DERIVED BEILINSON-BERNSTEIN LOCALIZATION IN POSITIVE CHARACTERISTIC

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Contents

1. Introduction

This is the notes for the author's presentation in the student seminar. We follow the paper [BMR1] to talk about a derived equivalence between the representations of a semisimple Lie algebra in positive characteristic and the category of twisted D-modules supported generalized Springer fibers.

We work over a field k, and consider a semisimple Lie algebra $\mathfrak{g}$. Let $U = U(\mathfrak{g})$ be the enveloping algebra. If k has characteristic zero, one has the Beilinson-Bernstein Localization Theorem, which gives an equivalence of abelian categories. When k has positive characteristic, the sheaf of differential operators still makes sense. (There are different versions and we are using the version called crystalline differential operators.) In fact, fixing a character λ of the Cartan subalgebra $\mathfrak{h}$, one can consider the sheaf of twisted differential operators $\mathscr{D}^{\lambda}_{\mathscr{B}}$ on the flag variety $\mathscr{B}$. One can still consider the localization functor $\mathscr{L}$ and the global sections functor Γ. They are functors between module categories over $U(\mathfrak{g})$ with the fixed central character λ, denoted by $\operatorname{Mod} U(\mathfrak{g})^{\lambda}$, and the category of sheaves of coherent modules over $\mathscr{D}^{\lambda}_{\mathscr{B}}$, denoted by $\operatorname{Coh} \mathscr{D}^{\lambda}_{\mathscr{B}}$.

$$\begin{tikzcd} \operatorname{Mod} U(\mathfrak{g})^{\lambda} \arrow[r, bend left, "\mathscr{L}"] & \operatorname{Coh} \mathscr{D}^{\lambda}_{\mathscr{B}} \arrow[l, bend left, "\Gamma"] \end{tikzcd}$$

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The functor Γ is right adjoint to $\mathscr{L}$. In the positive characteristic case, we do not get an equivalence between abelian categories. However, if one pass to the derived categories, one still get an equivalence between triangulated categories.

The sheaf of differential operators of $\mathscr{B}$ behave better in the case k has characteristic $p > 0$, in the sense that it is an Azumaya algebra on $T^*\mathscr{B}^{(1)}$, the Frobenius twist of the cotangent bundle $T^*\mathscr{B}$. When restricting to a formal neighborhood of the Springer fiber, this Azumaya algebra splits, which means its module category is equivalent to the category of coherent sheaves.

Comparing different characters, one gets an affine braid group action. The generators of the affine braid group act by (composition of) translation functors. We will describe this action on the level of $\operatorname{Mod} U(\mathfrak{g})$, on the level of D-modules, and also on the level of coherent sheaves. In order to fully understand the effect of translation functors on the coherent sheaf level, we need a version of the derived Beilinson-Bernstein localization theorem for singular central characters of the Lie algebra. This is described in [BMR2]. Again the Azumaya algebra of twisted differential operators splits on the formal neighborhood of the parabolic Springer fibers. We will use this singular version of localization theorem to describe the affine braid group action.

2. Modules over twisted differential operators in prime characteristic

For any character of $U(\mathfrak{t})$, say λ, one can consider the central reduction $\mathscr{D}_X^\lambda := \tilde{\mathscr{D}}_X \otimes_{\mathscr{O}_{\mathfrak{t}^*}} \lambda$. Note that this definition differs from [SR] by a $-\rho$.

Let $\mathscr{B}$ be the flag variety G/B and $\tilde{\mathscr{B}}$ the trivialization of $\operatorname{Pic}(\mathscr{B})$. It can be described as G/U where U is the unipotent radical of B. Then $\tilde{\mathscr{B}} \to \mathscr{B}$ is a torsor under the Cartan subgroup H. Consider the sheaf of vector fields $\mathscr{T}_{\tilde{\mathscr{B}}}$ on $\tilde{\mathscr{B}}$. Take the sections in $\pi_*\mathscr{T}_{\tilde{\mathscr{B}}}$ on $\mathscr{B}$ invariant under the translation action of H, i.e., $(\pi_*\mathscr{T}_{\tilde{\mathscr{B}}})^H =: \tilde{\mathscr{T}}_{\mathscr{B}}$. It is a sheaf of Lie algebras which is a central extension of $\mathscr{T}(\mathscr{B})$ by $\mathfrak{h}$, the lie algebra of H. The enveloping sheaf of algebras of the Lie algebroid $\tilde{\mathscr{T}}_{\mathscr{B}}$ is denoted by $\tilde{\mathscr{D}}_{\mathscr{B}}$. The enveloping algebra of $\mathfrak{h}$, which is $\mathscr{O}(\mathfrak{h}^*)$, lies in the center of $\tilde{\mathscr{D}}_{\mathscr{B}}$. This part of center will be referred to as the Harish-Chandra center.

