} \to 0,$$

and the result for $j_!$ follows. The statment for j_* is proved similarly. □

Definition 4.6.11 For any $\mathcal{D}_U$-module $\mathcal{M}$ we define a submodule

$$\Psi(\mathcal{M}) = \frac{j_*\,\mathcal{M}f^s[[s]]}{j_!\,\mathcal{M}f^s[[s]]}$$

supported on the divisor $Y = f^{-1}(0)$.

Proposition 4.6.12 $\Psi(\mathcal{M})$ is a holonomic module.

Proof. It suffices to show that, for some $N \gg 0$,

$$s^N j_*\,\mathcal{M}f^s[[s]] \subset j_!\,\mathcal{M}f^s[[s]]$$

(by the previous corollary). Note that:

$$s^N\left(\mathcal{D}_X[[s]](f^{s-k}\mathcal{M}_0)\right) \subset \mathcal{D}_X[[s]](f^{s+k}\mathcal{M}_0).$$

Assume for simplicity that $\mathcal{M}_0$ is generated by one section m. By the lemma on b-function, there exists an operator $u(\partial, s)$ such that $u(\partial, s)(m \cdot f^{s+1}) = b(s)\cdot(mf^s)$. Then $u(\partial, s+k-i)(m\cdot f^{s+1+k-i}) = b(s+k-i)m\cdot f^{s+k-i}$. Hence we can find an operator Q such that

$$Q(\partial, s)(m\cdot f^{s+k}) = b(s+k-1)\cdot b(s+k-2)\ldots b(s-k)(m\cdot f^{s-k}).$$

Now take N to be equal to the number of integral roots of b-finction. (Since $s-\alpha$ is invertible in $\mathbb{C}[[s]]$.) (**That's was someone's notes and I don't understand this last part**). □

Corollary 4.6.13 The functor $\Psi : \mathcal{H}ol(\mathcal{D}_U) \to \mathcal{H}ol(\mathcal{D}_Y)$ is exact.

Proof. The functor $D_{hol}(\mathcal{D}_U) \to D_{hol}(\mathcal{D}_Y)$ preserves the abelian core of these derived categories. (**same comment as before**). □

Corollary 4.6.14 $\Psi(\mathbb{D}\mathcal{M}) = \mathbb{D}(\Psi\mathcal{M})$.

Proof. One has the following short exact sequence

$$0 \to j_!\,\mathcal{M}f^s[[s]] \to j_!\,\mathcal{M}f^s((s)) \to \frac{j_!\,\mathcal{M}f^s((s))}{j_!\,\mathcal{M}f^s[[s]]} \to 0$$

where the quotient $\frac{j_!\,\mathcal{M}f^s((s))}{j_!\,\mathcal{M}f^s[[s]]}$ is isomorphic to $j_!(\mathcal{M}\otimes\mathcal{E}_f^{ind})$. Now

$$\mathbb{D}\,\Psi\,\mathbb{D}(\mathcal{M}) = \mathbb{D}\,i^*j_*(\mathbb{D}\mathcal{M}\otimes\mathcal{E}_f^{proj}) =$$

$$= \mathbb{D}\,i^*j_*\,\mathbb{D}\,(\mathcal{M}\otimes\mathcal{E}_f^{ind}) = i^!j_!(\mathcal{M}\otimes\mathcal{E}_f^{ind}) = \Psi(\mathcal{M}).$$ □

4.6.15 Remark From now on all the arguments will makes sense not only in the category of $\mathcal{D}$-modules, but also in the category of $\mathcal{D}$–,odules with regular singularities, mixed Hodge modules, perverse sheaves, etc.

Beilinson in his (**reference**) introduced another functor that we are about to define.

Definition 4.6.16 For any $\mathcal{D}_U$-module $\mathcal{M}$ define

$$\Xi(\mathcal{M}) = \frac{j_* \mathcal{M} f^s[[s]]}{j_! \mathcal{M} f^s s[[s]]}$$

The functor Ξ is called *the maximal extension* (and sometimes the "double tail")

It follows from the definition that $\Xi(\mathcal{M})$ has a submodule $\frac{j_! \mathcal{M} f^s[[s]]}{j_! \mathcal{M} f^s s[[s]]}$ with the quotient module isomorphic to $\Psi(\mathcal{M})$:

$$0 \to j_! \mathcal{M} \xrightarrow{\alpha_-} \Xi(\mathcal{M}) \xrightarrow{\beta_-} \Psi(\mathcal{M}) \to 0. \tag{4.6.17}$$

We can also consider a submodule $\frac{j_* \mathcal{M} f^s s[[s]]}{j_! \mathcal{M} f^s s[[s]]}$ which is itself isomorphic to $\Psi(\mathcal{M})$ since mulitplication by s is an isomorphism. This leads to a short exact sequence

$$0 \to \Psi(\mathcal{M}) \xrightarrow{\beta_+} \Xi(\mathcal{M}) \xrightarrow{\alpha_+} j_* \mathcal{M} \to 0. \tag{4.6.18}$$

Lemma 4.6.19 The functor Ξ has the following properties:

(i) Ξ is exact,

(ii) Ξ commutes with Verdier duality,

(iii) The two exact sequences (4.6.17) and (4.6.18) get interchanged by the Verdier duality,

(iv) The composition $\alpha_+ \circ \alpha_- : j_! \mathcal{M} \to j_* \mathcal{M}$ coincides with the canonical map form $j_! \mathcal{M}$ to $j_* \mathcal{M}$,

(v) The composition $\beta_- \circ \beta_+ : \Psi(\mathcal{M}) \to \Psi(\mathcal{M})$ is multiplication by s.

Proof. The proof of exactness is the same as for Ψ. **The rest was omitted.** □

4.6.20 Example Condider a $\mathcal{D}$-module on the punctured affine line $\mathbb{C} \setminus \{0\}$ given by the ring of functons $R = \mathbb{C}[x, x^{-1}]$. One has

$$\Xi(R) \rightsquigarrow \begin{pmatrix} \delta \\ \mathcal{O} \\ \delta \end{pmatrix} \qquad \text{so called '}\textit{two-sided } \delta\textit{-tail}\text{'}.$$

In this case we can see that $\Xi(R)$ is indeed self-dual and also that this extension is *maximal* (since we know all the indecomposable objects, it follows that the only way the make $\Xi(R)$ bigger is to add at least one subquotient isomorphic to $\mathcal{O}$ which would change the restriction to the open part $\mathbb{C} \setminus \{0\}$).

Let $\mathcal{M}_X \in \mathcal{D}_X$ and $\mathcal{M}_U = \mathcal{M}_X|_U$. Consider the following complex

$$j_! \mathcal{M}_U \xrightarrow{(\alpha_- \oplus \gamma_-)} \Xi(\mathcal{M}_U) \oplus \mathcal{M}_X \xrightarrow{(\alpha_+ \oplus -\gamma_+)} j_* \mathcal{M}_U$$

where the γ's are the canonical maps. It follows immediately that the first map is injective and the second map is surjective.

Definition 4.6.21

$$\Phi(\mathcal{M}_X) = \frac{Ker\big(\alpha_+ \oplus -\gamma_+\big)}{Im\big(\alpha_- \oplus \gamma_-\big)}.$$

Lemma 4.6.22 The module $\Phi(\mathcal{M}_X)$ is supported on $Y = f^{-1}(0)$.

Proof. On the open part U the complex above restricts to

$$0 \to \mathcal{M}_U \to \mathcal{M}_U \oplus \mathcal{M}_U \to \mathcal{M}_U \to 0. \quad \square$$

Proposition 4.6.23 The functor Φ has the following properties

(i) Φ takes $\mathcal{D}$-modules to $\mathcal{D}$-modules (i.e. not just to complexes of $\mathcal{D}$-modules),

(ii) Φ commutes with the Verdier Duality functor $\mathbb{D}$,

(iii) Φ is exact.

Proof. The first assertion follows from the definition, the second from self-duality of the complex defining Φ and the third is proved by diagram chase. $\square$

4.6.24 Remark One can also give alternative definitions of the functors Ψ and Φ:

$$\Psi(\mathcal{M}_X) = i^*(\Delta^{ind} \otimes_{\mathcal{O}_X} \mathcal{M}_X), \quad \Phi(\mathcal{M}_X) = i^*(\nabla^{ind} \otimes_{\mathcal{O}_X} \mathcal{M}_X).$$

These definitions will not be used on these notes.

Note that the maps $(\beta_+, 0) : \Psi(\mathcal{M}_U) \to \Xi(\mathcal{M}_U) \oplus \mathcal{M}_X$ and $(-\beta_-) : \Xi(\mathcal{M}_U) \oplus \mathcal{M}_X \to \Psi(\mathcal{M}_U)$ descend to a pair of maps $u : \Psi(\mathcal{M}_U) \to \Phi(\mathcal{M}_X)$ and $v : \Phi(\mathcal{M}_X) \to \Psi(M_U)$, respectively. Moreover, the composition $v \circ u = \beta_- \circ \beta_+$ is equal to the monodromy map s. We will also use the following exact sequence:

$$0 \to i^! \mathcal{M}_X \to \Phi(\mathcal{M}_X) \to \Psi(\mathcal{M}_U) \to \mathcal{H}^1 i^! \mathcal{M}_X \to 0,$$

and the one obtained from it by Verdier duality:

$$0 \to i^* \mathcal{M}_X \to \Psi(\mathcal{M}_U) \to \Phi(\mathcal{M}_X) \to \mathcal{H}^1 i^* \mathcal{M}_X \to 0.$$

Now we apply this formalism to the following situation:

4.6.25 Problem. Let $Y = f^{-1}(0) \stackrel{i}{\hookrightarrow} X \stackrel{j}{\hookleftarrow} U$ be our usual diagram and suppose that Y is a smooth divisor. One can ask the following question: how can we glue the category $\mathcal{H}ol(\mathcal{D}_X)$ from the two categories $\mathcal{H}ol(\mathcal{D}_U)$ and $\mathcal{H}ol(\mathcal{D}_Y)$?

The answer to this question in the case of sheaves is that the gluing conditin is given by the specialization map. This data, however, is not "finite" (i.e. cannot be expressed in the language of finite-dimensional vector spaces) since the category of all sheaves on X is not artinian (IS THERE A BETTER EXPLANATION ?).

The problem for $\mathcal{D}$-modules was solved independently by Deligne, Verdier and Beilinson (and all the constructions can be also applied to perverse sheaves, mixed Hodge modules, etc.) Here we will give the construction due to Beilinson. One advantage of it is that it works even when Y is singular.

4.6.26 Gluing data. Consider the quadruples $(\mathcal{M}_U, \mathcal{M}_Y, u, v)$ where $u : \Psi(\mathcal{M}_U) \to \mathcal{M}_Y$ and $v : \mathcal{M}_Y \to \Psi(\mathcal{M}_U)$ are two morphisms of $\mathcal{D}_Y$-modules satisfying $(u \circ v)^{n_1} = 0$, $(v \circ u)^{n_2} = 0$, for $n_1, n_2 \gg 0$.

4.6.27 REMARK If $\mathcal{M}$ is an object of $\mathcal{H}ol(\mathcal{D}_X)$ then $\mathcal{M}_U = M|U, \mathcal{M}_Y = \Phi(\mathcal{M})$ and the morphisms u, v described above form a quadriple $F(\mathcal{M}) = (\mathcal{M}_U, \Phi(\mathcal{M}), u, v)$ satisfying the conditons imposed on gluing data.

Definition 4.6.28 Define the *gluing category* to be the category with objects being the gluing data quadruples and obvious morphisms.

THEOREM 4.6.28.1. *The category $\mathcal{H}ol(D_X)$ is equivalent to the gluing category.*

Proof. The remark above provides a functor F in one direction. Suppose a quadruple (M_U, M_Y, u, v) is given. Consider

$$\Psi(\mathcal{M}_U) \xrightarrow{(\beta_+ \oplus u)} \Xi(\mathcal{M}_U) \oplus \mathcal{M}_Y \xrightarrow{(\beta_- \oplus -v)} \Psi(\mathcal{M}_U)$$

One can see that the first arrow is injective (since β_+ is injective) and the second arrow is surjective (since β_- is surjective). Define

$$G(\mathcal{M}_U, \mathcal{M}_Y, u, v) = \frac{Ker\big(\beta_- \oplus -v\big)}{Im\big(\beta_+ \oplus u\big)}.$$

By direct diagram chase one shows that G defines an exact functor which is inverse to the functor F. $\square$

4.6.29 Remark The functor $\Xi(\mathcal{M})$ was introduced exactly because it is necessary to prove the theorem above **kosnoyazychno poluchilos'**.

4.6.30 Proof of (4.3.7.3).

(**the second proof uses functor Ξ. Maybe one could give the first proof right after the statement of the theorem and then give the second one much later**).

To formulate a general statement that implies the theorem, recall the definition of a *t-structure* on a triangulated category D. Let $h : D \to Ab$ be a functor from D to some abelian category Ab (one should think of h as a zero homology functor). For any $i \in \mathbb{Z}$, define the functors h^i by $h^i(M) = h(M[i])$, where M is an object of D and $[i]$ stands for shift in a triangulated category. We will say that h is *homological* if it $M \to \{h^i(M)\}_{i\in\mathbb{Z}}$ maps distinguished triangles to long exact sequences. Given such h, we can define

$$D^+ = \{M \mid h^i(M) = 0, \forall i < 0\}, \quad \text{and}$$

$$D^- = \{M \mid h^i(M) = 0, \forall i > 0\}.$$

Then we have an "exact sequence of categories": '$0 \to D^- \to D \to D^+ \to 0$.' Such a triple (D, D^+, D^-) is called a *t-structure* (one can show that any t-structure in the sense of the usual definition (cf. **???**) comes from some homological functor h, so this definiton is equivalent to the one of (cf. **???**)).

One defines a *heart* of t-structure to be $\mathcal{C} = D^+ \cap D^-$. It follows that $\mathcal{C}$ is an abelian category and there exists a functor $D^b(\mathcal{C}) \to D$ (**one should says smth abou iterated cones, and also that the axioms of the derived category are "bad"**). We will answer the following general question: is it true that the functor $D^b(\mathcal{C}) \to D$ is an equivalence of categories?

Of course, for any abelian heart $\mathcal{C}$ the categories $D^b(\mathcal{C})$ and D have the same *objects*. However, in general there is no reason why the *Hom*-groups in D should coincide with the Yoneda *Ext*'s in the category of complexes. We will outline two approaches to detecting in which case one in fact has such coincidence. The first approach is due to Beilinson:

Lemma 4.6.31 The natural functor $D^b(\mathcal{C}) \to D$ is an equivalence of categories if and only if one of the following equivalent conditions holds:

(i) $Ext^*_{\mathcal{C}}(M, N) \simeq Ext^*_D(M, N), \quad \forall M, N \in Ob(\mathcal{C})$.

