NOTES ON PROCESI BUNDLES AND THE SYMPLECTIC MCKAY EQUIVALENCE

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Contents

1. Introduction

We work over a separably closed field k. Let G be GL_n and $\mathfrak{g}$ its Lie algebra. Let $\mathbb{A}^{2n}$ be endowed with the usual $\mathfrak{S}_n$-action. Recall that for a suitable choice of stability condition θ, we have

$$\begin{tikzcd} T^*(\mathfrak{g}\oplus\mathbb{A}^n)///^\theta G \arrow[r, "\cong"] \arrow[d] & \mathrm{Hilb}^n(\mathbb{A}^2) =: X \arrow[d, "\pi"] & \\ T^*(\mathfrak{g}\oplus\mathbb{A}^n)///G \arrow[r, "\cong"] & V/\mathfrak{S}_n & V \arrow[l, "\eta"'] \end{tikzcd}$$

Note that the map π is equivariant under the $\mathbb{G}_m$-action, where the $\mathbb{G}_m$-action on $\mathbb{A}^2$ is by dilation.

When k is a field of positive characteristic, for simplicity, we denote $\mathbb{A}^{2n(1)}$ simply by V and denote $\mathrm{Hilb}^n(\mathbb{A}^2)^{(1)}$ by X. As has been explained in [Vain14], for any integral weight λ of G, on X there is a quantization $\mathbb{O}_X^\lambda$, coming from quantum Hamiltonian reduction of $\mathcal{D}(\mathfrak{g}\oplus\mathbb{A}^n)$, the sheaf of differential operators on $\mathfrak{g}\oplus\mathbb{A}^n$. For simplicity, we will denote $\mathbb{O}_X^0$ by $\mathbb{O}_X$. Let $\mathcal{W}$ be the Weyl algebra on the vector space $\mathbb{A}^n$. The ring of global sections of $\mathbb{O}_X$ is isomorphic to $\mathcal{W}^{\mathfrak{S}_n}$.

We have already seen in [Vain14] that $\mathbb{O}_X$ is an Azumaya algebra on X. We will show that

$$R\Gamma : D^b(\mathrm{Coh}\,\mathbb{O}_X) \to D^b(\mathrm{Mod}\,\mathcal{W}^{\mathfrak{S}_n})$$

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is an equivalence of derived categories. Let $(V/\mathfrak{S}_n)^\wedge_0$ be the formal neighborhood of 0 in $(V/\mathfrak{S}_n)$. Let $X^\wedge_0$ be $\pi^{-1}((V/\mathfrak{S}_n)^\wedge_0)$. Note that $X^\wedge_0$ is a projective algebraic scheme over $\operatorname{Spec} k[\![V]\!]^{\mathfrak{S}_n}$.

Let $\mathcal{W}^{\mathfrak{S}_n\wedge}$ be the restriction of the Azumaya algebra $\mathcal{W}^{\mathfrak{S}_n}$ to $(V/\mathfrak{S}_n)^\wedge_0$. The above equivalence restricts to an equivalence

$$D^b(\operatorname{Coh} \mathbb{O}|_{X^\wedge_0}) \cong D^b(\operatorname{Mod} \mathcal{W}^{\mathfrak{S}_n\wedge}).$$

When k has large enough characteristic, then $\mathcal{W}^{\mathfrak{S}_n}$ is Morita equivalent to $\mathcal{W}\#\mathfrak{S}_n$. It is almost evident that $\mathcal{W}$ viewed as an Azumaya algebra on $V^{(1)}$ admits an $\mathfrak{S}_n$-equivariant splitting when restricted to the formal neighborhood of the origin. Let $\mathcal{W}^\wedge$ be the restriction of $\mathcal{W}$ to the formal neighborhood. Therefore, there is an equivalence of categories

$$\operatorname{Mod} \mathcal{W}^\wedge\#\mathfrak{S}_n \cong \operatorname{Mod} k[\![V]\!]\#\mathfrak{S}_n.$$

We will also show that $\mathbb{O}_X$ splits on $X^\wedge_0$. The splitting of the Azumaya algebra yields a Morita equivalence

$$\operatorname{Coh} \mathbb{O}|_{X^\wedge_0} \cong \operatorname{Coh} X^\wedge_0.$$