Example 2.1. When $\mathfrak{g} = \mathfrak{sl}_2$, $\tilde{\mathscr{B}} = G/U \cong \mathbb{A}^2 - \{0\}$, and the extension $0 \to \mathfrak{h} \to \tilde{\mathscr{T}}_{\mathscr{B}} \to \mathscr{T}_{\mathscr{B}} \to 0$ becomes the Eular sequence

$$0 \to \mathscr{O} \to \mathscr{O}(1)^2 \to \mathscr{T}_{\mathbb{P}^1} \to 0.$$

Let $\tilde{T}^*\mathscr{B}$ be the total space of the locally free coherent sheaf $\tilde{\mathscr{T}}^*\mathscr{B}$. Recall this is the Grothendieck simultaneous resolution $\tilde{\mathfrak{g}}^*$. The adjoint action of H on $\mathfrak{h}$ induces an action on $\tilde{T}^*\mathscr{B}$, with moment map $\mathrm{pr}_2 : \tilde{T}^*\mathscr{B} \to \mathfrak{h}^*$. The G-action has moment map $\mathrm{pr}_1 : \tilde{T}^*\mathscr{B} \to \mathfrak{g}^*$. The sheaf of algebras $\tilde{\mathscr{D}}_{\mathscr{B}}$ on $\mathscr{B}$ has a Frobenius center, i.e., $(\mathscr{O}_{T^*\mathscr{B}})^p$. This means $\tilde{\mathscr{D}}_{\mathscr{B}}$ can be considered as a quasi-coherent sheaf on $\mathscr{T}^*\mathscr{B}^{(1)} =: \tilde{\mathfrak{g}}^{*(1)}$. $\tilde{T}^*\mathscr{B}^{(1)} =: \tilde{\mathfrak{g}}^{*(1)}$.

$$\begin{tikzcd}
 & & & & & \tilde{T}^*\mathscr{B}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^* \arrow[dll] \arrow[dr] & \\
 & \tilde{\mathcal{N}}^{(1)} \arrow[r, hook] \arrow[dl, "\mu"'] & & \tilde{\mathfrak{g}}^{*(1)} = \tilde{T}^*\mathscr{B}^{(1)} \arrow[dl, "\mathrm{pr}_1^{(1)}"] \arrow[drr, "\mathrm{pr}_2^{(1)}"'] & & & \mathfrak{h}^* \arrow[dl, "\mathrm{AS}"] \\
\mathcal{N}^{(1)} \arrow[rr, hook] \arrow[dr] & & \mathfrak{g}^{*(1)} \arrow[dr] & & & \mathfrak{h}^{*(1)} \arrow[dll] & \\
 & \{0\} \arrow[rr] & & \mathfrak{h}^{*(1)}/W & & &
\end{tikzcd}$$

Proposition 2.2. (1) *In $\tilde{\mathscr{D}}_{\mathscr{B}}$, we have $\mathscr{O}_{\tilde{T}^*\mathscr{B}^{(1)}} \cap \mathscr{O}(\mathfrak{h}^*) = \mathscr{O}(\mathfrak{h}^{*(1)})$.*
(2) *The sheaf $\tilde{\mathscr{D}}_{\mathscr{B}}$ is an Azumaya algebra on $\tilde{T}^*\mathscr{B}^{(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$.*

We consider D-modules set-theoretically supported on a subscheme. (We will omit later on the word *set-theoretically.*) We say a $\mathscr{D}$ (resp. $\tilde{\mathscr{D}}$) module $\mathcal{M}$ is coherent if it is a coherent sheaf over the center. Let λ be an integral $\mathfrak{h}$ character (a point in $\mathfrak{h}^*$ that comes from $\mathrm{Hom}(H, \mathbb{G}_m)$). Let $\mathcal{M}$ be a coherent $\tilde{\mathscr{D}}$ module. The action of the Frobensius center endows $\mathcal{M}$ with the structure of a coherent sheaf on $\tilde{T}^*\mathscr{B}^{(1)}$.

For any integral character $\lambda \in \mathfrak{h}^*$, let $\mathrm{Coh}_\lambda \tilde{\mathscr{D}}$ be the full subcategory of coherent $\tilde{\mathscr{D}}_{\mathscr{B}}$-modules consisting of module killed by a power of the maximal ideal λ in $U(\mathfrak{h})$. Note that these modules are supported on the cotangent bundle $T^*_\lambda\mathscr{B}^{(1)} = \tilde{\mathfrak{g}}^{*(1)} \times_{\mathfrak{h}^{*(1)}} \lambda \subseteq \tilde{\mathfrak{g}}^{*(1)} \times_{\mathfrak{h}^{*(1)}} \mathfrak{h}^*$. Note that $T^*_\lambda\mathscr{B}^{(1)}$ is canonically identified with $T^*\mathscr{B}^{(1)}$.

The global sections of the Frobenius center $\Gamma(\mathscr{B}, \mathscr{O}_{\tilde{T}^*\mathscr{B}^{(1)}})$ is the coordinate ring of $\mathfrak{g}^{*(1)}$. For any closed point χ in $\mathfrak{g}^{*(1)}$, we denote the fiber of χ in $\tilde{T}^*\mathscr{B}^{(1)}$ by $\mathscr{B}_\chi$. Think of χ as a character of $\tilde{\mathscr{D}}_{\mathscr{B}}$, the central reduction $\mathscr{D}_\chi := \tilde{\mathscr{D}} \otimes_{\mathscr{O}_{\tilde{T}^*\mathscr{B}^{(1)}}} \chi$ is a sheaf of algebras on $\mathscr{B}_\chi$. The category of coherent $\mathscr{D}_\chi$-modules, denoted by $\mathrm{Coh}\, \mathscr{D}_\chi$ is naturally a subcategory of $\mathrm{Coh}\, \tilde{\mathscr{D}}$.

Let $\mathrm{Coh}_\chi \tilde{\mathscr{D}}$ be the full subcategory of coherent $\tilde{\mathscr{D}}_{\mathscr{B}}$-modules consisting of module killed by a power of the maximal ideal χ in $\mathscr{O}_{\tilde{T}^*\mathscr{B}^{(1)}}$. Note that these modules are supported on $\mathscr{B}_\chi$.