(ii) For all $M, N \in Ob(\mathcal{C})$, $i > 0$ and $e \in Ext^i_{\mathcal{C}}(M, N)$ there exists an embedding $N \hookrightarrow N'$ (depending on e) such that the image of e under the map $Ext^i_{\mathcal{C}}(M, N) \to Ext^i_{\mathcal{C}}(M, N')$ is equal to zero.

(**Maybe one has to put finiteness conditions that are satisfied in examples**) $\square$

4.6.32 EXAMPLE Let us consider the situation when D is the derived category of complexes of sheaves with locally constant cohomology on some topological space X and $\mathcal{C}$ is the category Loc of local systems on X. Then we have to establish whether or not the groups $Ext^i_D(\mathcal{L}_1, \mathcal{L}_2) = H^i(X, \mathcal{L}_1^* \otimes \mathcal{L}_2)$ are isomorphic to $Ext^i_{Loc}(\mathcal{L}_1, \mathcal{L}_2) = Ext^i_{Loc}(\mathbf{1}, \mathcal{L}_1^* \otimes \mathcal{L}_2)$ (wheref $\mathbf{1}$ denotes the trivial one-dimensional local system. Denote the tensor product $\mathcal{L}_1^* \otimes \mathcal{L}_2$ by $\mathcal{L}$. Recall that the category Loc is equivalent to the category $Rep(\pi_1(X))$ of finite-dimensional representations of the fundamental group $\pi_1(X)$.

Now we want to know under what conditions the groups $H^i(X, \mathcal{L})$ are isomorphic to Yoneda groups $Ext_{Rep(\pi_1(X))}(\mathbf{1}, \mathcal{L})$. In general this is a dificult question but one can give a sufficient condition. First note that we have to use the Yoneda Ext's since the category $Rep(\pi_1(X))$ of *finite-dimensional* representations does not have enough projective objects. To use projective resolutions one has to consider a larger category $Rep^\infty(\pi_1(X))$ if infinite-dimensional representations of $\pi_1(X)$. With this understood, we can formulate the sufficient conditions on X:

(1) X is a $K(\pi_1(X), 1)$-space and

(2) Ext-groups in the category $Rep^\infty(\pi_1(X))$ coincide with the Ext-groups in the category $Rep(\pi_1(X))$.

Note that the second condition depends on the group $\pi_1(X)$ only. It is known to hold in the following cases:

(a) $\pi_1(X)$ is a finite group.

(b) $\pi_1(X)$ is a finitely generated free abelian group.

(c) $\pi_1(X)$ is a finitely generated free (non-abelian) group.

The condition (c) is of particular importance to us since it is satisfied when X is, for example, an affine algebraic curve. Hence we have proved the following proposition

Proposition 4.6.33 If X is a $K(\pi, 1)$-space and π satisfies one of the conditions above then the derived category D^b_{Loc} is equivalent to the derived category $D^b(Loc)$. $\square$

Now we are ready to prove that the category $D^b_{hol}(\mathcal{D}_X)$ is equivalent to the derived category of $\mathcal{H}ol(\mathcal{D}_X)$.

In fact, let $X = \coprod X_\alpha$ be an algebraic stratification of X such that:

(1) Each stratum X_α is a $K(\pi, 1)$-space and $\pi_1(X_\alpha)$ is a free finitely generated group, and

(2) For any α the morphism $j_\alpha : X_\alpha \hookrightarrow X$ is affine (this condition ensures that all the higher derived functors of $(j_\alpha)_*$ and $(j_\alpha)_!$ vanish).

Consider the category $\mathcal{C} \subset \mathcal{H}ol(\mathcal{D}_X, \{X_\alpha\})$ of holonomic $\mathcal{D}_X$-modules on X smooth along the strata.

THEOREM 4.6.33.2. *If $X = \coprod X_\alpha$ is a smooth algebraic stratification as above and and $\mathcal{M}, \mathcal{N} \in \mathcal{C}$ then, for any i,*

$$Ext^i_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) = Ext^i_{D^b_{hol}}(\mathcal{M}, \mathcal{N}).$$

Proof. For $i = 1$ the statement follows from the fact that he category $\mathcal{C}$ is stable under extensions. For the general case we need the following

Proposition 4.6.34 The following properties of a $\mathcal{D}$-module $\mathcal{N} \in Ob(\mathcal{C})$ are equivalent:

(a) For any α and any local system $\mathcal{L}$ on X_α one has

$$Ext^1_{\mathcal{D}^b_{hol}}((j_\alpha)_!\mathcal{L}, \mathcal{N}) = 0,$$

(b) The same property holds for Ext^i with any positive i.

(c) N has a finite filtration $\{N_i\}$ such that $N_i/N_{i-1} = (j_\beta)_* I$, where I_β is an injective object of the category of local systems on X_β.

Comments on proof. (c) implies (b) since

$$Ext^i((j_\alpha)_!\mathcal{L}, (j_\beta)_* I) = Ext^i(j^*_\beta (j_\alpha)_!\mathcal{L}, I) = Ext^i(0, I) = 0$$

if $\beta \neq \alpha$ and if $\beta = \alpha$ we use injectivity of I. The implication (b) $\Rightarrow$ (a) is obvious. Finally, (a) $\Rightarrow$ (c) is proved by categorical diagram chasing using induction on strata and one trick due to Jantzen. $\square$

If all the fundamental groups of the strata are finite (e.g. trivial) then the category $\mathcal{C}$ has enough injective and projective objects. Hence we can proceed as follows:

Step 1. Using long exact sequences and the 5-lemma we can reduce to the case when $\mathcal{M}$ is simple (since any holonomic module is of finite length).

Step 2. Choose an injective resolution $0 \to \mathcal{N} \to I_0 \to I_1 \to \dots$ in $\mathcal{C}$. This resolution gives two spectral sequences for computing Ext-groups in $\mathcal{C}$ and $D^b_{hol}(X)$, respectively. Since the differentials in the spectral sequences commute with the maps between them it is enough to prove the theorem in the case when $\mathcal{N}$ is injective.

Step 3. Argue by induction on dim $Supp\mathcal{M}$. Suppose $Supp\,\mathcal{M} = \overline{X_\alpha}$. Then

$$0 \to Ker u \to (j_\alpha)_!(\mathcal{M}|_{X_\alpha}) \xrightarrow{u} \mathcal{M} \to 0,$$

where the map u is surjective by simplicity of M and $Ker(u)$ is supported on the strata of smaller dimension. It follows from the long exact sequence of Ext's that it suffices to show that the groups $Ext^i(j_\alpha)_!(\mathcal{M}|_{X_\alpha}, \mathcal{N})$

are the same in $\mathcal{C}$ and $D^b_{hol}(X)$. Since we assumed $\mathcal{N}$ to be injective both these groups are trivial by the proposition above.

In the general case (when the fundamental groups are not finite) we cannot assum that $\mathcal{N}$ is injective since $\mathcal{C}$ doesnot have enough injectives. Instead, given any particular $e \in Ext^i_{D^b_{hol}}(\mathcal{M}, \mathcal{N})$, we want ot find an inclusion $\mathcal{N} \subset \mathcal{N}'$ such that the image of e in $Ext^i_{D^b_{hol}}(\mathcal{M}, \mathcal{N})$ is equal to zero. Of course, if $\mathcal{N}$ embeds into injective module $\mathcal{N}'$ then $\mathcal{N}'$ will work for *all* e. If this is not the case we can still find a module $\mathcal{N}'$ which annihilates each *particular* e. (**details still to be worked out**) $\square$

This result suffices to prove (4.3.7.3) since, given any particular pair of $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, we can first choose a stratification of X such that $\mathcal{M}$ and $\mathcal{N}$ are smooth along the strata of it. By removing finitely many divisors from each stratum we can guarantee that the inculsion of this strarum in X is affine and the stratum itself is an iterated locally trivial fibration with fibers isomorphic to affine curves (hence a $K(\pi, 1)$-space. Hence, for any particular pair $\mathcal{M}$ and $\mathcal{N}$ we can choose a stratification such that the conditions (1) and (2) are satisfied. $\square$

We now outline a second approach to (4.3.7.3) due to Beilinson. Recall that it suffices to find an embedding $\mathcal{N} \hookrightarrow \mathcal{N}'$ annihilating some particular element $e \in Ext^i_{D^b_{hol}}(\mathcal{M}, \mathcal{N})$.

Step 1. Choose a Zariski open subset $U \subset X$ that has a smooth morphism to some variety Z with fibers isomorphic to affine curves. We will find a module $\mathcal{N}'$ which solves the problem fo $Ext^i_{\mathcal{D}^b_{hol}(U)}$. Once we know how to do that, we can choose an open cover $X = \cup U_\alpha$, find a module $\mathcal{N}'_\alpha$ on each U_α and take $\mathcal{N}'$ to be equal to $\bigoplus (j_\alpha)_* \mathcal{N}'_\alpha$. To that end, note that the Ext-groups on U can be computed via some Ext-groups on Z using Leray spectral sequence. Shrinking U if necessary we can assume that all the direct image sheaves on Z involved in the spectral sequence are locally free. We will deduce the statement for U from the statement for Z, inducting on dimension.

Step 2. Induction on $min(\dim Supp\, \mathcal{M}, \dim Supp\, \mathcal{N})$. Suppose that both dimensions are less then $\dim X$. Then there exists a divisor Y containing both $Supp\, \mathcal{M}$ and $Supp\, \mathcal{N}$ (such Y may not be smooth and irreducible). We represent Y (locally) as a zero set of some function f and consider our usual diagram $Y \hookrightarrow X \hookleftarrow V$. We obviously have $Ext^i_{D^b_{hol}(X)}(\mathcal{M}, \mathcal{N}) = Ext^i_{D^b_{hol}(Y)}(\mathcal{M}, \mathcal{N})$ since both sides of the equality have topological meaning. Hence it suffices to prove $Ext^i_{\mathcal{C}(X)}(\mathcal{M}, \mathcal{N}) =$

$Ext^i_{\mathcal{C}(Y)}(\mathcal{M},\mathcal{N})$, where the category $calHol(Y)$ of holonomic modules on Y is taken to be the full subcategory of $\mathcal{H}ol(X)$ generated by modules supportedon Y. We always have a map from the LHS of the equality to the RHS of it. Since the functor $\Phi : \mathcal{H}ol(X) \to \mathcal{H}ol(Y)$ restricts to the identity on $\mathcal{H}ol(Y)$, the map $Ext^i_{\mathcal{C}(X)}(\mathcal{M},\mathcal{N}) \to Ext^i_{\mathcal{C}(Y)}(\mathcal{M},\mathcal{N})$ is injective.

To show the surjectivity recall that the Yoneda Ext groups classify all the extensions

$$0 \to M \to \ldots \to \mathcal{N} \to 0$$

modulo the equivalence relation generated by all commutative diagrams

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{M} \arrow[r] & \cdots \arrow[r] & \mathcal{N} \arrow[r] & 0. \end{tikzcd}$$

Hence our goal is to connect a representative of the Yoneda Ext-group on X to a representative of a Yoneda Ext-group on Y by a chain of such commutative diagrams. To that end, assume that we have an exact sequence

$$0 \to \mathcal{M} \to \mathcal{A}^\bullet \to \mathcal{N} \to 0.$$

Then one also has an exact sequence

$$0 \to \mathcal{M} \to \Phi(\mathcal{A}^\bullet) \to \mathcal{N} \to 0.$$

Now the chain of diagrams in question is generated by the following diagram of maps:

$$\mathcal{A}^\bullet \xrightarrow{qis} \mathcal{A}^\bullet \oplus \Xi(\mathcal{A}^\bullet|_U) \xrightarrow{qis} \frac{\mathcal{A}^\bullet \oplus \Xi(\mathcal{A}^\bullet|_U)}{j_!\mathcal{A}^\bullet} \leftarrow \Phi(\mathcal{A}^\bullet)$$

(Here we use the fact that $\Xi(\mathcal{A}^\bullet|_U)$ is acyclic.)

Step 3. As in the first proof we may assume that M is simple. Suppose also that $\dim Supp\,\mathcal{N} < \dim X$ and let $U = X \setminus Supp\,\mathcal{N} \overset{j}{\hookrightarrow} X$. Consider the diagram

$$0 \to Ker(u) \to j_!\,\mathcal{M}|U \xrightarrow{u} \mathcal{M} \to 0,$$

where the morphism u is again surjective by simplicity of $\mathcal{M}$. Since

$$Ext^i(j_!\,\mathcal{M}|U, N) = Ext^i(\mathcal{M}|_U, \mathcal{N}|_U) = 0,$$

the long exact sequence associated with the short exact sequence above reduces us to the case where the support of $\mathcal{M}$ is strictly smaller than X.

Step 4. Hence we can assume that $\mathcal{M}$ and $\mathcal{N}$ are simple an that both *Supp* $\mathcal{M}$ and *Supp* $\mathcal{N}$ are strictly smaller than X. In this case the sequence

$$0 \to Ker(u) \to j_! \mathcal{M}|U \xrightarrow{u} \mathcal{M} \to 0,$$

and a similar sequence for $\mathcal{N}$

$$0 \to N \xrightarrow{v} j_* N_U \to Coker(v) \to 0$$

allow us to induct on the dimensions of the supports. □

4.6.35 Malgrange construction.

Let $Y = f^{-1}(0) \hookrightarrow X \hookleftarrow U$ be as before and $\mathcal{M}$ be a $\mathcal{D}_U$-module. Consider an embedding $X \overset{\varepsilon}{\hookrightarrow} X \times \mathbb{C}$ given by $\varepsilon(x) = (x, t = f(x))$. Note that the (co)-normal bundle to $\varepsilon(X)$ is trivial, hence we do not have to worry about left/right modules when we take direct images under ε. Consider $\varepsilon_* \mathcal{M} = \mathcal{D}_{X\times\mathbb{C}} \otimes_{\mathcal{D}^{\varepsilon(X)}_{X\times\mathbb{C}}} \mathcal{M}$. The ring $\mathcal{D}_{X\times\mathbb{C}}$ is generated by the operators in $\mathcal{D}_X$, multiplication by the coordinate t on $\mathbb{C}$ and the partial derivative ∂_t.

Proposition 4.6.36 (Malgrange) The assignment $mf^s \mapsto 1 \otimes m$, $s \mapsto -t\,\partial_t$, defines an isomorphism $\mathcal{D}_X[s]\,(\mathcal{M}f^s) \to \varepsilon_* \mathcal{M}$ □

4.6.37 Remarks

(1) The the operator $t \in \mathcal{D}_{X\times\mathbb{C}}$ corresponds under the isomorphism above to the operator $\tilde{t} : \sum u_i(s) f^s m_i \mapsto \sum u_i(s+1) f \cdot f^s m_i$.