Putting all the equivalences together, we have

$$D^b(\operatorname{Coh} X^\wedge_0) \cong D^b(\operatorname{Coh} \mathbb{O}|_{X^\wedge_0}) \cong D^b(\operatorname{Mod} \mathcal{W}^{\mathfrak{S}_n\wedge}) \cong D^b(\operatorname{Mod} \mathcal{W}^\wedge\#\mathfrak{S}_n) \cong D^b(\operatorname{Coh}_{\mathfrak{S}_n} V^\wedge_0).$$

The objects $\hat{\mathcal{E}} \in D^b(\operatorname{Coh} X^\wedge_0)$ corresponding to $\mathcal{O}_{V^\wedge_0}$ is a vector bundle on $X^\wedge_0$, as will be proved. We will show that this vector bundle $\hat{\mathcal{E}}$ extends to a vector bundle $\mathcal{E}$ on the entire X. The extension $\mathcal{E}$ is called *the Procesi bundle*. Also we will show that $R\operatorname{Hom}_X(\mathcal{E}, -)$ induces an equivalence $D^b(\operatorname{Coh} X) \cong D^b(\operatorname{Mod} \mathcal{O}[V]\#\mathfrak{S}_n)$, which is called *the symplectic McKay equivalence*.

The main goal of this notes it to show the existence of a Procesi bundle following the arguments in [BK04]. The classification has been carried out in [Los13].

2. What is a Procesi bundle

Definition 2.1. A Procesi bundle on X is an $\mathbb{G}_m$-equivariant vector bundle $\mathcal{E}$, satisfying

(1) $\operatorname{End}_X(\mathcal{E}) \cong k[V]\#\mathfrak{S}_n$ as $\mathbb{G}_m$-equivariant $k[V]$-algebras;
(2) $H^i(X, \mathscr{E}\mathrm{nd}_X(\mathcal{E})) = 0$ for all $i > 0$;
(3) $\mathcal{E}^{\mathfrak{S}_n} = \mathcal{O}_X$.

Lemma 2.2. *By (1) every fiber of $\mathcal{E}$ carries an action of $\mathfrak{S}_n$, which is isomorphic to the regular representation.*

Proof. Let V^0 be the open subset of V on which $\mathfrak{S}_n$ acts freely. The map η is a Galois cover when restricted to V^0. By (1) we have a morphism of sheaves $\mathcal{O}(\eta_* V^0)\#\mathfrak{S}_n \to \mathscr{E}\mathrm{nd}_{X^0}(\mathcal{E}|_{X^0})$. This implies $\mathcal{E}|_{X^0}$ is a module over the sheaf of algebras $\eta_* V^0$, which is equivariant with respect to the $\mathfrak{S}_n$-action. Therefore, there is a vector bundle $\bar{\mathcal{E}}$ on V^0, equivariant with respect to $\mathfrak{S}_n$, such that $\mathcal{E}|_{X^0} \cong \eta_*\bar{\mathcal{E}}$. Counting the rank we know that $\bar{\mathcal{E}}$ is a line bundle on V^0. Since

V^0 is obtained from V by removing a codimension-2 locus, the Piccard group of V^0 is trivial. Hence, $\bar{\mathcal{E}} \cong \mathcal{O}_{V^0}$. So, $\mathcal{E}|_{X^0} \cong \eta_*\mathcal{O}_{V^0}$. In particular, on X^0, for any irreducible $\mathfrak{S}_n$-representation N, the component $\mathrm{Hom}_{\mathfrak{S}_n}(N, \mathcal{E})$ has rank the same as the dimension of N. Then by upper-semi-continuity, and the fact that $\mathcal{E}$ is a vector bundle, we know that $\mathrm{Hom}_{\mathfrak{S}_n}(N, \mathcal{E})$ has rank the same as the dimension of N everywhere on X. □

Note that by this Lemma, (3) makes sense, as for any x in X, the fiber of $\mathcal{E}$ at x is isomorphic to $k\mathfrak{S}_n$ as $\mathfrak{S}_n$-modules, hence $\mathcal{E}^{\mathfrak{S}_n}$ is a line bundle.