For $\chi \in \mathcal{N}^{(1)} \subseteq \mathfrak{g}^{*(1)}$, the twisted Springer fiber $(\mu^{(1)})^{-1}(\chi) = \mathscr{B}^{(1)}_\chi \cap T^*_\lambda\mathscr{B}^{(1)} \subseteq \tilde{T}^*\mathscr{B}^{(1)}$ is denoted by $\mathscr{B}_{\lambda,\chi}$. For any integral character $\lambda \in \mathfrak{h}^*$, let $\mathrm{Coh}_{\lambda,\chi} \tilde{\mathscr{D}}$ be the full subcategory of coherent $\tilde{\mathscr{D}}_{\mathscr{B}}$-modules consisting of module killed by a power of the maximal ideal (λ, χ). Note that these modules are supported on $\mathscr{B}_{\lambda,\chi}$.

For any integral $\lambda \in \mathfrak{h}^*$, and any $\chi \in \mathcal{N}^{(1)} \subseteq \mathfrak{g}^{*(1)}$, the central reduction of $\mathscr{D}^\lambda$ with respect to χ, denoted by $\mathscr{D}^\lambda_\chi$, is a sheaf of algebras on $\mathscr{B}_{0,\chi}$. The category $\mathrm{Coh}\, \mathscr{D}^\lambda_\chi$ is naturally a subcategory of $\mathrm{Coh}\, \tilde{\mathscr{D}}_{\mathscr{B}}$.

3. Derived localization theorem

A central character of U can be considered as a pair (λ, χ) where $\lambda \in \mathfrak{h}^*/W$ and $\chi \in \mathfrak{g}^{*(1)}$. We can consider various subcategories of $\operatorname{Mod} U(\mathfrak{g})$. Let $\operatorname{Mod}_\lambda U$ be the subcategory of $\operatorname{Mod} U$ consisting of modules killed by certain power of the maximal ideal λ in the Harish-Chandra center (i.e., the Harish-Chandra center acts via the generalized character λ). Let $\operatorname{Mod} U^\lambda$ be the subcategory of $\operatorname{Mod} U$ consisting of U-modules killed by the maximal ideal λ of the Harish-Chandra center. Similarly we have $\operatorname{Mod}_{\chi,\lambda} U$ and $\operatorname{Mod} U^\lambda_\chi$, etc.

Proposition 3.1. *Assume p is sufficiently large, then we have*

- $H^i(\mathscr{B}, \tilde{\mathscr{D}}) = 0$ *for* $i > 0$ *and* $H^0(\mathscr{B}, \tilde{\mathscr{D}}) \cong \tilde{U}$;
- $H^i(\mathscr{B}, \mathscr{D}^\lambda) = 0$ *for* $i > 0$ *and* $H^0(\mathscr{B}, \mathscr{D}^\lambda) \cong U^\lambda$.

This is similar to [SR].

Let mod-$_\lambda\tilde{U}$ be the category of not-necessarily-finitely-generated $\tilde{U}$-modules with generalized character λ. Now we consider the derived global section functor $R\Gamma_{\tilde{\mathscr{D}},\lambda} : D^b(\operatorname{Coh}_\lambda \tilde{\mathscr{D}}) \to D^-(\text{mod-}_\lambda\tilde{U})$.

Lemma 3.2. *The image of $R\Gamma_{\tilde{\mathscr{D}},\lambda}$ lands in the subcategory $D^b(\operatorname{Mod}_\lambda U)$.*

Proof. In fact, observe that taking a finite affine covering of $\mathscr{B}$, the functor $R\Gamma_{\tilde{\mathscr{D}},\lambda}$ sends any module $\mathcal{M}$ to the Cěch complex. The derived global section functors respect the forgetful functors. In other words, the following diagram commutes

$$\begin{tikzcd} D^b(\operatorname{Coh}_\lambda \tilde{\mathscr{D}}) \arrow[r, "\text{Forget}"] \arrow[d, "R\Gamma_{\tilde{\mathscr{D}},\lambda}"'] & D^b(q\operatorname{Coh}\mathscr{B}) \arrow[d, "R\Gamma"] \\ D^-(\operatorname{Mod}_\lambda U) \arrow[r, "\text{Forget}"'] & D^-(\text{Vect}) \end{tikzcd}.$$

The image of the functor $R\Gamma_{\tilde{\mathscr{D}},\lambda}$ lands actually in $D^b(\text{mod-}_\lambda\tilde{U}$.

Now it remains to show for any complex $\mathcal{M}^\bullet$, the complex $R\Gamma_{\tilde{\mathscr{D}},\lambda}\mathcal{M}$ has finitely generated cohomology. In fact, the map $\tilde{U} \to \tilde{\mathscr{D}}$ is filtered. The associated graded map is $\mathscr{O}_{\mathfrak{g}^* \times_{\mathfrak{h}^*//W} \mathfrak{h}^*} \to \mathscr{O}_{G \times_B \mathfrak{n}^\perp}$. We have the spectral sequence $R^*\mu_*(\operatorname{gr}\mathcal{M}^\bullet) \Rightarrow \operatorname{gr}(R^*\Gamma_{\tilde{\mathscr{D}},\lambda}\mathcal{M}^\bullet)$. This shows $\operatorname{gr}(R^*\Gamma\mathcal{M}^\bullet)$ is a bounded complex of coherent sheaves on $\mathfrak{g}^* \times_{\mathfrak{h}^*//W} \mathfrak{h}^*$. As each $\operatorname{gr}(R^i\Gamma\mathcal{M}^j)$ is a finitely generated module over $\operatorname{gr}\tilde{U}$, we know that $R^i\Gamma\mathcal{M}^j$ is finitely generated over $\tilde{U}$. So $R^*\Gamma_{\tilde{\mathscr{D}},\lambda}\mathcal{M}^\bullet$ is a complex over $\tilde{U}$, hence can be represented by a bounded complex of finitely generated $\tilde{U}$-modules. $\square$

The main theorem of this section is the following.