(2) One can easily check that $(f - t) \cdot (1 \otimes m) = 0$ and, more generally $(f - t)^n \cdot (u \otimes m) = 0$ for $n \gg 0$ depending on u.

(3) The proposition above allows us to think of $\mathcal{D}_X[s]\,(\mathcal{M}f^s)$ as a $\mathcal{D}_{X\times\mathbb{C}}$-module. Alternatively, we can view it as a $\mathcal{D}_X[s, \tilde{t}, \tilde{t}^{-1}]$-module.

Denote $\mathcal{D}_X[s]\,(\mathcal{M}f^s)$ by $\mathcal{N}$ and choose a $\mathcal{D}_X[t]$-lattice $L \subset \mathcal{N}$. Then the quotient L/tL is naturally a $\mathcal{D}_X$-module.

THEOREM 4.6.37.3. *For any lattice L, L/tL is a holonomic $\mathcal{D}_X$-module.*

We will prove this theorem later and now we will state one corollary of it. Note that s still acts on L/tL and this action commutes with that of $\mathcal{D}_X$. Since L/tL is holonomic, the space of its endomorphisms as a $\mathcal{D}_X$-module is finite dimensional. Hence s has a *minimal polynomial* on L/tL.

Corollary 4.6.38 There exists a polynomial $b_L \in \mathbb{C}[s]$ such that $b_L(s)$ acts on L/tL by zero. □

Definition 4.6.39 We denote by $Spec_L(s)$ the set of roots of b_L.

Proposition 4.6.40 There exists a unique lattice L_0 such that

$$-1 < Re\,Spec_{L_0}(s) \leq 0.$$

Definition 4.6.41 If L_0 is the lattice given by the proposition above we denote the quotient L_0/tL_0 by $\Psi^{tot}(\mathcal{M})$. The reason fo this notation is that $\Psi(\mathcal{M})$ is isomorphic to the submodule in $\Psi^{tot}(\mathcal{M})$ on which the action of s is nilpotent.

Proof of (4.6.37.3). First of all, proving that the class of L/tL in K^+ does not depend on L, we have shown that any lwo lattices have isomorphic subquotients. Since the holonomicity is preserved under extensions, it suffices to prove the result for one particular lattice.

Choose an $\mathcal{O}$-coherent subsheaf $\mathcal{M}_0$ of $\mathcal{M}$ such that $\mathcal{M} = \mathcal{D}_X \cdot \mathcal{M}_0$. Then the lattice $L = \mathcal{D}_X[s](\mathcal{M}_0 f^s)$ is stable under s (and of course t). Then the quotient L/tL is isomorphic to $\frac{\mathcal{D}_X[s]\mathcal{M}_0 f^s}{\mathcal{D}_X[s]\mathcal{M}_0 f^{s+1}}$. Assume for simplicity that $\mathcal{M}_0$ is generated by one section m (and induct in general on number of generators). By the lemma on b-function we can find an operator $u(s)$ such that $u(s)(mf^{s+1} = b(s)mf^s$. Hence $b(s) \cdot \mathcal{D}_X[s]mf^s \subset \mathcal{D}_X[s]mf^{s+1}$. This implies that L/tL is a quotient of $\frac{\mathcal{D}_X[s]mf^s}{b(s)\mathcal{D}_X[s]mf^s}$ and it suffices to show that $\frac{\mathcal{D}_X[s]mf^s}{b(s)\mathcal{D}_X[s]mf^s}$ is holonomic. If $b(s) = \prod(s-\beta_i)^{k_i}$ then

$$\frac{\mathcal{D}_X[s]mf^s}{b(s)\mathcal{D}_X[s]mf^s} = \bigoplus_i \frac{\mathcal{D}_X[s]f^s}{(s-\beta_i)^{k_i} \cdot \mathcal{D}_X[s]nf^s}.$$

Since holonomicity is preserved under extensions, we can assume that all k_i are equal to 1. Then for $\beta = 0$ the statement was proved before and for $\beta \neq 0$ the proof follows the same pattern. □

4.6.42 Construction of the lattice L_0.

First choose some lattice L and let $[a,b]$ be the interval of minimal length containing $Re\,Spec_L(s)$. Let l be the biggest integer satisfying $b + l \leq 0$, respectively and k be any integer satisfying $a + k < 0$, $l > k$. Then the module $t^l L/t^k L$ is holonomic. Decompose it with respect to the action of s:

$$t^l L/t^k L = \bigoplus_\lambda V_\lambda,$$

where V_λ is the λ-generalized eigenspace.

Since multiplication by t^l shifts eigenvalues by t^l, all λ in the decomposition above are non-positive. Define $\bar{L}_0$ to be the subspace $\bar{L}_0 = \bigoplus_{-1<\lambda} V_\lambda$ and let L_0 be the inverse image of $\bar{L}_0$ under the projection map $t^l L \to t^l L / t^l L$. Then L_0 satisfies all the properties required from it. $\square$

4.6.43 Remarks.

(1) The construction of L_0 above is independent of the original choice of L.

(2) The operation of taking eigenvalues of $s = t\partial_t$ on the $\mathcal{D}_X[s,t]$-module $\mathcal{N} = \mathcal{M}f^s[s]$ itself would not make sense since the action of s is not locally finite. Hence we mod out a "small" lattice, which cannot affect th eigenvalues in the $(-1,0]$-range. Once we do that, choosing the eignespaces corresponding to all $\lambda \in (-1,0]$ becomes an exact functor.

(3) Instead of the interval $(-1,0]$ we could choose any bounded set G representing the points of $\mathbb{R}/\mathbb{Z}$.

Proposition 4.6.44 The zero weight component in L_0/tL_0 is isomorphic to $\Psi(\mathcal{M})$. $\square$

4.6.45 Remark One can also take the λ-component of L_0/tL_0 and obtain an exact functor $\Psi^\lambda(\mathcal{M})$ corresponding to $\mathcal{M}f^s[[s-\lambda]]$ instead of $\mathcal{M}f^s[[s]]$.

4.7. Verdier specialization.

Let Y be a submanifold of X (not necessarily a divisor). We will construct the Verdier specialization map

$$Sp_{X/Y} : \mathcal{H}ol(\mathcal{D}_X) \to \mathcal{H}ol(\mathcal{D}_{T_Y X}).$$

This fonctor is connected (**unproved?**) with our previous constructions in the following way. Suppose $Y = f^{-1}(0)$ for some function f on X. Such a function is not defined uniquely an choosing a particular f amounts to trivializing the normal bundle $T_Y X$ induced by the trivialization of $T^*_Y X$ via the section df. Once this bundle is trivialized we can write

$$\varepsilon : Y \hookrightarrow Y \times \mathbb{C} = T_Y X, \quad y \mapsto y \times 1,$$

(in invariant terms one can say that the dual map $\varepsilon : Y \hookrightarrow T^*_Y X$ is given by df). In this particular case one has

$$\varepsilon^!(Sp_{X/Y}\mathcal{M}) = \Psi^{tot}_f \mathcal{M}.$$

So, if g is another function vanishing along Y then $\Psi^{tot}_f \mathcal{M}$ and $\Psi^{tot}_g \mathcal{M}$ are isomorphic but non-canonically. Thus the functor Ψ has an advantage of being defined even for singular divisors Y (but it depends on the choice of f), while Sp is canonical and works in any codimension.

Consrtuction of $Sp_{X/Y}$ uses the standard deformation to normal bundle diagram:

$$
\begin{tikzcd}
T_Y X \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d, "g"'] & X \times \mathbb{C}^* \arrow[l, hook'] \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & X \\
\{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}^* \arrow[l, hook']
\end{tikzcd}
$$

If X is affine then $\mathcal{X}$ is defined to be $Spec\big(\bigoplus_{i\geq 0} I_Y^i\big)$ and in general $\mathcal{X}$ is obtained by gluing the spectra of this type. Note that the projection g is a *flat* map. For any holonomic $\mathcal{D}_X$-module $\mathcal{M}$ we define

$$Sp_{X/Y}(\mathcal{M}) := \Psi_g^{tot} pr_1^+ \mathcal{M}.$$

It follows that in fact the image of Verdier specialization functor belongs to the subcategory of *monodromic* (i.e. $\mathbb{C}^*$-invariant) holonomic $\mathcal{D}$-modules on $T_Y X$.

4.7.1 Verdier gluing.

If $\mathcal{M}$ is a holonomic module on $X \setminus Y$ then, applying the same procedure as above, we obtain a module $Sp_{X/Y}(\mathcal{M})$ on $T_Y X \setminus Y$. Given a holonomic module $\mathcal{M}$ on $X \setminus Y$ and an monodromic module $\mathcal{M}_Y$ we define the *glueing data* to the pair $(\mathcal{M}, \mathcal{M}_Y)$ together with an isomorphism $u : Sp_{X/Y}(\mathcal{M}) \simeq \mathcal{M}_Y|_{T_Y X \setminus Y}$. This gluing data is equivalent to Beilinson's glueing data when Y is a divisor since then bundle $T_Y X$ has rank one and giving a monodromic module on $T_Y X$ "almost" amounts to giving a module on Y itself. As before, from any glueing data we can obtain a $\mathcal{D}$-module on X itself (**was it explained how?**).

To provide another view on Verdier specialization, let $Y \subset X$ be a submanifold and denote the conormal bundle $T_Y^* X$ by Λ. We introduce a *non-separating* $\mathbb{Z}$-filtration $F_i^\Lambda \mathcal{D}_X$ on $\mathcal{D}_X$. We will in fact give three equivalent definitions. In all of them we take $F_0^\Lambda \mathcal{D}_X$ to be $\mathcal{D}_X^Y$ (the subring generated by $\mathcal{O}_X$ and the sheaf T_X^Y of vector fields tangent to Y at all points of Y).

4.7.2 First definition.

$$F_i^\Lambda \mathcal{D}_X = \{u \,|\, u(I_Y^k) \subset I_Y^{k-i}, \ \forall k \geq i\}.$$

4.7.3 Second definition.

$$F_i^\Lambda \mathcal{D}_X = \sum_{j-k\leq i} I_Y^k \mathcal{D}_X^j \mathcal{D}_X^Y.$$

4.7.4 Third definition. Recall that $\mathcal{B}_{Y|X}$ denotes the sheaf

$$\mathcal{H}^{codim\,Y}_{[Y]}(\mathcal{O}_X) = i_*\,\mathcal{O}_Y,$$

where $i : Y \hookrightarrow$ is the embedding. The sheaf $\mathcal{B}_{Y|X}$ has a natural $\mathcal{O}_X$-module filtration (by the number of derivatives in directions transversal to Y). We now define

$$F_i^{\Lambda}\mathcal{D}_X = \{u\,|u(\mathcal{B}^k_{Y|X}) \subset \mathcal{B}^{k+i}_{Y|X}\ \forall k\}.$$

Let p be the projection $T_Y X \to Y$. The geometrical significance of the filtration introduced above is that, for any $k \in \mathbb{Z}$, there exists a canonical isomorphism

$$F_k^{\Lambda}\mathcal{D}_X/F^{\Lambda}_{k-1}\mathcal{D}_X \simeq p_\cdot\Big(\mathcal{D}_{T_Y X}(k)\Big),$$

where $\mathcal{D}_{T_Y X}(k)$ stands for the sheaf of differential operators of homogeneous degree k along the fibers, i.e. if Eu is the Euler vector field generating the $\mathbb{C}^*$ action on the fibers of p then $u \in \mathcal{D}_{T_Y X}(k)$ if and only if $[Eu, u] = k\,u$.

4.7.5 Example If Y is adivisor in X and t denotes a local coordinate along the fibers of p then $t \in \mathcal{D}_{T_Y X}(-1)$, $\partial_t \in \mathcal{D}_{T_Y X}(1)$ and $t\,\partial_t \in \mathcal{D}_{T_Y X}(0)$.

Recall that in general $T_Y X = Spec(\oplus I^i_Y/I^{i+1}_Y)$, so any section of I_Y gives rise to a section of $p_\cdot\,\mathcal{D}_{T_Y X}(-1)$. If $\xi \in T^Y_X$ then $\xi \cdot I^i_Y \subset I^i_Y$ hence ξ acts on $\oplus I^i_Y/I^{i+1}_Y$ and therefore gives a vector field on $T_Y X$ of homogeneous degree -1. This establishes the isomorphism

$$\mathrm{gr}^{F^{\Lambda}} \simeq p_\cdot\,\mathcal{D}_{T_Y X}.$$

One has a notion of a good filtration with respect to F^{Λ} on a $\mathcal{D}_X$-module $\mathcal{M}$. Any such filtration is obtained by choosing an $\mathcal{O}_X$-coherent submodule $\mathcal{M}_0$ generating $\mathcal{M}$ and defining $\mathcal{M}^{\Lambda}_i = F^{\Lambda}_i \cdot \mathcal{M}_0$. Then $\mathrm{gr}^{\Lambda}\mathcal{M}$ is a $\mathrm{gr}^{F^{\Lambda}}\mathcal{D}_X$-module or, equivalently, a $\mathcal{D}$-module on $T_Y X$. However, since $p_\cdot\,\mathcal{D}_{T_Y X(k)}$ is not $\mathcal{O}_Y$-coherent, the action of the vector filed Eu on $\mathrm{gr}^{\Lambda}\mathcal{M}$ may not be locally finite. We will say that a filtration on $\mathcal{M}$ is *very good* if the action Eu *is* locally finite.

Theorem 4.7.5.4. *(Kashiwara) If $\mathcal{M}$ is holonomic then there exists a unique very good filtration on $\mathcal{M}$ such that*

$$i - 1 < Re\,Spec\,\mathcal{M}^{\Lambda}_i/\mathcal{M}^{\Lambda}_{i-1} \le i,$$

for any $k \in \mathbb{Z}$. □

4.7.6 REMARK The main difficulty of the proof is in finding at least one very good filtration. Once this is done, the restriction on the spectrum will be easy to achieve **koryavo**.

Definition 4.7.7 The unique filtration provided by the theorem above is called *Kashiwara filtration*.

From now on we will only consider the KAshiwara filtration on $\mathcal{M}$.

Corollary 4.7.8 The fuctor $\mathcal{M} \to \mathrm{gr}^{\Lambda}\mathcal{M}$ is exact. □

THEOREM 4.7.8.5. *There exists a natural isomorphism of $\mathcal{D}_{T_Y X}$-modules*

$$\mathrm{gr}^{\Lambda}\mathcal{M} \simeq Sp_{X/Y}\mathcal{M}.$$

4.7.9 REMARK This theorem was first proved by several people in 1984-85 (unpublished). The assertion of this theorem is a generalization of the coincidence of two definitions of Ψ.