Let $e = \frac{1}{n!}\sum_{g\in\mathfrak{S}_n} g$. Then the global sections of $\mathcal{E}$ can be calculated as

$$H^0(X, \mathcal{E}) \cong H^0(X, \mathscr{E}\mathrm{nd}_X(\mathcal{E})e) \cong H^0(X, \mathscr{E}\mathrm{nd}_X(\mathcal{E}))e \cong (k[V]\#\mathfrak{S}_n)e \cong k[V].$$

It worth mentioning that the requirement for $\mathcal{E}$ to be equivariant under the $\mathbb{G}_m$-action is automatic, given the vanishing of its self-extension and the structure of its endomorphism ring. Also this $\mathbb{G}_m$-action can be modified such that the induced action on $\mathrm{End}_X(\mathcal{E})$ is compatible with the one on $k[V]\#\mathfrak{S}_n$.

Remark 2.3. Let X^0 be the locus where π is an isomorphism. Conditions (1) and (2) in the definition can be reformulated as

a) $\mathcal{E}|_{X_0} \cong \pi^*\eta_*\mathcal{O}|_{X^0}$;
b) $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) = 0$ for all $i > 0$, and $\mathrm{End}(\mathcal{E}) \cong \mathrm{End}(\mathcal{E}|_{X^0})$ as $\mathbb{G}_m$-equivariant $\Gamma(\mathcal{O}_X)$-algebras.

Proposition 2.4. *Suppose there is a vector bundle $\hat{\mathcal{E}}$ on $X_0^\wedge$ such that*

$$H^i(X_0^\wedge, \mathscr{E}\mathrm{nd}_{X_0^\wedge}(\hat{\mathcal{E}})) = 0$$

for all $i > 0$, and

$$\mathrm{End}_{X_0^\wedge}(\hat{\mathcal{E}}) \cong k[\![V]\!]\#\mathfrak{S}_n$$

as $k[\![V]\!]^{\mathfrak{S}_n}$-algebras. Then this $\hat{\mathcal{E}}$ extends to a Procesi bundle on X.

We need some lemmas to prove this Proposition.

Lemma 2.5. *Let Z be an affine variety over k, carrying a $\mathbb{G}_m$-action which contracts Z to $0 \in Z$. Let $\pi : Y \to Z$ be a projective morphism which is equivariant with respect to $\mathbb{G}_m$. Let $Z_0^\wedge$ be the formal neighborhood of $0 \in Z$, and let $Y_0^\wedge$ be the preimage of $Z_0^\wedge$ in Y. Assume there is a vector bundle $\hat{\mathcal{E}}$ on $Y_0^\wedge$ with* $\mathrm{Ext}^1(\hat{\mathcal{E}}, \hat{\mathcal{E}}) = 0$.

Then, there exist a $\mathbb{G}_m$-equivariant vector bundle $\mathcal{E}$ on Y whose restriction to $Y_0^\wedge$ is isomorphic to $\hat{\mathcal{E}}$.

We need a standard fact the proof of which can be found in, e.g., [BFG06, Proposition 11.1.3]. Let $Y \to Z$ be a proper morphism. Let $Z_0^\wedge$ be the formal neighborhood of $0 \in Z$, and let $Y_0^\wedge$ be the preimage of $Z_0^\wedge$ in Y. Let $\mathcal{E}$ be a vector bundle on $Y_0^\wedge$ such that $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) = 0$. If $Y_0^\wedge$ is endowed with a $\mathbb{G}_m$-action, then this action can be lifted to $\mathcal{E}$. If $\mathcal{E}$ is indecomposable, then the $\mathbb{G}_m$-equivariant structure on $\mathcal{E}$ is unique up to a twisting.

Proof of Lemma 2.5. As Y is projective over Z, there is a graded algebra $B^\bullet$, such that $Z = \operatorname{Spec} B^0$ and $Y = \operatorname{Proj} B^\bullet$. The fact that $\pi : Y \to Z$ is equivariant with respect to the $\mathbb{G}_m$-action implies that $B^\bullet$, as a module over the graded ring B^0 with a non-trivial grading defined by the weights of the $\mathbb{G}_m$-action, is also a graded module.