Theorem 3.3 ([BMR1])**.** *Assume p is sufficiently large and $\lambda \in \mathfrak{h}^*$ is integral regular. Then, the global section functor*

$$R\Gamma_{\tilde{\mathscr{D}},\lambda} : D^b(\operatorname{Coh}_\lambda(\tilde{\mathscr{D}})) \to D^b(\operatorname{Mod}_\lambda U)$$

is an equivalence of triangulated categories.

As an immediate consequence, we get the following Corollary.

Corollary 3.4. *Similarly, we have*

$$R\Gamma_{\mathscr{D}^\lambda} : D^b(\mathrm{Coh}(\mathscr{D}^\lambda)) \cong D^b(\mathrm{Mod}\, U^\lambda);$$
$$R\Gamma_{\mathscr{D}^\lambda,\chi} : D^b(\mathrm{Coh}_\chi\, \tilde{\mathscr{D}}^\lambda) \cong D^b(\mathrm{Mod}_\chi\, U^\lambda);$$
$$R\Gamma_{\tilde{\mathscr{D}}_{\lambda\chi}} : D^b(\mathrm{Coh}_{\lambda,\chi}\, \tilde{\mathscr{D}}) \cong D^b(\mathrm{Mod}_{\lambda,\chi}\, U).$$

The proof of Theorem 3.3 is the same as in the case of characteristic zero in [SR]. We include a sketch of the proof is in the Appendix.

4. Baby Verma and point modules

For the moment being, let us work in the general set-up where we have a smooth variety X, with its sheaf of differential operators $\mathscr{D}_X$ which is an Azumaya algebras on $T^*X^{(1)}$. Let $b \in X$ be a closed point and let $\zeta = (b, \chi)$ be a closed point in $T^*X^{(1)}$ which lies in the fiber over b. Let

$$\mathscr{D}_{X,\zeta} := \mathscr{D}_X \otimes_{\mathscr{O}_{T^*X^{(1)}}} \mathscr{O}_\zeta$$

be the central reduction of $\mathscr{D}$ with respect to the point $\zeta \in T^*X^{(1)}$ thought of as a character of $\mathfrak{Z}_X$, the center of $\mathscr{D}_X$. Here $\mathscr{O}_\zeta$ is the structure sheaf (residue field) of ζ.

Let $a \in X$ be such that $\mathrm{Fr}(a) = b$. Let ξ be the closed point $\xi := (a, \chi) \in T^{*,(1)}X := T^*X^{(1)} \times_{X^{(1)}} X$.

$$\begin{tikzcd}
\zeta \in T^*X^{(1)} \arrow[d] & T^{*,(1)}X \ni \xi \arrow[l] \arrow[d] \\
b \in X^{(1)} & X \ni a \arrow[l, "\mathrm{Fr}"']
\end{tikzcd}$$

Let $\delta(a)$ be the Dirac-delta distributions at the point a, which sends a local function f to $f(a)$. Any vector field $\partial \in \mathscr{D}_X$ acts on $\delta(a)$ by $\partial\delta(a)(f) := -\partial f(a)$. Let δ_a^o be the submodule in the module of distributions generated by $\delta(a)$. Note that δ_a^o is a right D-module. Let δ_a be the associated left D-module, i.e., $\delta_a := \delta_a^o \otimes_{\mathscr{O}} \omega_X^*$ with ω_X being the canonical sheaf of X. The central reduction of this module with respect to $\zeta = (b, \chi)$, denoted by $\delta^\xi := \delta_a \otimes_{\mathfrak{Z}_X} \mathscr{O}_\zeta$ is called the *point module.* Since

$$\mathfrak{Z}_X \otimes_{\mathscr{O}_{T^{*,(1)}X}} \mathscr{O}_\xi \cong \mathfrak{Z}_X \otimes_{\mathfrak{Z}_X \otimes_{\mathscr{O}_{X^{(1)}}} \mathscr{O}_X} \mathscr{O}_\xi \cong \mathscr{O}_\zeta,$$

we have

$$\delta^\xi := \delta_a \otimes_{\mathfrak{Z}_X} \mathscr{O}_\zeta \cong \delta_a \otimes_{\mathscr{O}_{T^{*,(1)}X}} \mathscr{O}_\zeta.$$

Lemma 4.1. *We have*

$$\mathscr{D}_{X,\zeta} \cong \mathscr{E}\mathrm{nd}_{\mathscr{O}_\zeta}(\delta^\xi).$$

Proof. This is by local calculation. Let $x_1, \dots, x_n$ be the local coordinates around a. Then $\mathscr{D}_{X,\zeta}$ has basis $x^I \partial^J$ where $I,\, J \in \{0, 1, \dots, p-1\}^n$. And δ^ξ has basis ∂^I with $I \in \{0, 1, \dots, p-1\}^n$. $\square$

Now we consider the twisted differential operators. Let $\tilde{X} \to X$ be a torsor under a torus T. Let $\mathfrak{t} := \operatorname{Lie} T$. For integral $\lambda \in \mathfrak{t}^*$, we have the sheaf of twisted differential operators $\mathscr{D}^\lambda$ which is an Azumaya algebra on $T^*X^{(1)}$. Let $\zeta = (b, \chi)$ be a closed point in $T^*X^{(1)}$, and let $a \in X$ be such that $\operatorname{Fr}(a) = b$. We have the central reduction $\mathscr{D}^\lambda_{X,\zeta} := \mathscr{D}^\lambda_X \otimes_{\mathfrak{Z}} \mathscr{O}_\zeta$ of the algebra, acting on the central reduction $\delta^\xi := \delta_a \otimes_{\mathfrak{Z}} \mathscr{O}_\zeta$ of the module.