4.8. Fourier transform and second microlocalization. Note that one can also consider the dual situation: let $p^{\vee} : T^*_Y X \to Y$ be the projection of the conormal bundle to Y. Then we can formulate the dual counterpart of Proposition **???**:

Proposition 4.8.1 There exists a natural isomorphism of algebras

$$\mathrm{gr}^{F^{\Lambda}}\mathcal{D}_X \simeq p^{\vee}_{\cdot}\,\mathcal{D}_{T^*_Y X}.$$

The isomorphism in this case has to be constructed differently: any section ξ of T_X defines a function on T^*X. If ξ is in fact a section of T_X^Y then ξ vanishes on $\Lambda = T^*_Y X$ hence the hamiltonian vectore field v_ξ associated to ξ is tangent to Λ (this is a property of any coisotropic variety).

Hence, given any $\mathcal{D}_X$-module $\mathcal{M}$, we can construct a $\mathcal{D}$-module on $T_Y X$ and another $\mathcal{D}$-module on $T^*_Y X$. To understand the relationship between these two modules we will need a notion of Fourier transform.

Let us suppose first that E is a (complex) vector space and $E^{\vee}$ is the dual vector space. One has a natural isomorphism of algebras

$$\mathcal{D}_E \simeq \mathcal{D}_{E^{\vee}},$$

which interchanges multiplication by the coordinate functions with partials. Therefore, any $\mathcal{D}_E$-module can be also viewed as a $\mathcal{D}_{E^{\vee}}$.

If $p : E \to X$ is a vector bundle over an algebraic variety X, and $p^{\vee} : E^{\vee} \to X$ is the dual bundle, then the isomorphism above becomes an isomorphism of sheaves

$$p^{\vee}_{\cdot}\,\mathcal{D}_{E^{\vee}} = det^{?}E \otimes p_{\cdot}\,\mathcal{D}_E \otimes det^{?-1}E.$$

Therefore, given any $\mathcal{D}_E$-module $\mathcal{N}$, we can consider $p.\mathcal{N}$ as a $p.\mathcal{D}_E$-module and hence $det^? E \otimes p.\mathcal{N}$ becomes a $p^\vee_. \mathcal{D}_{E^\vee}$-module. Finally, we obtain a $\mathcal{D}_{E^\vee}$ which is called the *Fourier transform* of $\mathcal{N}$.

Definition 4.8.2 We denote by $\Phi^\wedge(\mathcal{M})$ the $\mathcal{D}$-module on $T^*_Y X$ obtained from a $\mathcal{D}_X$-module $|M$ via the procedure described above.

Proposition 4.8.3 The $\mathcal{D}_{T^*_Y X}$-module $\Phi^\wedge(\mathcal{M})$ is isomorphic to the Fourier transform of $Sp_{X/Y}(\mathcal{M})$.

Now suppose that Y is defined by a single equation $\{f = 0\}$. This gives a trivialization on both normal and conormal bundles to Y and we define the maps $\varepsilon : Y \to T_Y X$ and $\varepsilon^\vee : Y \to T^*_Y X$ by $y \mapsto (y, 1)$. In this notations we have the followng important formulas:

$$\varepsilon^! Sp_{X/Y}(\mathcal{M}) = \Psi_f(\mathcal{M}), \quad (\varepsilon^\vee)^! \Phi^\wedge(\mathcal{M}) = \Phi_f(\mathcal{M}).$$

4.8.4 Comments.

(1) Last time we've had a fixed $p \in \mathbb{C}[x_1, \dots, x_n]$ and we used the $\mathcal{D}$-module $\mathcal{M} = \mathbb{C}[x_1, \dots, x_n]_p \cdot p^\lambda$.

Now $V = \{v \in \mathbb{C}^n \mid p(v) \neq 0\}$ is affine Zariski open in $\mathbb{C}^n$ and $\mathcal{M} = \mathbb{C}[V] \cdot p^\lambda$. In case $n = 1$ this is the exact analogue of $\mathcal{M} = \mathbb{C}[x, x^{-1}] \cdot x^\lambda$ for $p(x) = x$.

(2) For any fixed $\varphi \in C^\infty_.(\mathbb{R}^n)$ and $Re\,\lambda > 0$, the assignment

$$\lambda \longmapsto \int_U \varphi(x) p(x)^\lambda \, dx$$

is holomorphic w.r.t. λ, in fact,

$$\frac{d}{d\lambda} \int_U \varphi(x) p(x)^\lambda \, dx = \int_U \varphi(x) [\log p(x)] p(x)^\lambda \, dx$$

4.8.5 Example On $\mathbb{R}^n$, consider a quadratic form $p(x) = \sum_{i=1}^k x_i^2 - \sum_{j=k+1}^n x_j^2$, and let $U = \{p(x) > 0\}$. We are looking for an algebraic differential operator $u \in D(\mathbb{C}^n)[\lambda]$, such that

$$u(p^{\lambda+1}) = b(\lambda) p^\lambda, \tag{$*$}$$

where b is a b function.

[last time: $p(x) = x$, $\frac{d}{dx} x^{\lambda+1} = (\lambda + 1) x^\lambda$, $b(\lambda) = \lambda + 1$.]

Note that for equation $(*)$, the signs do not matter because we can make a complex-linear change of coordinates.

So we assume $p(x) = \sum_{i=1}^n x_i^2$. Then $u = \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

4.8.6 Claim.

$$\Delta(p^{\lambda+1}) = 4(\lambda+1)(\lambda+\frac{n}{2})p^{\lambda}$$

4.8.7 Hint. The operators p, Δ, $Eu = \sum x_i \frac{\partial}{\partial x_i}$ form an $\xi l_2(\mathbb{C})$.

Proposition 4.8.8 For $X = \mathbb{C}^n$, the original definition of $\mathcal{D}(\mathbb{C}^n)$ $\Longleftrightarrow$ Grothendieck's definition.

4.9. Additional Comments on Grothendieck's Definition. A is a commutative ring, $S \subset A$ a multiplicatively closed subset which does not contain any zero divisors $\Longrightarrow$ get $S^{-1}A$.

Moreover, for any A-module M, we get $S^{-1}M$.

Proposition 4.9.1 If M, N are A-modules and $u \in \mathit{Diff}_A(M,N)$, then u extends canonically to a differential operator $u \in \mathit{Diff}_{S^{-1}A}(S^{-1}M, S^{-1}N)$.

4.9.2 Example If $\xi : A \longrightarrow N$ is a derivation, we extend it by

$$\xi(\frac{a}{s}) = \frac{s \cdot \xi(a) - a \cdot \xi(s)}{s^2}.$$

Corollary 4.9.3 If X is a smooth affine variety with $A = \mathbb{C}[X]$ and $U \subseteq X$ is a Zariscki open affine, then any differential operator u on X "restricts" canonically to U.

4.9.4 Notation $\forall$ A-bimodule D, have $(\mathit{ad}\, a)(d) = a \cdot d - d \cdot a$, $\quad \forall a \in A$, $d \in D$.

Lemma 4.9.5 [Ore] Let D be an A-bimodule. Assume that every $\mathit{ad}\, s$, $s \in S$, is locally nilpotent on D. Then

(1) Any left fraction $s^{-1}d$ can be written as a right fraction ut^{-1}, i.e., $(\forall s \in S,\ d \in D)(\exists t \in S,\ u \in D)(su = dt)$.

(2) Similarly, any right fraction can be written as a left fraction.

In fact we can choose $t = s^k$, some $k \in \mathbb{N}$.

Proof. In general, we have an identity

$$(\mathit{ad}\, s)^n(d) \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} s^r d s^{n-r} \quad \text{(ind. on } n\text{)}.$$

Choose n so that $(\mathit{ad}\, s)^n d = 0$. Hence

$$(-1)^{n-1} d s^n = \sum_{r=1}^{n} (-1)^{n-r} \binom{n}{r} s^r d s^{n-r}.$$

Corollary 4.9.6 There exists an unique A-bimodule isomorphism φ making the following diagram commute:

$$\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ S^{-1}D \arrow[rr, "\simeq", "\varphi"'] & & DS^{-1} \end{tikzcd}$$

In particular, via this identification, $S^{-1}D = DS^{-1}$ is an $S^{-1}A$-bimodule.

Last time: If X is any connected smooth algebraic variety, we define a sheaf $\mathcal{D}_X$ by $U \rightsquigarrow D_{Groth}(U)$.

By the above, $\mathcal{D}_X$ is a presheaf on the affine opens of X. To proof that it is a sheaf, we must show that for any smooth affine X, we have $D_{Groth}(X) = \Gamma(X, \mathcal{D}_X) =$ global sections of the associated sheaf.

Proposition 4.9.7 If X is smooth affine, then $\mathcal{D}_X^k$ is a sheaf for each $k = 0, 1, 2, \dots$.

4.10. Comments on the definition of $\mathcal{D}_X$.

1) $\mathcal{D}_X \subseteq End_{\mathbb{C}}\mathbb{C}[X]$ s.t. on each open $U \subset X$, $u(\mathbb{C}[U]) \subset \mathbb{C}[U]$ and $u|_{\mathbb{C}[U]} \in D_{Groth}(U)$.

2) More elegantly: $\mathcal{D}_X \subseteq \mathcal{E}nd_{\mathbb{C}}\mathcal{O}_X$.

3) If $X = \mathbb{C}^n$, we have defined $\mathcal{D}(\mathbb{C}^n)$ as the subalgebra generated by $\mathbb{C}[\mathbb{C}^n]$ and vector fields. For general X, let us assign to any open affine $U \subset X$ the subalgebra of $End_{\mathbb{C}}(\mathbb{C}[U])$ generated by $\mathbb{C}[U]$ and vector fields on U. However, we can not even prove that this is a presheaf.

4) In holomorphic geometry, we have holomorphic local coordinates, therefore can define differential operator locally by $\sum u_k \partial^k$.

4.11. Principal symbol map.

Special case $X = \mathbb{C}^n$:

$$\begin{aligned} \mathcal{D}^k(\mathbb{C}^n) \ni u = \sum_{k_1+\ldots+k_n \leqslant k} u_{k_1 \ldots k_n} \partial^{k_1} \cdots \partial^{k_n} &\longmapsto \sigma_k(u) \\ = \sum_{k_1+\ldots+k_n = k} u_{k_1 \ldots k_n}(x) \xi_1^{k_1} \cdots \xi_n^{k_n} & \\ \in \mathbb{C}[x_1, \dots, x_n][\xi_1, \dots, \xi_n] \text{ homogeneous in } \xi\text{'s}. & \end{aligned}$$

In general, let X be a smooth affine variety, $A = \mathbb{C}[X]$. Then

$$X \ni x \longleftrightarrow m_x = \{f \in A | f(x) = 0\} \subset A \text{ maximal ideal.}$$

Have $\mathcal{D}^k(X) = \mathcal{D}^k(A)$, the Grothendieck's differential operator of order $\leqslant k$ on X.

Lemma 4.11.1 If $u \in \mathcal{D}^k(A)$ and $f \in m_x^{k+1} \subset A$, then $(uf)|_x = (uf)(x) = 0$.

Proof. Induction on k. We have $f \in m_x^{k+1} \Longrightarrow f =$ sum of terms of the form $f_0 \cdot f_1 \cdot \ldots \cdot f_k$, where $f_i \in m_x$. By definition, $[f_0, u] \in \mathcal{D}^{k-1}(A)$. By induction, $[f_0, u](f_1 f_2 \ldots f_k)|_x = 0$.

Hence $(uf)|_x = [u, f_0](f_1 f_2 \ldots f_k)|_x + f_0(x) \cdot u(f_1 f_2 \ldots f_k)|_x = 0$ because $f_0 \in m_x$.

First construction of the principal symbol.

We want to define

$$\mathcal{D}^k(X) \ni u \mapsto \sigma_k(u)|_x = \text{ a degree } k \text{ homogeneous polynomial on } T_x^*X$$

4.11.2 Reminder. $\mathbb{C}[x]/m_x \cong \mathbb{C}$ via $f \mapsto f(x)$.

$m_x/m_x^2 \cong T_x^*X$ via $f \mapsto df|_x$.

x is smooth $\Leftrightarrow$ the canonical map.

$S^k(m_x/m_x^2) \longrightarrow m_x^k/m_x^{k+1}$ is an isomorphism $\forall\, k$ ($S^k = k$-th. symm. power).

The definition of $\sigma_k(u)|_x$ is as follows. Define a $\mathbb{C}$-linear map $m_x^k \longrightarrow \mathbb{C}$ by $f \mapsto (uf)|_x$. By the lemma, this descends to a $\mathbb{C}$-linear map $m_x^k/m_x^{k+1} \longrightarrow \mathbb{C}$. By the above, this is the same as a map $\sigma_k(u)|_x \in \mathit{Hom}_{\mathbb{C}}(S^k(m_x/m_x^2), \mathbb{C}) = \mathit{Hom}_{\mathbb{C}}(S^k T_x^*X, \mathbb{C})$.

But to give a degree k homogeneous polynomial on a vector space V is the same as to give a linear map $S^kV \longrightarrow \mathbb{C}$.

4.11.3 Reformulation. If $\varphi \in m_x$, then $f = \varphi^k \in m_x^k$, and $(uf)|_x$ depends only on $d\varphi|_x$ and the dependence is polynomial of degree k.

Varying $x \in X$, we get $\sigma_k(u) \in \mathbb{C}[T^*X]$.

Lemma 4.11.4 [Useful formula] $u \in \mathcal{D}^k(X),\ f \in \mathbb{C}[X] = A \ \Rightarrow e^{tf} = \sum\limits_{k=0}^{\infty} t^k \dfrac{f^k}{k!} \in A[[t]]$.

Then

$$e^{-tf} \cdot u(e^{tf}) = t^k \cdot (\sigma_k(u) \cdot df) + \text{ lower powers of } t.$$

[here, $X \xrightarrow{df} T^*X \xrightarrow{\sigma_k(u)} \mathbb{C}$].

Proof. (DX).

This lemma is the second construction of the principal symbol.

Third construction of the principal symbol.

If $u \in \mathcal{D}^k(A)$, then by definition, for all $f_0, f_1, \ldots, f_k \in A = \mathbb{C}[X]$, we have $[f_0, [f_1, \ldots, [f_k, u] \ldots]] = 0$. We define a map

$$\sigma : A \otimes_{\mathbb{C}} A \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} A \longrightarrow A$$

by

$$(f_1, \ldots, f_k) \mapsto [f_1, [f_2, \ldots, [f_k, u] \ldots]] \in A.$$

In fact, σ is symmetric. If suffices to note that $[f_i, [f_{i+1}, v]] = [f_{i+1}, [f_i, v]]$ for any operator v, because $[f_i, f_{i+1}] = 0$. Thus σ induces

$$\sigma : S^k_{\mathbb{C}} A \longrightarrow A.$$

Moreover, $\sigma(f_1 \otimes \ldots \otimes f_k)$ depends only on the differentials df_i, and so it gives a map $S^k T^* X \longrightarrow A$.