Define $B^{0\wedge}$ to be the completion of B^0 at the maximal ideal corresponding to $0 \in Z$. Define $B^{\bullet\wedge} := B^\bullet \otimes_{B^0} B^{0\wedge}$, which is a graded algebra over $B^{0\wedge}$. The projective scheme $Y_0^\wedge$ is $\operatorname{Proj} B^{\bullet\wedge}$. The vector bundle $\hat{\mathcal{E}}$ on $Y_0^\wedge$ corresponds to a graded $\mathbb{G}_m$-equivariant locally free module over $B^{\bullet\wedge}$. Taking the $\mathbb{G}_m$-finite vectors, we get a graded locally free module over $B^\bullet$. This module corresponds to a vector bundle $\mathcal{E}$ on $\operatorname{Proj} B^\bullet$ which is Y, whose restriction to $Y_0^\wedge$ is $\hat{\mathcal{E}}$. $\square$

Proof of Proposition 2.4. By assumption we have a vector bundle $\hat{\mathcal{E}}$ on $X_0^\wedge$ with $\operatorname{Ext}^1(\hat{\mathcal{E}}, \hat{\mathcal{E}}) = 0$. By Lemma 2.5, $\hat{\mathcal{E}}$ extends to a $\mathbb{G}_m$-equivariant vector bundle $\mathcal{E}$ on X. We only need to check that $\mathcal{E}$ satisfies (1) and (2) in Definition 2.1.

Lemma B.1 implies that the $\mathbb{G}_m$-action on $\hat{\mathcal{E}}$ can be modified so that $\operatorname{End}_{X_0^\wedge}(\hat{\mathcal{E}}) \cong k[\![V]\!]\#\mathfrak{S}_n$ as $\mathbb{G}_m$-equivariant $k[\![V]\!]^{\mathfrak{S}_n}$-algebras. Not to interrupt the flow of this notes, we put the proof of Lemma B.1 in to an Appendix.

We need to verify that $R^i\pi_* \mathscr{E}\mathrm{nd}(\mathcal{E}) = 0$ for all $i > 0$. As $\mathcal{E}$ is equivariant with respect to $\mathbb{G}_m$, the support of $R^i\pi_* \mathscr{E}\mathrm{nd}(\mathcal{E})$ need to contain a $\mathbb{G}_m$-stable closed point. The only such point is 0. We only need to show that origin is not in the support of $R^i\pi_* \mathscr{E}\mathrm{nd}(\mathcal{E})$. Indeed, as base change to $(V/\mathfrak{S}_n)_0^\wedge$ is flat, we have $R^i\pi_* \mathscr{E}\mathrm{nd}(\mathcal{E})_0^\wedge = 0$.

Again by flatness of the base change, $R^0\pi_* \mathscr{E}\mathrm{nd}(\mathcal{E})_0^\wedge \cong k[\![V]\!]\#\mathfrak{S}_n$. As $\mathcal{E}$ is equivariant with respect to $\mathbb{G}_m$, so is $R^0\pi_* \mathscr{E}\mathrm{nd}(\mathcal{E})$. By taking $\mathbb{G}_m$-finite vectors, we get the desired isomorphism $R^0\pi_* \mathscr{E}\mathrm{nd}(\mathcal{E}) \cong k[V]\#\mathfrak{S}_n$.

$\square$

3. Derived equivalences from exceptional objects

Definition 3.1. An object M in an abelian category is said to be almost exceptional if it is nonzero and $\operatorname{Ext}^i(M, M) = 0$ for all $i > 0$, and the algebra $\operatorname{End}(M)$ has finite homological dimension.

The goal of this section is to show the following proposition.

Proposition 3.2. *Let Y be a smooth irreducible variety, projective over some affine variety. Let $\mathcal{A}$ be an Azumaya algebra on Y. If $\mathcal{E}$ is an almost exceptional object in* $\operatorname{Coh}\mathcal{A}$*, then the functor*

$$R\operatorname{Hom}_{\mathcal{A}}(\mathcal{E}, -) : D^b(\operatorname{Coh}\mathcal{A}) \to D^b(\operatorname{Mod}\operatorname{End}(\mathcal{E}))$$

is an equivalence.