Lemma 4.2. *Similar to the non-twisted case,* $\mathscr{D}^\lambda_{X,\zeta} \cong \mathscr{E}\mathrm{nd}_{\mathscr{O}_\zeta}(\delta^\xi)$.

Now we take a closer look at the module structure of δ^ξ locally. Suppose we have a Lie algebra $\mathfrak{a}$ acting on X (a morphism of sheaves of lie algebras $\mathfrak{a} \to \mathscr{T}_X$) with $\tilde{X}$ being equivariant under $\mathfrak{a}$ (meaning the morphism $\mathfrak{a} \to \mathscr{T}_X$ can be lifted to $\mathfrak{a} \to \tilde{\mathscr{T}}_X$). We express the action of $\mathfrak{a}$ on δ^ξ in terms of the generator $v = 1 \otimes 1 \in \delta^\xi$. Then we have two extremal situations.

Case 1: $a \in X$ is a fixed point of $\mathfrak{a}$, i.e., when $\mathfrak{a}$ maps trivially to T_aX. In this case the map $\mathfrak{a} \to \tilde{\mathscr{T}}_X$ comes from a map $\mathfrak{a} \to \mathfrak{t}$. Let $\lambda_a \in \mathfrak{a}^*$ be the character induced via the map $\mathfrak{a} \to \mathfrak{t}$ from the character $\lambda \in \mathfrak{t}^*$. Let ω_a be the character by which $\mathfrak{a}$ acts on the canonical sheaf ω_X. Then the action of $\mathfrak{a}$ on v is by the character $\lambda_a - \omega_a$.

Case 2: $a \in X$ is a point where $\mathfrak{a}$ acts simply transitively. By this we mean the map $\mathfrak{a} \to T_X$, yeilds an isomorphism $\mathfrak{a} \cong T_aX$ when taking the fiber at $a \in X$. The point $\chi \in T^*_aX$ can be thought of as a character of the Lie algebra T_aX. We use the notation $\chi_a \in \mathfrak{a}^*$ to mean the character of $\mathfrak{a}$ induced via the isomorphism $\mathfrak{a} \cong T_aX$. Then the map $U_{\chi_a}(\mathfrak{a}) \to \delta^\xi$ sending f to $f(v)$ is an isomorphism.

Now take $X = \mathscr{B}$, and $\tilde{X} = \tilde{\mathscr{B}}$. We consider the differential operators twisted by an integral $\lambda \in \mathfrak{h}^*$. Let $\mathfrak{b}$ be the Borel subalgebra corresponding to the point $b \in \mathscr{B}$. Decompose $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. Let $\mathfrak{n}^-$ be the opposite $\mathfrak{n}$. We have $\mathfrak{n}^{-*} \cong T^*_{\mathfrak{b}}\mathscr{B}$. Consider the pair $(\mathfrak{b}, \chi)$ where $\chi \in \mathfrak{g}^{*(1)}$ such that $\chi|_{\mathfrak{b}^{(1)}} = 0$. As χ restricts to $\chi|_{\mathfrak{n}^-} \in \mathfrak{n}^{-*} \cong T^*_{\mathfrak{b}}\mathscr{B}$, the pair $(\mathfrak{b}, \chi)$ can be thought of as a closed point $(\mathfrak{b}, \chi)$ in $T_{\mathfrak{b}}\mathscr{B}$.

We look at the point module δ^ξ correspond to $\xi = (\mathfrak{b}, \chi, \lambda)$. Consider the $\mathfrak{b}$ action on $\mathscr{B}$. It fixes b, hence $\mathfrak{b}$ acts on v by $\lambda + 2\rho$. Then look at the $\mathfrak{n}^-$-action which is simply transitive at $\mathfrak{b}$. Its action on v induces an isomorphism $U_\chi(\mathfrak{n}^-) \cong \delta^\xi$ as modules over $\mathfrak{n}^-$. To summarize, as a module over U^λ, the point module δ^ξ is isomorphic to $U_\chi(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_{\lambda+2\rho}$. It is called the *baby Verma module.*

Also since the support of δ^ξ is one point, its global section is δ^ξ itself with the natural U^λ action, which is again the baby Verma module.

Example 4.3. Let $\mathfrak{g} = \mathfrak{sl}_2 = \langle e, h, f \rangle$. Then $\pi : \tilde{\mathscr{B}} \to \mathscr{B}$ is the natural projection $\mathbb{A}^2 - \{0\} \to \mathbb{P}^1$. Now we consider $\mathfrak{g} \to \tilde{\mathscr{T}} \cong 2\mathscr{O}(1)$. Clearly e maps trivially and h maps to the subsheaf $\mathscr{O} \to 2\mathscr{O}(1)$ generated by a global $\mathbb{G}_m$-invariant vector field on $\mathbb{A}^2 - \{0\}$. Up to scalar, there is only one such global non-vanishing $\mathbb{G}_m$-invariant vector field, i.e., $x\partial_x + y\partial_y$. Locally f acts as a vector field on $\mathbb{P}^1$.

Let s be a local section of $\mathscr{O}(n)$, then $h = x\partial_x + y\partial_y$ acts on s by scaler n. In particular, h acts on the anti-canonical sheaf $\mathscr{O}(2)$ by 2.