Indeed, it suffices to show that if $df_i|_x = 0$ for at least one i, then $\sigma(f_1 \otimes \ldots \otimes f_k)|_x = 0$. By symmetry, we may assume that $i = 1$. We may also assume that $f_i(x) = 0$, and therefore $f_1 \in m_x^2$. By linearity, we may then assume that $f = \varphi \cdot \psi$, where φ, $\psi \in m_x$. But now

$$\sigma(f_1 \otimes \ldots \otimes f_k)|_x = [\varphi\psi, ?]|_x = \varphi|_x \cdot [\psi, ?]|_x + \psi|_x \cdot [\varphi, ?]|_x = 0.$$

This way we have defined σ_k as a sheaf map

$$\mathcal{D}^k_X \longrightarrow \pi.\mathcal{O}_{T^*X}(k)$$

where $\pi : T^*X \to X$ is the projection and $\pi.$ is pushforward, and "(k)" means "homogeneous of degree k along the fibers".

Theorem 4.11.4.6. *We have a short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{D}^{k-1}_X \longrightarrow \mathcal{D}^k_X \xrightarrow{\sigma_k} \pi.\, \mathcal{O}_{T^*X}(k) \longrightarrow 0.$$

In fact, for any smooth affine X, we have a short exact sequence

$$0 \longrightarrow \mathcal{D}^{k-1}(X) \longrightarrow \mathcal{D}^k(X) \xrightarrow{\sigma_k} \pi.\, \mathbb{C}^k_{homog}[T^*X] \longrightarrow 0.$$

Proof. By Grothendieck's definition, we have $\mathcal{D}^{k-1}(X) = Ker(\sigma_k)$. So we must prove that σ_k is surjective.

Recall that we've had a naive definition of $\mathcal{D}^k(X)$ as the subspace $\mathcal{D}^k_{orig}(X) \subseteq End_{\mathbb{C}}(\mathbb{C}[X])$ generated by all elements of the form $f_0 \xi_1 \ldots \xi_l$, where $l \leqslant k$, $f_0 \in \mathbb{C}[X]$, ξ_i are vector fields. It is clear that $\mathcal{D}^k_{orig}(X) \subseteq \mathcal{D}^k_{Groth}(X)$.

We will prove surjectivity of σ_k together with the following

Proposition 4.11.5 $\mathcal{D}_{orig} = \mathcal{D}_{Groth}$.

Indeed, by induction, w.m.a. $\mathcal{D}^{k-1}_{orig} = \mathcal{D}^{k-1}_{Groth}$. Then it suffices to prove that $\sigma_k : \mathcal{D}^k_{orig} \longrightarrow \mathbb{C}^k_{homog}[T^*X]$ is surjective.

Lemma 4.11.6 Let $A = \mathbb{C}[X]$ and M a locally free f.g. A-module. Then

$$Hom_A(S^k_A M,\, A) = S^k_A Hom_A(M,\, A).$$

Proof. This is true for $M = A \Longrightarrow$ true for any $M = A^{\oplus n} \Longrightarrow$ true for any direct summand of $A^{\oplus n}$. But locally free f.g. $\Longrightarrow$ projective f.g.

Now we have $\mathbb{C}^k_{homog}[T^*X] = Hom_A(S^k_A \underbrace{\Gamma(X,\, T^*X)}_{=:M},\, A) = S^k Hom_A(M,\, A)$ because M is locally free (since X is smooth).

Also, if M is locally free, then it is projective, and hence

$$Hom_A(Hom_(M,\, A),\, A) \cong M \text{ canonically.}$$

Now $\Gamma(X,\, T^*X) = Hom_A(\Gamma(X,\, TX),\, A)$,whence

$$\mathbb{C}^k_{homog}[T^*X] = S^k\Gamma(X,\, TX).$$

Hence every element of this can be written as a sum of products of the form $\xi_1, \ldots, \xi_k$ where $\xi_i \in \Gamma(X,\, TX)$.

Let X be an algebraic variety and $U \subseteq X$ open affine. For any $f \in \mathbb{C}[U]$, put $U_f = U \setminus \{f = 0\}$. For any $\mathbb{C}[U]$-module M, we have M_f.

Lemma 4.11.7 For an $\mathcal{O}_X$-sheaf $\mathcal{F}$, the following are equivalent:

(1) $\mathcal{F} = \varinjlim \mathcal{F}_i$ (union) of coherent subsheaves $\mathcal{F}_i \subseteq \mathcal{F}$.

[E.g. $\mathcal{D}_X = \varinjlim \mathcal{D}^k_X$]

(2) For any open affine $U \subseteq X$ and any $f \in \mathbb{C}[U]$, we have

$$\Gamma(U,\, \mathcal{F})_f = \Gamma(U_f,\, \mathcal{F}).$$

If (1) and (2) hold, we say $\mathcal{F}$ is *quasi-coherent.*

4.11.8 Homework.

1) $p \in \mathbb{C}[x_1, \ldots, x_n] \Longrightarrow$ show that $D(\mathbb{C}^n) \cdot e^p$ is holonomic.

2) (a) For which $\lambda, \mu \in \mathbb{C}$ do we have $D(\mathbb{C}) \cdot e^{\lambda x} \cong D(\mathbb{C}) \cdot e^{\mu x}$ as D-modules.

(b) Is $D(\mathbb{C}) \cdot e^{1/x} \cong D(\mathbb{C}) \cdot e^{1/x^2}$?

3)* Let M be a holonomic $D(\mathbb{C}^n)$-module.

(a) Prove that $\dim_{\mathbb{C}} End_{D(\mathbb{C}^n)}(M) < \infty$

(b) Suppose we have a filtration on M such that $h_M(t) = \dfrac{c}{n!}t^n +$ lower order terms, $c \in \mathbb{N}$. Let $a : M \longrightarrow M$ be a $D(\mathbb{C}^n)$-module map. Show that the elements $Id_M, a, a^2, \ldots, a^c$ are linearly depended over $\mathbb{C}$.

4.12. Prehomogeneous vector spaces.

V = finite dimensional vector space over $\mathbb{C}$, $G \subseteq GL(V)$ is a closed connected reductive algebraic subgroup.

Definition 4.12.1 The pair (V, G) is called a *regular pre-homogeneous vector spae* if G has an open dense orbit V^{reg} such that $V \setminus V^{reg}$ is an irreducible hypersurface, i.e., there exists irreducible $f \in \mathbb{C}[V]$ such that $V \setminus V^{reg} = \{f = 0\}$.

4.12.2 Examples.
(1) $\mathbb{C}^n$ with $f(x) = x_1^2 + x_2^2 + \ldots + x_n^2$ (assume $n > 2$).
Take $G = SO(n) \cdot \mathbb{C}^*$.
(2) Counterexample: $G = GL(V)$ for $\dim V > 1$.
(Prehomogeneous but not regular)
(3) $G = GL(V)$ acting on $End(V)$ by left multiplication.

Elementary properties:

(1) f is a G-*semiinvariant*, i.e., there exists an algebraic group homomorphism $\chi : G \longrightarrow \mathbb{C}^*$ such that $f(gx) = \chi(g)f(x)$ $\forall x \in V$, $g \in G$.

(2) f is homogeneous of degree $d := deg\, f$. [Indeed, the G-action commutes with the $\mathbb{C}^*$-action $\Longrightarrow$ the $\mathbb{C}^*$-action must preserve V^{reg}.]

Theorem 4.12.2.7. *There exists $b(\lambda) \in \mathbb{C}[\lambda]$ such that $deg\, b = deg\, f$ and*

$$f(\partial)f^{\lambda+1} = b(\lambda)f^{\lambda}$$

($f(\partial)$ = differential operator with constant coefficients obtained by replacing the x_i's by ∂_i's in f).

Corollary 4.12.3 $\det(\partial) \cdot \det(x)^{\lambda+1} = b(\lambda) \cdot \det(x)^{\lambda} \longleftarrow$ related to the "Capelli identity".

In fact, it is known that

$$b(\lambda) = \prod_{1 \leqslant k \leqslant n} (\lambda + k).$$

4.12.4 Claim. T is a D-endomorphism of M.
Proof. Straightforward.

Hence T is well defined as an endomorphism of

$$M|_1 = M/(t-1)M.$$

Also, since

$$T : t^n \mapsto \exp(2\pi i n) \cdot t^n = t^n,$$

it is clear that T is unipotent.

Conversely, to define the functor in the other direction, it suffices (by the Jordan decomposition) to consider the case

$$T = \begin{pmatrix} 1 & 1 & & & 0 \\ & 1 & 1 & & \\ & & 1 & & \\ & & & \ddots & 1 \\ 0 & & & & 1 \end{pmatrix}$$

For this T, the corresponding D-module is

$$(\log t)^{n-1} \cdot R \oplus (\log t)^{n-2} \cdot R \oplus \ldots \oplus (\log t) \cdot R \oplus R.$$

To prove that the functor $\mathcal{C}^* \to \{(V, T)\}$ constructed above is an equivalence, it is enough to show that it is fully faithful and essentially surjective. Last time we have shown essential surjectivity.

Let $\mathcal{C}_2$ be the category of pairs (V, T) as above, and let

$$Mon : \mathcal{C}^* \to \mathcal{C}_2$$

be the monodromy functor. Let

$$\mathcal{J}ord_n = \left(V = \mathbb{C}^n, T = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & \ddots & 1 \\ & & & & 1 \end{pmatrix} \right) \in \mathcal{C}_2,$$

and let

$$\mathcal{J}_n = R \cdot (\log t)^{n-1} \oplus R \cdot (\log t)^{n-2} \oplus \ldots \oplus R \cdot (\log t) \oplus R \ \in \mathcal{C}^*.$$

4.12.5 Claim. $Mon(\mathcal{J}_n) \cong \mathcal{J}ord_n$.

Proof. $\mathcal{J}_n$ has a $\mathbb{C}$-basis $\{t^k \cdot (\log t)^l \mid k \in \mathbb{Z},\ 0 \leqslant l \leqslant n-1\}$, and the action of $t\,\partial_t$ is given by

$$t\,\partial_t : t^k \cdot (\log t)^l \mapsto kt^k \cdot (\log t)^l + lt^k \cdot (\log t)^{l-1}$$

For a fixed k, the matrix of $t\,\partial_t$ is thus

$$\begin{pmatrix} k & n-1 & & & \\ & k & n-2 & & \\ & & & \ddots & 1 \\ & & & & k \end{pmatrix},$$

and we see that

$$T = e^{2\pi i t\,\partial_t} = \begin{pmatrix} 1 & 1 & & * \\ & 1 & & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix},$$

which is what we want.

Next we have to prove that

$$Hom_{D_{\mathbb{C}^*}}(\mathcal{J}_n,\, \mathcal{J}_m) \cong Hom_{\mathcal{C}_2}(\mathcal{J}ord_n,\, \mathcal{J}ord_m)$$

Geometrically,

$$\mathcal{J}ord_n : \overset{e_0}{\cdot} \longrightarrow \overset{e_1}{\cdot} \longrightarrow \overset{e_2}{\cdot} \longrightarrow \ldots \longrightarrow \overset{e_{n-1}}{\cdot} \longrightarrow 0$$

$$\mathcal{J}ord_m : \overset{e_0}{\cdot} \longrightarrow \overset{e_1}{\cdot} \longrightarrow \overset{e_2}{\cdot} \longrightarrow \ldots \longrightarrow \overset{e_{m-1}}{\cdot} \longrightarrow 0$$

Using this, we can compute the dimensions of both homomorphisms and thus prove that they are isomorphic.

Alternative interpretation.

Let $M \in \mathcal{C}^*$, than we can write $M = \bigoplus\limits_{i\in\mathbb{Z}} M_i$, $\dim M_i < \infty$, where each M_i is $t\,\partial_t$-stable. [$M_i =$ the i-th eigenspace of $t\,\partial_t$]. Picture:

$$\begin{tikzcd}
\cdots & \underset{\cdot}{M_{-1}} \arrow[r, shift left, "t"] & \underset{\cdot}{M_0} \arrow[l, shift left, "\partial_t"] \arrow[r, shift left, "t"] \arrow[loop below, "t\partial_t"] & \underset{\cdot}{M_1} \arrow[l, shift left, "\partial_t"] \arrow[r, shift left, "t"] \arrow[loop below, "t\partial_t"] & \underset{\cdot}{M_2} \arrow[l, shift left, "\partial_t"] & \cdots
\end{tikzcd}$$

Since we are working in $\mathcal{C}^*$, t is clearly an isomorphism; in particular, all the M_i have the same dimension.

The category $\mathcal{C}^*$ has a single simple object, namely R. It does not have projective objects, but it does have a pro-object $\mathcal{E}^{proj}$.

Note that $\mathcal{J}_n$ has a filtration will all quotients $\cong R$

$$\begin{array}{ccccc}
\mathcal{J}_n \sim & \left(\begin{array}{c} R \\ \hline R \\ \hline \vdots \\ \hline R \end{array}\right) & \longleftarrow & \left(\begin{array}{c} R \\ \hline R \\ \hline \vdots \\ \hline R \\ \hline R \end{array}\right) & \sim \mathcal{J}_{n+1} \\
& 0 & \longleftarrow & &
\end{array}$$

We take

$$\mathcal{E}^{proj} := \varprojlim \mathcal{J}_n \sim \begin{pmatrix} R \\ \hline R \\ \hline R \\ \hline \vdots \end{pmatrix}$$

Lemma 4.12.6 The functor $M \rightsquigarrow Hom_{\mathcal{D}}(\mathcal{E}^{proj}, M)$ is given by

$$Hom_{\mathcal{D}}(\mathcal{E}^{proj}, M) = M_0;$$

in particular, $\mathcal{E}^{proj}$ is projective.

4.12.7 REMARK We can also define $\mathcal{E}^{ind} = \varinjlim \mathcal{J}_n$ with respect to the natural inclusions $\mathcal{J}_n \hookrightarrow \mathcal{J}_{n+1}$, and then $\mathcal{E}^{ind}$ is the injective hull of R.

To prove the lemma above, we have the following

4.12.8 Claim. $\mathcal{J}_n \cong \mathcal{D}/\mathcal{D} \cdot (t\,\partial_t)^n$.

Proof. We have a natural map $\mathcal{D}/\mathcal{D} \cdot (t\,\partial_t)^n \to \mathcal{J}_n$ given by $1 \mapsto (\log t)^{n-1}$, and it is clearly surjective.

Since t is invertible in R, we have

$$\begin{aligned} \mathcal{D} &= \mathbb{C}[t, t^{-1}][\partial_t] \\ &= \mathbb{C}[t, t^{-1}][t\,\partial_t]. \end{aligned}$$

Thus $\mathcal{D}/\mathcal{D}\cdot(t\,\partial_t)^n$ is a free R-module with basis $1, (t\,\partial_t), (t\,\partial_t)^2, \ldots, (t\,\partial_t)^{n-1}$, completing the proof.