Now we examine the structure of a point module as module over $U(\mathfrak{g})^\lambda$. Assume without loss of generality, we take $0 \in \mathbb{P}^1$. On local chart, we get

$$\begin{tikzcd} k[w,\xi] \arrow[r, "w\mapsto z^p", "\xi\mapsto\xi"'] & k[z,\xi] \\ k[w] \arrow[u] \arrow[r, "w\mapsto z^p"] & k[z] \arrow[u] \end{tikzcd}$$

where $\xi = \partial_w$. Let ∂ be ∂_z, then $\partial^p = \xi$. We consider the point module defined by $0 \in \mathbb{P}^1$ and $\chi \in T_0^*\mathbb{P}^{1(1)}$. This point module is generated by one element $v = \delta(0)dz\chi$. Clearly $z\delta(0) = 0$, and $\partial^p(v) = \xi v = \chi v$, and $hv = h(\delta(0))dz\chi + \delta(0)h(dz)\chi = \lambda v + 2v$. Therefore, $U(\mathfrak{b})$ acts by the $\mathfrak{h}$-character $\lambda + 2$. As module over $U(\mathfrak{n}^-) \cong k[f]$, this module is isomorphic to $k[\partial]/(\partial^p - \chi)$ where $f = \partial$.

Appendix A. Differential operators twisted by an arbitrary character

Suppose we have a line bundle $\mathscr{L}$ on a smooth variety X, we can consider differential operators twisted by this line bundle. We would like to fit it into a family. Or in other words we would like to define differential operators twisted by a scalar (in the field) multiple of this line bundle. (If we can do this for one line bundle, we can do it simultaneously for several line bundles.) Let $\tilde{X}$ be the total space of the line bundle $\mathscr{L}$ with the zero section removed. This is a torsor under the torus $T = \mathbb{G}_m$. Let $\pi : \tilde{X} \to X$ be the natural projection. The twisted product $\tilde{X} \times_T \mathbb{A}^1$ is the line bundle $\mathscr{L}$. Consider the sheaf of vector fields $\mathscr{T}_{\tilde{X}}$ on $\tilde{X}$. Take the sections in $\pi_*\mathscr{T}_{\tilde{X}}$ on X invariant under the translation action of T, i.e., $(\pi_*\mathscr{T}_{\tilde{X}})^T =: \tilde{\mathscr{T}}_X$. It is a Lie algebroid which is a central extension of $\mathscr{T}(X)$ by $\mathfrak{t}$, the lie algebra of T. (A Lie algebroid is a sheaf $\mathscr{L}$ of Lie algebras, with a morphism $a : \mathscr{L} \to \mathscr{T}(X)$ of sheaves of lie algebras, which satisfies the Leibniz rule $[x, fy] = a(x)(f)y - f[x,y]$ for local function f and local sections x and y of $\mathscr{L}$.) The enveloping sheaf of algebras of the Lie algebroid $\tilde{\mathscr{T}}_X$ is denoted by $\tilde{\mathscr{D}}_X$. The enveloping algebra of $\mathfrak{t}$, which is $\mathscr{O}(\mathfrak{t}^*)$, lies in the center of $\tilde{\mathscr{D}}$. For any character of $U(\mathfrak{t})$, say λ, one can consider the central reduction $\mathscr{D}_X^\lambda := \tilde{\mathscr{D}}_X \otimes_{\mathscr{O}_{\mathfrak{t}^*}} \lambda$. Note that this definition differs from [SR] by a $-\rho$.

So far everything has been characteristic free. Now suppose the underlying field k has positive characteristic. For any variety Y of finite type, the relative Frobenius of Y is denoted by $\mathrm{Fr} : Y \to Y^{(1)}$. Just like the sheaf of ordinary differential operators has a large Frobenius center, so does the sheaf of twisted differential operators. We denote the total space of the locally free sheaf $\tilde{\mathscr{T}}_X$ which is $\operatorname{Spec}\operatorname{Sym}\tilde{\mathscr{T}}^*$ by $\tilde{T}^*X$. On $\tilde{X}$, there is a central embedding $\mathscr{O}_{\mathscr{T}^*\tilde{X}^{(1)}} \hookrightarrow \mathscr{D}_{\tilde{X}}$.

Applying $(\pi_* -)^T$ one get a central embedding $\mathscr{O}_{\tilde{\mathscr{T}}^* X^{(1)}} \hookrightarrow \tilde{\mathscr{D}}_X$. Locally, this map sends a section ξ of $\tilde{\mathscr{T}}_X$ to $\xi^p - \xi^{[p]} \in \tilde{\mathscr{D}}_X$, where $\xi^{[p]}$ is the vector field acting on a function f via $\xi \circ \cdots \circ \xi$ p-times. We call $\mathscr{O}_{\tilde{\mathscr{T}}^* X^{(1)}}$ the Frobenius center of $\tilde{\mathscr{D}}$.

Now we get two parts of the center of $\tilde{\mathscr{D}}$, i.e., $U(\mathfrak{t})$ and $\mathscr{O}_{\tilde{\mathscr{T}}^* X^{(1)}}$. How do they intersect? The intersection is $\mathscr{O}_{\mathfrak{t}^{*(1)}}$, embedded in $\mathscr{O}_{\mathfrak{t}^*}$ by the Artin-Schreier map ($\xi \mapsto \xi^p - \xi^{[p]}$) and embedded in $\mathscr{O}_{\tilde{T}^* X^{(1)}}$ in the natural way. The upshot is: $\tilde{\mathscr{D}}_X$ is a coherent sheaf of algebras over $\tilde{T}^* X^{(1)} \times_{\mathfrak{t}^{(1)}} \mathfrak{t}$.