Now we immediately get the tautological projections

$$\mathcal{J}_{n+1} \longrightarrow \mathcal{J}_n.$$

Moreover, we see that

$$\mathcal{E}^{proj} = \varprojlim_n \mathcal{D}/\mathcal{D} \cdot (t\,\partial_t)^n.$$

This implies that

$$Hom(\mathcal{E}^{proj}, M) = Hom\big(\varprojlim_n \mathcal{D}/\mathcal{D} \cdot (t\,\partial_t)^n,\ M\big) = M_0$$

Now we have

$$\mathcal{J}ord^{proj} = \varprojlim \mathcal{J}ord_n = \underset{e_0}{\cdot} \longrightarrow \underset{e_1}{\cdot} \longrightarrow \underset{e_2}{\cdot} \longrightarrow \ldots$$

It remains to prove that our functor takes

$$End(\mathcal{E}^{proj}) \xrightarrow{\simeq} End(\mathcal{J}ord^{proj})$$

We claim that both are isomorphic to $\mathbb{C}[[u]]$.

$$\boxed{\begin{array}{l}\mathcal{E}^{proj} = \varprojlim \mathcal{D}/\mathcal{D}\cdot(t\,\partial_t)^n = \varprojlim \mathcal{J}_n \\ \mathcal{E}^{ind} = R[\log t] = \varinjlim \mathcal{J}_n\end{array}}$$

4.12.9 Remark We have duality $\mathbb{D}$ on holomorphic modules. Note that $\mathbb{D}(R) = R$. Hence $\mathbb{D} : \mathcal{C}^* \longrightarrow \mathcal{C}^*$ observe that

$$\mathbb{D} : \begin{pmatrix} A_1 \\ \hline A_2 \\ \hline A_3 \end{pmatrix} \longmapsto \begin{pmatrix} \mathbb{D}A_1 \\ \hline \mathbb{D}A_2 \\ \hline \mathbb{D}A_3 \end{pmatrix}$$

We have $\mathbb{D}\big(\mathcal{E}^{proj}\big) \cong \mathcal{E}^{ind}$.

Recall the diagram

$$\{0\} \hookrightarrow \mathbb{C} \overset{j}{\hookleftarrow} \mathbb{C}^*$$

Let $\delta = \mathcal{D}/\mathcal{D}t$ and $\mathcal{O} = \mathbb{C}[t] = \mathcal{D}/\mathcal{D}\,\partial$.

Definition 4.12.10 Let $\mathcal{C}$ be the category of $\mathcal{D}$-modules of finite length on $\mathbb{C}$ with all subquotients isomorphic to either δ or $\mathcal{O}$.

We have j_*R. As a space, $j_*R = R = \mathbb{C}[t, t^{-1}]$.

We have a short exact sequence of $\mathcal{D}_{\mathbb{C}}$-modules

$$0 \to \mathcal{O} \to R \to \delta \to 0.$$

Thus

$$R = \mathbb{C}[t, t^{-1}] \sim \begin{pmatrix} \delta \\ \hline \mathcal{O} \end{pmatrix}$$

Now we have

$$j_!R = \mathbb{D}j_*\mathbb{D}R = \mathbb{D}(j_*R) \sim \begin{pmatrix} \mathcal{O} \\ \hline \delta \end{pmatrix}$$

There is a canonical morphism $j_!R \to j_*R$; it looks like

$$\begin{tikzcd} \mathcal{O} \arrow[dr, dashed] & \delta \\ \delta \arrow[dr, dashed] & \mathcal{O} \\ & 0 \end{tikzcd}$$

[because this map has to restrict to the identity on $\mathbb{C}^*$]

Explicitly,

$$j_*R = \mathcal{D}/\mathcal{D}(\partial t) \quad \text{and} \quad j_!R = \mathcal{D}/\mathcal{D}(t\partial),$$

and the map $j_!R \to j_*R$ is multiplication on the right by t.

Now we have $\mathcal{D}$-modules $j_*\mathcal{J}_n$, $j_!\mathcal{J}_n$. Pictorially,

$$j_*\mathcal{J}_n \sim \begin{pmatrix} \delta \\ \mathcal{O} \\ \delta \\ \mathcal{O} \\ \vdots \\ \delta \\ \mathcal{O} \end{pmatrix} \quad \text{and} \quad j_!\mathcal{J}_n \sim \begin{pmatrix} \mathcal{O} \\ \delta \\ \mathcal{O} \\ \delta \\ \vdots \\ \mathcal{O} \\ \delta \end{pmatrix}.$$

We can do four limit constructions. We set

$$\begin{aligned} \nabla^{proj} &:= j_!\mathcal{E}^{proj}, \qquad \Delta^{proj} := j_*\mathcal{E}^{proj}, \\ \nabla^{ind} &:= j_!\mathcal{E}^{ind}, \qquad \Delta^{ind} := j_*\mathcal{E}^{ind}. \end{aligned}$$

Pictorially,

$$\nabla^{proj} = \begin{pmatrix} \mathcal{O} \\ \delta \\ \vdots \end{pmatrix}, \quad \Delta^{proj} = \begin{pmatrix} \delta \\ \mathcal{O} \\ \vdots \end{pmatrix}, \quad \Delta^{ind} = \begin{pmatrix} \vdots \\ \delta \\ \mathcal{O} \end{pmatrix}, \quad \nabla^{ind} = \begin{pmatrix} \vdots \\ \mathcal{O} \\ \delta \end{pmatrix}.$$

For future reference:

$$\nabla^{proj} = \mathbb{D}(\Delta^{ind}), \quad \Delta^{proj} = \mathbb{D}(\nabla^{ind}), \text{ etc.}$$

An important thing is to calculate

$$j_!\mathcal{E}^{proj} \longrightarrow j_*\mathcal{E}^{proj}.$$

We have a short exact sequence

$$0 \to j_!\mathcal{E}^{proj} \to j_*\mathcal{E}^{proj} \to \delta \to 0,$$

and

$$0 \to \delta \to j_*\mathcal{E}^{ind} \to j_!\mathcal{E}^{ind} \to 0.$$

Proposition 4.12.11 The category $\mathcal{C}$ is equivalent to the category of diagrams of finite dimensional vector spaces

$$\cdot \underset{v}{\overset{u}{\rightleftarrows}} \cdot \quad \text{with } (uv)^N = 0 \text{ for some } N \geqslant 1$$

Proof. The functor is given by

$$\mathcal{M} \longmapsto \Big(Hom\big(\Delta^{proj},\ M\big) \rightleftarrows Hom\big(\nabla^{proj},\ M\big) \Big)$$

where the maps are induced by

$$
\begin{tikzcd}
\delta \arrow[dr] & \mathcal{O} \arrow[dl] \\
\mathcal{O} \arrow[dr] & \delta \arrow[dl] \\
\delta \arrow[dr] & \mathcal{O} \arrow[dl] \\
\mathcal{O} & \delta \\
\vdots & \vdots
\end{tikzcd}
$$

Another proof. We still have the picture

$$
\begin{tikzcd}
\underset{-1}{M_{-1}} \arrow[r, shift left, "t"] & \underset{0}{M_0} \arrow[l, shift left, "\partial_t"] \arrow[r, shift left, "t"] \arrow[loop below, "t\partial_t"] & \underset{1}{M_1} \arrow[l, shift left, "\partial_t"] \arrow[r, shift left, "t"] & \underset{2}{M_2} \arrow[l, shift left, "\partial_t"]
\end{tikzcd}
\qquad M = \bigoplus_{i \in \mathbb{Z}} M_i,
$$

but t is no longer invertible. However, since $\partial t = t\partial + 1$, both ∂t and $t\partial$ are invertible on M_i for $i \neq 0, -1$, which implies that $t : M_i \to M_{i+1}$ is an isomorphism whenever $i \neq -2, -1, 0$.

For $M = \mathcal{O}$, we have $M_0 = \mathbb{C}$ and $M_{-1} = (0)$.

For $M = \delta$, we have $M_0 = (0)$ and $M_{-1} = \mathbb{C}$.

In general we get

$$Hom(\nabla^{proj}, M) = M_0 \text{ and } Hom(\Delta^{proj}, M) = M_{-1}$$

We have a category $\mathcal{C}$, and we have shown that

$$\mathcal{C} \simeq \left\{ M_{-1} \underset{u}{\overset{v}{\rightleftarrows}} M_0 \mid (uv)^N = 0 \text{ for some } N \geqslant 1 \right\}$$

A quasi-inverse can be constructed as follows. Given $F \underset{u}{\overset{v}{\rightleftarrows}} E$, we set $M = \mathbb{C}[t] \otimes_{\mathbb{C}} E \oplus \mathbb{C}[\partial] \otimes_{\mathbb{C}} F$, and define

$$t(1 \otimes f) := 1 \otimes v(f),$$

$$\partial(1 \otimes e) := 1 \otimes u(e).$$

Relation to duality:

$$\mathbb{D} : \mathcal{C} \to \mathcal{C} \quad (\text{since } \mathbb{D}(\mathcal{O}) = \mathcal{O},\ \mathbb{D}(\delta) = \delta)$$

By definition,

$$\mathbb{D} = R\ Hom_{\mathcal{D}}(-, \mathcal{D}).$$

We have a free resolution

$$0 \to N \xrightarrow{p} N \xrightarrow{a} M \to 0,$$

where

$$N = \mathcal{D} \otimes_{\mathbb{C}} E \oplus \mathcal{D} \otimes_{\mathbb{C}} F,$$

$a =$ the obvious action map, and P is given by

$$P : (a \otimes e, b \otimes f) \mapsto (a\partial \otimes e, -au(e)) + (-bv(f), bt \otimes f).$$

Obviously,

$$Hom_{\mathcal{D}}(\mathcal{D} \otimes E, \mathcal{D}) = E^* \otimes \mathcal{D} \quad (E^* = Hom_{\mathbb{C}}(E,\ \mathbb{C}))$$

Now applying $Hom_{\mathcal{D}}(-, \mathcal{D})$ to the resolution above, we get

$$0 \longrightarrow N^* \xrightarrow{p^*} N^* \longrightarrow 0$$

where

$$N^* = E^* \otimes_{\mathbb{C}} \mathcal{D} \oplus F^* \otimes_{\mathbb{C}} \mathcal{D},$$

and

$$P^* : (e^* \otimes a, f^* \otimes b) \mapsto (e^* \otimes \partial a, -v^*(e^*) \otimes a) + (-u^*(e^*) \otimes b, f^* \otimes tb).$$

We see from this that $\mathbb{D}(M)$ corresponds to the diagram

$$\begin{tikzcd} E^* \arrow[r, shift left, "u^*"] & F^* \arrow[l, shift left, "v^*"] \end{tikzcd}$$

[We have to use the involution to switch from right to left modules.]

Now on the category $\mathcal{C}$, we have two functors:

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\Psi"] \arrow[r, shift right, "\Phi"'] & \mathcal{V}ect \end{tikzcd}$$

where

$$\Psi(M) = Hom(\nabla^{proj}, M) = M_0,$$
$$\Phi(M) = Hom(\Delta^{proj}, M) = M_{-1}.$$

4.12.12 Remark $\Psi(M)$ is the geometric fiber of M at the point $t = 1$.

We want to produce maps

$$\begin{tikzcd} \Psi(M) \arrow[r, shift left, "can"] & \Phi(M) \arrow[l, shift left, "var"] \end{tikzcd}$$

with

$$T - Id : \Psi \xrightarrow{can} \Phi \xrightarrow{var} \Psi, \quad T = \exp(-2\pi i t\partial)|_{M_0}$$

We will define

$$can = u, \quad var = \frac{e^{2\pi i(vu)-1}}{vu} \cdot v$$

4.12.13 Remark The universal cover of $\mathbb{C}^*$ is

$$\mathbb{C} \longrightarrow \mathbb{C}^*$$
$$z \longmapsto t = e^{2\pi i z}$$

Then we have $t^n = e^{2\pi i n z}$, $\log t = z$. So, geometrically,

$R[\log] = \mathbb{C}[t, t^{-1}, \log t] =$ "quasi-polynomials" on the universal cover.

We have

$$R[\log] = \Delta^{ind}.$$

We can define, equivalently,

$$\Psi(M) = Tor^{\mathcal{D}}(M, \Delta^{ind}).$$

If $M = \mathbb{C}[t] \otimes_{\mathbb{C}} E \oplus \mathbb{C}[\partial] \otimes_{\mathbb{C}} F$ as above, then we get

$$\Psi(M) = \left\{(\widetilde{e}, \widetilde{f}) \in E \otimes R[\log] \oplus F \otimes R[\log] \mid \partial\widetilde{e} = -v(\widetilde{f}),\ t\widetilde{f} = u(\widetilde{e})\right\}.$$

But t is invertible on Δ^{ind}, so

$$\left.\begin{array}{l} \partial\widetilde{e} = -v(\widetilde{f}) \\ t\widetilde{f} = u(\widetilde{e}) \end{array}\right\} \iff t\partial\widetilde{e} = -vu\widetilde{e},$$

and a solution of the last equation is

$$\widetilde{e} = \exp(-vu \log t) \cdot e,\ \forall e \in E.$$

We have

$$\widetilde{f} = \frac{1}{t} u(\widetilde{e}) = \frac{1}{t} \exp(-uv \log t) u(e).$$

Recall that on $\mathbb{C}^*$, we've had $\mathcal{J}_n$. Let us introduce a new variable s. We have a short exact sequence

$$0 \to \mathbb{C}[[s]] \to \mathbb{C}((s)) \to \frac{\mathbb{C}((s))}{\mathbb{C}[[s]]} \to 0.$$

We also have the residue map

$$Res : \mathbb{C}((s)) \longrightarrow \mathbb{C}$$

given by

$$a(s) \overset{Res}{\longmapsto} a_{-1} = \text{“}\int_C a(s)\, ds\text{''}$$

Note that

- $Res(\mathbb{C}[[s]]) = (0)$
- If $a \in \mathbb{C}((s))$ is such that $Res(a \cdot \mathbb{C}[[s]]) = (0)$, then $a \in \mathbb{C}[[s]]$.

This can also be done for any ring:

$$0 \to R[[s]] \to R((s)) \to \frac{R((s))}{R[[s]]} \to 0.$$

We take $R = \mathbb{C}[t, t^{-1}]$. We also consider

$$0 \to Rt^2[[s]] \to Rt^2((s)) \to \frac{Rt^2((s))}{Rt^2[[s]]} \to 0.$$

Lemma 4.12.14 There is a $\mathcal{D}_{\mathbb{C}}$-module isomorphism (where coordinate on $\mathbb{C}$ is t):

$$\frac{Rt^2((s))}{Rt^2[[s]]} \xrightarrow{\simeq} R[\log t].$$

Proof. Formally, we have

$$t^s = e^{(\log t)s} = \sum_{i=0}^{\infty} \frac{s^i}{i!}(\log t)^i.$$

We define our map as the composition

$$\sum_{i=-N}^{\infty} a_i s^i t^s \longmapsto \sum_{i,j} a_i s^i \frac{s^j (\log t)^j}{j!} \overset{Res}{\longmapsto} ? \in R[\log t],\ a_i \in Rt^s((S)).$$

Clearly, the kernel of this composition is precisely $Rt^s[[s]]$, and the map is surjective.

4.12.15 Remark The universal cover of $\mathbb{C}^*$ is

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C}^* \\ z &\longmapsto t = e^{2\pi i z} \end{aligned}$$

Then we have $t^n = e^{2\pi i n z}$, $\log t = z$. So, geometrically,

$R[\log] = \mathbb{C}[t, t^{-1}, \log t] =$ "quasi-polynomials" on the universal cover.

We have

$$R[\log] = \Delta^{ind}.$$

We can define, equivalently,

$$\Psi(M) = Tor^{\mathcal{D}}(M, \Delta^{ind}).$$

If $M = \mathbb{C}[t] \otimes_{\mathbb{C}} E \oplus \mathbb{C}[\partial] \otimes_{\mathbb{C}} F$ as above, then we get

$$\Psi(M) = \left\{ (\widetilde{e}, \widetilde{f}) \in E \otimes R[\log] \oplus F \otimes R[\log] \mid \partial\widetilde{e} = -v(\widetilde{f}),\ t\widetilde{f} = u(\widetilde{e}) \right\}.$$

But t is invertible on Δ^{ind}, so

$$\left.\begin{aligned} \partial\widetilde{e} &= -v(\widetilde{f}) \\ t\widetilde{f} &= u(\widetilde{e}) \end{aligned}\right\} \iff t\partial\widetilde{e} = -vu\widetilde{e},$$

and a solution of the last equation is

$$\widetilde{e} = \exp(-vu\log t)\cdot e,\ \forall e \in E.$$

We have

$$\widetilde{f} = \frac{1}{t}u(\widetilde{e}) = \frac{1}{t}\exp(-uv\log t)u(e).$$

Recall that on $\mathbb{C}^*$, we've had $\mathcal{J}_n$. Let us introduce a new variable s. We have a short exact sequence

$$0 \to \mathbb{C}[[s]] \to \mathbb{C}((s)) \to \frac{\mathbb{C}((s))}{\mathbb{C}[[s]]} \to 0.$$

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$$Res : \mathbb{C}((s)) \longrightarrow \mathbb{C}$$

given by

$$a(s) \overset{Res}{\longmapsto} a_{-1} = \text{``}\int_C a(s)\, ds''$$

Note that

- $Res(\mathbb{C}[[s]]) = (0)$
- If $a \in \mathbb{C}((s))$ is such that $Res(a \cdot \mathbb{C}[[s]]) = (0)$, then $a \in \mathbb{C}[[s]]$.

This can also be done for any ring:

$$0 \to R[[s]] \to R((s)) \to \frac{R((s))}{R[[s]]} \to 0.$$

We take $R = \mathbb{C}[t, t^{-1}]$. We also consider

$$0 \to Rt^2[[s]] \to Rt^2((s)) \to \frac{Rt^2((s))}{Rt^2[[s]]} \to 0.$$

Lemma 4.12.16 There is a $\mathcal{D}_{\mathbb{C}}$-module isomorphism (where coordinate on $\mathbb{C}$ is t):

$$\frac{Rt^2((s))}{Rt^2[[s]]} \xrightarrow{\simeq} R[\log t].$$

Proof. Formally, we have

$$t^s = e^{(\log t)s} = \sum_{i=0}^{\infty} \frac{s^i}{i!}(\log t)^i.$$

We define our map as the composition

$$\sum_{i=-N}^{\infty} a_i s^i t^s \longmapsto \sum_{i,j} a_i s^i \frac{s^j(\log t)^j}{j!} \overset{Res}{\longmapsto} ? \in R[\log t],\ a_i \in Rt^s((S)).$$

Clearly, the kernel of this composition is precisely $Rt^s[[s]]$, and the map is surjective.

Corollary 4.12.17

$$\begin{tikzcd}
j_*R[\log t] \arrow[r, phantom, "="] \arrow[d, phantom, "\|"] & \Delta^{ind} \arrow[r, phantom, "="] & \frac{Rt^s((s))}{Rt^s[[s]]} \\
j_*(\varinjlim \mathcal{J}_n) & & \\
& \mathcal{J}_n \arrow[ul, hook'] &
\end{tikzcd}$$

Question. How can we see $\mathcal{J}_n$ inside RHS?
Answer. We have

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}_n \arrow[r] \arrow[d, phantom, "\wr|"] & R[\log t] \arrow[r] \arrow[d, phantom, "\wr|"] & R[\log t]/\mathcal{J}_n \arrow[r] & 0 \\
& \frac{s^{-n}\cdot Rt^s((s))}{Rt^s[[s]]} \arrow[r, hook] & \frac{Rt^s((s))}{Rt^s[[s]]} & &
\end{tikzcd}$$

Corollary 4.12.18

$$\mathcal{J}_n \cong \frac{Rt^s[[s]]}{s^n Rt^s[[s]]} = \frac{Rt^s[s]}{s^n Rt^s[s]} \tag{$*$}$$

We have, on $\mathbb{C}^*$,

$$\mathcal{E}^{proj} = \varprojlim \mathcal{J}_n = Rt^s[[s]]$$

(this is obvious from $(*)$). Hence

$$\Delta^{proj} = j_*\mathcal{E}^{proj} = Rt^s[[s]].$$

Now we get

$$0 \longrightarrow \Delta^{proj} \longrightarrow j_*Rt^s((s)) \longrightarrow \Delta^{ind} \longrightarrow 0.$$

Pictures.

$$\Delta^{ind} \quad = \quad j_*\mathcal{E}^{ind} \quad = \quad \begin{array}{c} \vdots \\ \mathcal{O} \\ \hline \delta \\ \hline \mathcal{O} \\ \hline\hline \end{array}$$

$$\Delta^{proj} \quad = \quad j_*\mathcal{E}^{proj} \quad = \quad \begin{array}{c} \hline \underline{\delta} \\ \underline{\mathcal{O}} \\ \underline{\delta} \\ \vdots \end{array}$$

$$j_*Rt^s((s)) \quad = \quad \begin{array}{c} \vdots \\ \underline{\mathcal{O}} \\ \underline{\delta} \\ \mathcal{O} \\ \hline \underline{\delta} \\ \underline{\mathcal{O}} \\ \underline{\delta} \\ \vdots \end{array}$$

$$\nabla^{ind} \quad = \quad j_!\mathcal{E}^{ind} \quad = \quad \begin{array}{c} \vdots \\ \underline{\delta} \\ \underline{\mathcal{O}} \\ \delta \\ \hline \end{array}$$

$$\nabla^{proj} \quad = \quad j_!\mathcal{E}^{proj} \quad = \quad \begin{array}{c} \hline \underline{\mathcal{O}} \\ \underline{\delta} \\ \mathcal{O} \\ \vdots \end{array}$$

$$j_!Rt^s((s)) \quad = \quad \begin{array}{c} \vdots \\ \underline{\delta} \\ \underline{\mathcal{O}} \\ \delta \\ \hline \underline{\mathcal{O}} \\ \underline{\delta} \\ \mathcal{O} \\ \vdots \end{array}$$

Last time we've have $j_!\mathcal{E}^{proj} \hookrightarrow j_*\mathcal{E}^{proj}$. It induces an *isomorphism* $j_!Rt^s((s)) \xrightarrow{\simeq} j_*Rt^s((s))$:

$$\begin{tikzcd}
\underline{\mathcal{O}} \arrow[dr] & \underline{\delta} \\
\delta \arrow[ddr] & \mathcal{O} \\
\rule{2cm}{0.6pt} & \rule{2cm}{0.6pt} \\
\underline{\mathcal{O}} \arrow[dr] & \underline{\delta} \\
\delta & \mathcal{O}
\end{tikzcd}$$

General situation

We have X = smooth variety, $f : X \to \mathbb{C}$ a regular function, $Y = f^{-1}(0)$ = divisor:

$$\begin{tikzcd}
Y = f^{-1}(0) \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] & U = X \setminus Y \arrow[l, hook', "j"'] \arrow[d] \\
\{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}^* \arrow[l, hook', "j"']
\end{tikzcd}$$

Assume X is affine. Let N_U be a holonomic $\mathcal{D}_U$-module. We have a map $j_!(N_U) \to j_*(N_U)$, and

$$Im\big(j_!(N_U) \to j_*(N_U)\big) = j_{!*}(N_U).$$

Proposition 4.12.19 Let $N_0 \subseteq N_U$ be a finite dimensional generating subspace, so that $\mathcal{D}_U \cdot N_0 = N_U$. Then

$$j_{!*}(N_U) = \mathcal{D}_X(f^k N_0)$$

for sufficiently large k.

Proof. Note that for $k \gg 0$,

$$\mathcal{D}_X(f^k N_0) \text{ does not depend on } k$$

(because j_*N_U is holonomic $\Longrightarrow$ has finite length). Let $M := j_{!*}(N_U)$. We have

- $M|_U = N_U$
- N_U/M is supported at $f^{-1}(0)$

Now there is exists $k \gg 0$ s.t. $f^k N_0 = (0)$ in N_U/M, whence $f^k N_0 \subseteq M$, so that $\mathcal{D}(X) f^k N_0 \subseteq M$. If this is not an equality, then

$$\Big(\frac{M}{\mathcal{D}(X)f^kN_0}\Big)\Big|_U = \frac{M|_U}{\mathcal{D}(X)f^kN_0|_U} = \frac{N_U}{N_U} = (0),$$

which contradicts the definition of M.

4.12.20 Homework. $\mathcal{J}_n = R\log^{n-1} + R\log^{n-2} + \ldots + R$, $R = \mathbb{C}[t, t^{-1}]$. This is a $\mathcal{D}$-module on $\mathbb{C}^*$ which is free of rank n as an R-module.

(1) Describe the corresponding connection on the trivial bundle R^n.

(2) Check that the dual connection on $Hom_R(\mathcal{J}_n, R)$ is isomorphic to the original connection.

We've proved that $\mathcal{C} \xrightarrow{\simeq} \left\{M_{-1} \rightleftarrows M_0\right\}$. Find the images of $j_*\mathcal{J}_n$ and $j_*\mathcal{J}_n$ under this equivalence.

Set-up.

$$\begin{tikzcd}
Y = f^{-1}(0) \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] & U = X \setminus f^{-1}(0) \arrow[l, hook'] \arrow[d] \\
\{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}^* \arrow[l, hook', "j"']
\end{tikzcd}$$

Let us replace $t \rightsquigarrow f$, $t^s \rightsquigarrow f^s$, $\log t \rightsquigarrow \log f$. On U, we have

$$\mathcal{J}_n(f) = f^*\mathcal{J}_n = \mathcal{O}_U \cdot (\log f)^{n-1} + \mathcal{O}_U \cdot (\log f)^{n-2} + \ldots + \mathcal{O}_U \cdot \log f + \mathcal{O}_U.$$

We can consider

$$\varinjlim_n \mathcal{J}_n(f) = \mathcal{O}_U[\log f] \cong \frac{\mathcal{O}_U f^s((s))}{\mathcal{O}_U f^s[[s]]}$$

as before (write $f^s = e^{(\log f)s}$). We have

$$\begin{aligned} f^*\Delta^{ind} &= \varinjlim{}_n \mathcal{J}_n(f), \\ f^*\Delta^{proj} &= \mathcal{O}_U f^s[[s]], \end{aligned}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_U f^s[[s]] \arrow[r] \arrow[d, equal] & \mathcal{O}_U f^s((s)) \arrow[r] & \frac{\mathcal{O}_U f^s((s))}{\mathcal{O}_U f^s[[s]]} \arrow[r] \arrow[d, equal] & 0 \\
& f^*\Delta^{proj} & & f^*\Delta^{ind} &
\end{tikzcd}$$

Remainder:

Lemma 4.12.21 [on the b-function] If M_U is holonomic on U and $M_U = \mathcal{D}_U \cdot M_0$, where M_0 is a finite-dimensional generating subspace, then there exists $b \in \mathbb{C}[s]$ with

$$b(s) f^s M_0 \subseteq \mathcal{D}_X[s]\big(f^{s+1} M_0\big).$$

Proposition 4.12.22 For any holonomic M_U on U, we have a canonical isomorphism

$$j_!\big(M_U f^s((s))\big) \xrightarrow{\simeq} j_*\big(M_U f^s((s))\big)$$

in the appropriate category.

Proof. Surjectivity. Image $= j_{!*}\big(M_U f^s((s))\big) \subset j_*\big(M_U f^s((s))\big)$. By the proposition proved last time, we have

$$j_{!*}(-) = \mathcal{D}_X\big(f^k \cdot \text{generating subspace}\big) \quad \text{for } k \gg 0.$$

Choose $M_0 \subset M_U$, a finite-dimensional generating subspace of M_U over $\mathcal{D}_X$ [we can do it because M_0 is holonomic]. We get

$$\begin{aligned} j_{!*}\big(M_U f^s((s))\big) &= \mathcal{D}_X\big(M_0 f^s f^k((s))\big) = \\ &= \mathcal{D}_X\big(M_0 f^{s+k}((s))\big) \supseteq \\ &\supseteq \mathcal{D}_X b(s+k-1) M_0 f^{s+k-1}((s)) \supseteq \end{aligned}$$

$$\supseteq \mathcal{D}_X \underbrace{b(s+k-1)b(s+k-2)\cdot \ldots \cdot b(s)}_{\text{because this is a unit in } \mathbb{C}((s))} M_0 f^s((s)) = j_*\big(M_U f^s((s))\big).$$

We write
$$\begin{aligned} \mathcal{E}_f^{proj} &= \varprojlim \mathcal{J}_n(f) = f^*\mathcal{E}^{proj} \text{ (on } U),\\ \mathcal{E}_f^{ind} &= \varinjlim \mathcal{J}_n(f) \end{aligned}$$

Lemma 4.12.23 On U, we have

$$\mathbb{D}\big(M_U f^s[[s]]\big) = \frac{\big(\mathbb{D}M_U\big) f^s((s))}{\big(\mathbb{D}M_U\big) f^s[[s]]},$$

i.e.

$$\mathbb{D}\big(\mathcal{E}_f^{proj} \otimes_{\mathcal{O}_U} M_U\big) = \mathcal{E}_f^{ind} \otimes_{\mathcal{O}_U} \big(\mathbb{D}M_U\big).$$

Note that $f^*\mathcal{J}_n =$ free $\mathcal{O}_U$-module of rank n.
Hence $f^*\mathcal{J}_n \otimes_{\mathcal{O}_U} M_U \cong M_U^{\oplus n}$ as $\mathcal{O}_U$-modules. Also,

$$f^*\mathcal{J}_n \otimes_{\mathcal{O}_U} M_U \sim \begin{pmatrix} M_U \\ \hline M_U \\ \hline \vdots \\ \hline M_U \end{pmatrix}$$

Step 1. $\mathbb{D}\mathcal{E}_f^{proj} \cong \mathcal{E}_f^{ind}$. It is enough to check that $\mathbb{D}\big(f^*\mathcal{J}_n\big) \cong f^*\mathcal{J}_n$.