Exercise A.1. About the Artin-Schreier map: We think of $\mathfrak{t}$ as global translation-invariant vector fields on T, then for $x \in \mathfrak{t}$, $x^{[p]}$ is the vector field by which $x \circ \cdots \circ x$ (p-times) acts on functions. In particular, $x^{[p]}$ and x^p act on functions in the same way.

Recall that for a variety X, an Azumaya algebra on X is just a coherent sheaf of algebras on X which locally under the locally finite flat topology (flat topology for short) or equivalently the étale topology is isomorphic to a matrix algebra. This is equivalence to being a locally free coherent sheaf of algebras, whose every geometric fiber is a matrix algebra.

The Azumaya algebras on X, up to Morita equivalence, are classified by the Brauer group $\mathrm{Br}(X) \subseteq H^2(X_{\acute{e}t}, \mathbb{G}_m)$. When R is a strict Henselian ring (only étale covers are disjoint unions of itself), e.g., complete local ring with separably closed residue field, the étale cohomology vanishes.

Proposition A.2.
(1) *As in the case of $\mathscr{D}_X$, the sheaf $\tilde{\mathscr{D}}_X$ is an Azumaya algebra on $\tilde{T}^* X^{(1)} \times_{\mathfrak{t}^{(1)}} \mathfrak{t}$, which splits on the flat cover coming from the base change* $\mathrm{Fr} : X \to X^{(1)}$.
(2) *The sheaf of twisted differential operators $\mathscr{D}_X^\lambda$ is an Azumaya algebra on* $\tilde{T}^* X^{(1)} \times_{\mathfrak{t}^{(1)}} \mathrm{AS}(\lambda)$ *(which will be called the twisted cotangent bundle and denotd by $T^*_{\mathrm{AS}(\lambda)} X^{(1)}$), which splits on the étale cover coming from the base change* $\mathrm{Fr} : X \to X^{(1)}$. *(This flat cover is called the twisted cotangent bundle and is denoted by $T^{*,(1)}_{\mathrm{AS}(\lambda)} X$.)*

The spaces involved are summarized in the following diagram.

$$\begin{tikzcd}
\tilde{T}^*_{\mathrm{AS}(\lambda)} X \times_{X^{(1)}} X \arrow[r] \arrow[d] & \tilde{T}^* X^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^* \times_{X^{(1)}} X \arrow[r] \arrow[d] & \tilde{T}^* X^{(1)} \times_{X^{(1)}} X \arrow[r] \arrow[d] & X \arrow[d, "\mathrm{Fr}"] \\
\tilde{T}^*_{\mathrm{AS}(\lambda)} X^{(1)} \arrow[r] \arrow[d] & \tilde{T}^* X^{(1)} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^* \arrow[r] \arrow[d] & \tilde{T}^* X^{(1)} \arrow[r] \arrow[d] & X^{(1)} \\
\{\lambda\} \arrow[r] & \mathfrak{t}^* \arrow[r, "\mathrm{AS}"'] & \mathfrak{t}^{*(1)} &
\end{tikzcd}$$

Example A.3. When λ is an integral character, i.e., coming from a character of T, then $\mathrm{AS}(\lambda) = 0$. In this case the sheaf $\mathscr{D}_X^\lambda$ becomes the usual differential operators values in a line bundle, and $\mathrm{AS}(\lambda) = 0$ which means the twisted cotangent bundle is the cotangent bundle.

Remark A.4. Let χ be a point in $\operatorname{Spec}\Gamma(X^{(1)}, \mathscr{O}_{\tilde{T}^*X^{(1)}})$, considered as a central character of global sections of $\tilde{\mathscr{D}}_X$ or $\mathscr{D}_X^\lambda$. One can also do central reduction of using the Frobenius center, and get a sheaf of algebras on the fiber of χ in $\tilde{T}^*X$.

Appendix B. The derived localization theorem

B.1. **Quantum comoment map.** Recall $\tilde{\mathscr{B}} = G/N$ has an action by $G\times H$. The quantum comoment map $\mu : U(\mathfrak{g})\otimes U(\mathfrak{h}) \to \mathscr{D}_{\tilde{\mathscr{B}}}$ induces a map $\bar{\mu} : U(\mathfrak{g})\otimes U(\mathfrak{h}) \to \pi_*(\mathscr{D}_{\tilde{\mathscr{B}}})^H \cong \tilde{\mathscr{D}}_{\mathscr{B}}$.

Lemma B.1.
1. *The map $\bar{\mu} : U(\mathfrak{g}) \otimes U(\mathfrak{h}) \to \tilde{\mathscr{D}}_{\mathscr{B}}$ factors through $\tilde{U} := U \otimes_{\mathscr{O}(\mathfrak{h}^*//W)} \mathscr{O}(\mathfrak{h}^*)$.*
2. *We have an isomorphism $U(\mathfrak{h}) \to \Gamma(\mathscr{B}, \mathscr{D})^G$. The embedding $i_{HC} : U^G \to \Gamma(\mathscr{B}, \mathscr{D})^G$ is the Harish-Chandra map with image $U(\mathfrak{h})^{W\bullet}$.*

$$\begin{tikzcd}
U(\mathfrak{h})^W \arrow[r, hook] \arrow[d, "\cong"] & U(\mathfrak{h}) \arrow[r] & \Gamma(\mathscr{B}, \mathscr{D}) \\
U(\mathfrak{g})^G \arrow[r] & U(\mathfrak{g}) \arrow[ur] &
\end{tikzcd}$$

Proof of Lemma B.1. Part (2) clearly follows from part (1). We only need to prove the two statements of part (2).

The first statement is a generalization of Proposition 1.13 in [SR]. The map $\mu_{\mathfrak{g}} : U(\mathfrak{g})\otimes_k \mathscr{O}_{\mathscr{B}} \to \mathscr{D}_{\mathscr{B}}$ is surjective. For any point $\mathfrak{b} \in \mathscr{B}$, the fiber of the map $\mu_{\mathfrak{g}}$ at $\mathfrak{b}$, $U(\mathfrak{g}) \to (\mathscr{D}_{\mathscr{B}})_{\mathfrak{b}}$, has kernel generated by $\mathfrak{n}$. In other words, the fiber of $\mathscr{D}_{\mathscr{B}}$ at the point $\mathfrak{b} \in \mathscr{B}$ is $U(\mathfrak{g})/\mathfrak{n}$. Therefore, we have $\Gamma(\mathscr{B}, \mathscr{D}_{\mathscr{B}})^G \cong (U/\mathfrak{n})^B$. Then we conclude $(U/\mathfrak{n})^B = (U/\mathfrak{n})^H \cong U(\mathfrak{h})$.

So far we have seen that $i_{HC} : U^G \hookrightarrow U \twoheadrightarrow U/\mathfrak{n}$ lands in $(U/\mathfrak{n})^B \cong U(\mathfrak{h}) \subseteq U/\mathfrak{n}$. To show it coincides with the Harish-Chandra homomorphism, we first describe the Harish-Chandra homomorphism as the composition $U^G \hookrightarrow U = (\mathfrak{n}U + U\mathfrak{n}^-)\oplus U(\mathfrak{h}) \twoheadrightarrow U(\mathfrak{t})$. Then we conclude the second claim using the diagram

$$\begin{tikzcd}
 & U \arrow[r, two heads] & U/\mathfrak{n} & \\
U^G \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & \mathfrak{n}U + U(\mathfrak{b}) \arrow[u, hook] \arrow[d, hook] \arrow[r, two heads] & (\mathfrak{n}U + U(\mathfrak{b}))/\mathfrak{n}U \arrow[u, hook] & U(\mathfrak{t}) \arrow[l, "\cong"'] \arrow[ul, hook] \\
 & U \arrow[r, equal] & (\mathfrak{n}U + U\mathfrak{n}^-) \oplus U(\mathfrak{h}) \arrow[ur, two heads] &
\end{tikzcd}.$$

∎

B.2. **Localization functors.** Now we define the localization functors. For an U-module M, define $\mathscr{L}(M) := \tilde{\mathscr{D}} \otimes^{\mathbb{L}}_U M \in D^b(\mathrm{Coh}\,\tilde{\mathscr{D}})$. For $\lambda \in \mathfrak{h}^*$, we start with $M \in D^b(\mathrm{Mod}_\lambda U)$, applying $\mathscr{L}$ to it. The complex $\mathscr{L}(M)$ decomposes according to the central characters of $\mathscr{D}$. All the central characters appearing in $\mathscr{L}(M)$ are in $W\lambda$. We denote this decomposition by $\mathscr{L}(M) = \oplus_{\mu \in W\bullet\lambda} \mathscr{L}^{\lambda\to\mu}(M)$. We define

$$\mathscr{L}^{\hat{\lambda}} := \mathscr{L}^{\lambda\to\lambda} : D^b(\mathrm{Mod}_\lambda U) \to D^b(\mathrm{Coh}_\lambda \tilde{\mathscr{D}}).$$

This localization functor is left adjoint to $R\Gamma_{\tilde{\mathscr{D}},\lambda}$.

Similarly we have the localizations for the other categories

$$\mathscr{L}^{\lambda} : D^b(\mathrm{Mod}\, U^\lambda) \to D^b(\mathrm{Coh}\, \mathscr{D}^\lambda)$$

sending M to $\mathscr{D}^\lambda \otimes^{\mathbb{L}}_{U^\lambda} M$. Note that this functor has bounded homological degree, hence is well-defined. This is because we have the following diagram:

$$\begin{tikzcd} D^-(\mathrm{Mod}\, U^\lambda) \arrow[r, "i"] \arrow[d, "\mathscr{L}^\lambda"'] & D^-(\mathrm{Mod}\, U) \arrow[d, "\mathscr{L}^{\hat{\lambda}}"] \\ D^-(\mathrm{Coh}\, \mathscr{D}^\lambda) \arrow[r, "\iota"] & D^-(\mathrm{Coh}_\lambda \mathscr{D}). \end{tikzcd}$$

The functors i and ι are clearly exact functors and hence send bounded complexes to bounded ones. As we have seen, $\mathscr{L}^{\hat{\lambda}}$ has bounded homological degree, therefore, so does $\mathscr{L}^\lambda$.

Corollary B.2. *For λ regular, the functor $R\Gamma_{\tilde{\mathscr{D}},\lambda} \circ \mathscr{L}^{\hat{\lambda}}$ is isomorphic to* id *on* $D^b(\mathrm{Mod}_\lambda U)$

The following lemma is proved the same way as in [SR].

Lemma B.3. *For regular λ, and for $\mathcal{M}^\bullet \in D^b(\mathrm{Coh}_\lambda \mathscr{D})$ such that $R\Gamma_{\tilde{\mathscr{D}},\lambda}(\mathcal{M}) = 0$, then $\mathcal{M} = 0$.*

This concludes the derived localization theorem.

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