4.12.24 Exercise Define a canonical pairing

$$f^*\mathcal{J}_n \otimes f^*\mathcal{J}_n \longrightarrow \mathcal{O}_U.$$

Step 2. General case. Write a free resolution

$$0 \to p^n \to p^{n-1} \to \ldots \to p^1 \to p^0 \to M_U \to 0$$

(the p^j are free $\mathcal{D}_U$-modules).

We get

$$0 \leftarrow \mathcal{D}M_U \leftarrow Hom_{\mathcal{D}_U}(p^n, \mathcal{D}_U) \leftarrow \ldots \leftarrow Hom_{\mathcal{D}_U}(p^0, \mathcal{D}_U) \leftarrow 0.$$

But now

$$0 \to \mathcal{E}_f^{proj} \otimes_{\mathcal{O}_U} p^n \to \ldots \to \mathcal{E}_f^{proj} \otimes_{\mathcal{O}_U} p^0 \to \mathcal{E}_f^{proj} \otimes_{\mathcal{O}_U} M_U \to 0$$

is a free resolution of $\mathcal{E}_f^{proj} \otimes_{\mathcal{O}_U} M_U$. But

$$Hom_{\mathcal{D}_U}\big(\mathcal{E}_f^{proj} \otimes_{\mathcal{O}_U} p^j, \mathcal{D}_U\big) \cong \big(\mathcal{E}_f^{proj}\big)^v \otimes_{\mathcal{O}_U} Hom_{\mathcal{D}_U}\big(p^j, \mathcal{D}_U\big),$$

which completes the proof.

In the proposition above, it remains to prove injectivity.

But we have

$$j_!\big[(\mathbb{D}M_U)f^s((s))\big] \twoheadrightarrow j_*\big[(\mathbb{D}M_U)f^s((s))\big]$$

Applying $\mathbb{D}$ gives

$$\begin{tikzcd}
\mathbb{D}j_![-] \arrow[d, equal] & \mathbb{D}j_*[-] \arrow[l, hook'] \arrow[d, equal, "\text{using the lemma}"] \\
j_*\big(M_U f^s((s))\big) & j_!\big(M_U f^s((s))\big) \arrow[l, hook']
\end{tikzcd}$$

Corollary 4.12.25 The map

$$j_!\big(M_U f^s[[s]]\big) \longrightarrow j_*\big(M_U f^s((s))\big)$$

is injective.

Definition 4.12.26 The *nearby cycle functor* $\psi(-) : \mathcal{H}ol\mathcal{D}_U\text{-}mod \longrightarrow \mathcal{H}ol\mathcal{D}_X\text{-}mod$ supported on $X \setminus U$ is defined by

$$\psi(\mathcal{M}) := \frac{j_*\mathcal{M}f^s[[s]]}{j_!\mathcal{M}f^s[[s]]}$$

This comes with a canonical $\mathcal{D}_X$-module endomorphism given by multiplication by s.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_U$-module; $\mathcal{M}_0$ finite dim'l such that $\mathcal{D}_U\mathcal{M}_0 = \mathcal{M}$.

Lemma 4.12.27 We have the following isomorphisms of $\mathcal{D}_X[[s]]$-modules:

(a) $j_!\big(\mathcal{M}f^s[[s]]\big) \cong \mathcal{D}_X[[s]]\big(f^{s+k}\mathcal{M}_0\big)$

(b) $j_*\big(\mathcal{M}f^s[[s]]\big) \cong \mathcal{D}_X[[s]]\big(f^{s-k}\mathcal{M}_0\big)$
for $k \gg 0$

Proof. (a) follows from the fact that $j_!\big(Mf^s[[s]]\big) \hookrightarrow j_*(-)$, whence $j_!(-) \cong j_{!*}(-)$.

(b) follows by duality.

Proposition 4.12.28 We have

$$j_*\mathcal{M} = \frac{\mathcal{D}_X[s]\big(f^{s-k}\mathcal{M}_0\big)}{s\cdot\mathcal{D}_X[s]\big(f^{s-k}\mathcal{M}_0\big)},$$

and

$$j_!\mathcal{M} = \frac{\mathcal{D}_X[s]\big(f^{s+k}\mathcal{M}_0\big)}{s\cdot\mathcal{D}_X[s]\big(f^{s+k}\mathcal{M}_0\big)},$$

for $k \gg 0$.

Proof. Look at the exact sequence on U:

$$0 \to \mathcal{M}f^s[[s]] \xrightarrow{s} \mathcal{M}f^s[[s]] \to \mathcal{M} \to 0.$$

Applying j_* and $j_!$, we get the result (because these functors are exact).

We go back to the nearby cycle functor ψ.

Theorem 4.12.28.8. *$\psi(\mathcal{M})$ is a holonomic $\mathcal{D}_X$-module supported on $f^{-1}(0)$, and the functor $\psi(-)$ is exact.*

Proof. Let us first prove exactness.

We have the following general nonsense result.

Lemma 4.12.29 Let F, G be exact functors $\mathcal{C} \to \mathcal{C}'$. Suppose we have in addition a morphism (natural transformation) $F \to G$, which is injective on all objects. Then the functor $M \rightsquigarrow G(M)/F(M)$ is also exact.

Proof. Obvious.

We have

$$\begin{tikzcd} Y = f^{-1}(0) \arrow[r, hook] \arrow[d] & X \arrow[d] & U \arrow[l, hook', "j"'] \arrow[d] \\ \{0\} \arrow[r, hook] & \mathbb{C} & \mathbb{C}* \arrow[l, hook'] \end{tikzcd}$$

If $\mathcal{M}$ is a $\mathcal{D}_U$-module, we get $j_!\mathcal{M} \to j_*\mathcal{M}$. Hence

$$\varprojlim_n j_!\left(\frac{\mathcal{M}[[s]]}{s^n\mathcal{M}[[s]]}\right) = j_!\mathcal{M}[[s]] \longrightarrow j_*\mathcal{M}[[s]] = \varprojlim_n j_*\left(\frac{\mathcal{M}[[s]]}{s^n\mathcal{M}[[s]]}\right)$$

Generalization of the general nonsense lemma from last time:

Let F', F, $F'' : \mathcal{C} \longrightarrow \mathcal{D}$ be three exact functors. Assume that we have morphisms of functors $F' \to F \to F''$, whose composition is zero, such that $F'(M) \hookrightarrow F(M)$ and $F(M) \twoheadrightarrow F''(M)$ for all $M \in \mathcal{C}$. Then the functor $M \rightsquigarrow Ker(F(M) \to F''(M))/Im(F'(M) \hookrightarrow F(M))$ is exact.

Now we want to prove that

$$\psi(M) = \frac{j_*(f^s\mathcal{M}[[s]])}{j_!(f^s\mathcal{M}[[s]])}$$

is a holonomic $\mathcal{D}_X$-module supported on Y. We claim that it is enough to show holonomicity. Indeed, if $\psi(M)$ is holonomic, then it has finite length, whence

$$\begin{aligned}\psi(M) &= Coker\left(\varprojlim j_!\left(\frac{\mathcal{M}[[s]]f^s}{s^n f^s\mathcal{M}[[s]]}\right) \longrightarrow \varprojlim j_*\left(\frac{\mathcal{M}[[s]]f^s}{s^n f^s\mathcal{M}[[s]]}\right)\right)\\ &= Coker\left(j_!\left(\frac{\mathcal{M}[[s]]f^s}{s^n f^s\mathcal{M}[[s]]}\right) \longrightarrow j_*\left(\frac{\mathcal{M}[[s]]f^s}{s^n f^s\mathcal{M}[[s]]}\right)\right)\end{aligned}$$

for some $n \gg 0$, and the latter is clearly supported on Y.

Main theorem of last time. If $K = \mathbb{C}((s))$, then

$$K \otimes j_!(\mathcal{M}f^s) \xrightarrow{\simeq} K \otimes j_*(\mathcal{M}f^s).$$

Proof. Choose a finite dimensional subspace $\mathcal{M}_0 \subset \mathcal{M}$ such that $\mathcal{M} = \mathcal{D}_U \cdot \mathcal{M}_0$. Then last time we have proved that

$$\mathcal{D}_X \cdot \big(\mathcal{M}_0 f^{s+k}\big)[[s]] \supseteq B(s)\mathcal{D}_X \cdot \big(\mathcal{M}_0 f^{s-l}\big)[[s]],$$

$$B(s) = b(s+k-1)b(s+k-2)\dots b(s-l).$$

Now $B(s)$ is invertible in K. Also note that

$$\mathcal{D}_X \cdot \big(\mathcal{M}_0 f^{s-l}\big)[[s]] \subseteq \mathcal{D}_X \cdot \big(\mathcal{M}_0 f^{s-l-1}\big)[[s]],$$

and this is an increasing chain of $\mathcal{D}_X[[s]]$-submodules of $j_*\mathcal{M}f^s[[s]]$. When we tensor them with K, they become the same. But

$$K \otimes j_*(\mathcal{M}f^s) = \bigcup_l K \otimes \mathcal{D}_X \cdot (\mathcal{M}_0 f^{s-l}),$$

which completes the proof.

Key lemma. There is exists $n \gg 0$ such that

$$s^n j_*\mathcal{M}f^s[[s]] \subseteq j_!\mathcal{M}f^s[[s]]$$

This implies that

$$\frac{j_*(f^s\mathcal{M}[[s]])}{j_!(f^s\mathcal{M}[[s]])} \quad \text{is a quotient of} \quad \frac{j_*(f^s\mathcal{M}[[s]])}{s^n j_*\mathcal{M}f^s[[s]]},$$

and the latter has a finite filtration with all quotients isomorphic to $j_*\mathcal{M}$, hence is holomorphic. This implies our theorem.

Proof of the key lemma. We know that

$$j_!\mathcal{M}f^s[[s]] = \mathcal{D}_X \cdot \mathcal{M}_0 f^{s+k}[[s]],$$

$$j_*\mathcal{M}f^s[[s]] = \mathcal{D}_X \cdot \mathcal{M}_0 f^{s-k}[[s]],$$

for $k \gg 0$. By the same argument as above, we have

$$B(s) \cdot \mathcal{D}_X \cdot \mathcal{M}_0 f^{s-k}[[s]] \subseteq \mathcal{D}_X \cdot \mathcal{M}_0 f^{s+k}[[s]].$$

We can write $B(s) = s^n \cdot \big(\text{unit of } \mathbb{C}[[s]]\big)$, and this gives the desired result.

4.12.30 Note. The proof above shows that we can choose k so that $[-k, k]$ contains all the integral roots of the b-functions, and $n =$ the number of integral roots of the b-function.

4.13. Digression. Given $\mathcal{N}$, a holonomic $\mathcal{D}_X$-module, we have $\mathcal{N}_{[Y]} =$ the maximal submodule of $\mathcal{N}$ supported on Y. We can define $\mathcal{N}^{[Y]} =$ the maximal quotient $\mathcal{N}$ supported on Y.

We also have the duality functor $\mathbb{D} : \mathcal{H}ol_X \longrightarrow \mathcal{H}ol_X$ which preserves supports (because $\mathbb{D}$ commutes with restrictions to open subsets). This implies that

$$\mathcal{N}^{[Y]} = \mathbb{D}\big((\mathbb{D}\mathcal{N})_{[Y]}\big).$$

Now we suppose $\mathcal{M}$ is a holonomic $\mathcal{D}_U$-module. By definition,

$$(j_*\mathcal{M})^{[Y]} = j_*\mathcal{M}/j_!\mathcal{M}.$$

Therefore

$$\psi(M) = \frac{j_*(f^s\mathcal{M}[[s]])}{j_!(f^s\mathcal{M}[[s]])} = \big(j_*\left(\mathcal{M} \otimes_{\mathcal{O}_U} \mathcal{E}_f^{proj}\right)\big)^{[Y]}$$

Lemma 4.13.1 There is a canonical isomorphism

$$\psi(M) \cong \big(j_!\left(\mathcal{M} \otimes_{\mathcal{O}_U} \mathcal{E}_f^{ind}\right)\big)_{[Y]}$$

Proof. Write

$$0 \to \mathcal{O}_U f^s[[s]] \hookrightarrow \mathcal{O}_U f^s((s)) \to \frac{\mathcal{O}_U f^s((s))}{\mathcal{O}_U f^s[[s]]} \to 0.$$

$\Longrightarrow$

$$0 \to \mathcal{M} \otimes \mathcal{E}_f^{proj} \to \mathcal{M}f^s((s)) \to \mathcal{M} \otimes \mathcal{E}_f^{ind} \to 0.$$

Let us apply $j_!$ and j_* to these short exact sequences.

$j_!$	$\mathcal{M} \otimes \mathcal{E}_f^{proj}$	$\mathcal{M} \otimes \mathcal{E}_f^{ind}$
j_*	$\mathcal{M} \otimes \mathcal{E}_f^{proj}$	$\mathcal{M} \otimes \mathcal{E}_f^{ind}$

$$\Longrightarrow \psi(\mathcal{M}) \cong Ker\left[j_! \left(\mathcal{M} \otimes \mathcal{E}_f^{ind}\right) \to j_* \left(\mathcal{M} \otimes \mathcal{E}_f^{ind}\right)\right] = \left(j_! \left(\mathcal{M} \otimes \mathcal{E}_f^{ind}\right)\right)_{[Y]}$$

Corollary 4.13.2 ψ commutes with $\mathbb{D}$.

Proof. We compute

$$\begin{aligned}\mathbb{D}_\psi \mathbb{D}(\mathcal{M}) &= \mathbb{D}\left(\left(j_* \left((\mathbb{D}\mathcal{M})\mathcal{E}_f^{proj}\right)\right)^{[Y]}\right) = \left(\mathbb{D} j_* \left((\mathbb{D}\mathcal{M})\mathcal{E}_f^{proj}\right)\right)_{[Y]} \\ &= \left(j_! \mathbb{D}\left((\mathbb{D}\mathcal{M})\mathcal{E}_f^{proj}\right)\right)_{[Y]} = \left(j_! \left(\mathcal{M}\mathcal{E}_f^{proj}\right)\right)_{[Y]} = \psi(\mathcal{M}).\end{aligned}$$

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