3.1. Exceptional objects on smooth varieties.

Lemma 3.3. *For any exceptional $\mathcal{E}$ in* $\operatorname{Coh}\mathcal{A}$*, the functor $- \otimes^{\mathbb{L}}_{\operatorname{End}_{\mathcal{A}}(\mathcal{E})} \mathcal{E}$ is fully-faithfull, i.e., it identifies $D^b(\operatorname{Mod}\operatorname{End}_{\mathcal{A}}(\mathcal{E}))$ with an admissible subcategory of $D^b(\operatorname{Coh}\mathcal{A})$.*

Note that the assumption $\mathrm{End}(\mathcal{E})$ having finite global dimension makes the functor $-\otimes^{\mathbb{L}}_{\mathrm{End}_{\mathcal{A}}(\mathcal{E})}\mathcal{E}$ well-defined on the level of bounded derived categories, i.e., sends bounded complexes to bounded ones.

Recall that for a triangulated category C, a subcategory I is said to be admissible if any object $\mathcal{F}$ in C fits into an exact triangle

$$\mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to \mathcal{F}_1[1]$$

with $\mathcal{F}_1 \in I$ and $\mathcal{F}_2 \in I^{\perp}$. Any triangulated subcategory $\iota : I \to C$, whose embedding functor ι admits a right adjoint τ is admissible. For any $\mathcal{F}$ in C, define $\mathcal{F}_1 := \tau\iota(\mathcal{F})$, and $\mathcal{F}_1 \to \mathcal{F}$ the adjunction morphism. One can easily check that the cone of $\mathcal{F}_1 \to \mathcal{F}$ lies in $I^{\perp}$ by adjointness.

Proof. The functor $-\otimes^{\mathbb{L}}_{\mathrm{End}_{\mathcal{A}}(\mathcal{E})}\mathcal{E}$ is the right adjoint of $R\,\mathrm{Hom}_{\mathcal{A}}(\mathcal{E},-)$. We would like to show that for any $M \in D^b(\mathrm{Mod}(\mathcal{E}))$, we have $R\,\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, M\otimes^{\mathbb{L}}_{\mathrm{End}(\mathcal{E})}\mathcal{E}) \cong M$. But this follows from the fact that $\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ for all $i \neq 0$. $\square$

3.2. **In the Calabi-Yau setting.** We say a triangulated category D is indecomposable if it can not be written as $D_1 \oplus D_2$ for non-zero D_1 and D_2.

Lemma 3.4. *Let Y be a connected quasi-projective variety over k, and let $\mathcal{A}$ be an Azumaya algebra on Y. Then the category $D^b(\mathrm{Coh}\,\mathcal{A})$ is indecomposable.*

Proof. For any $\mathcal{A}$-module P and any ample line bundle L on Y, the set $\{P\otimes L^k\}$ generates $D^b(\mathrm{Coh}\,\mathcal{A})$. Assume $D^b(\mathrm{Coh}\,\mathcal{A}) \cong D_1 \oplus D_2$, then take P to be an indecomposable direct summand of a free $\mathcal{A}$-module. Take L to be an ample line bundle such that $H^0(Y, L\otimes \mathscr{H}om_{\mathcal{A}}(P,P)) \neq 0$. For any k, the module $P\otimes L^k$ is also indecomposable, hence either in D_1 or in D_2. Assuming the former one. Note that $\mathrm{Hom}(P\otimes L^n, P\otimes L^m) \neq 0$ for all $n \leq m$, therefore for any k the object $P\otimes L^k$ lies in D_2. So, D_1 is zero as for any $M \in D_1$ we have $\mathrm{Ext}^i(P\otimes L^k, M) = 0$ for any i and any k. $\square$

Recall that on any smooth variety Y, proper over $\mathrm{Spec}\,R$, for any Azumaya algebra $\mathcal{A}$ on Y, there is a relative Serre functor $S : D^b(\mathrm{Coh}\,\mathcal{A}) \to D^b(\mathrm{Coh}\,\mathcal{A})$ given by $-\otimes K_Y[\dim Y]$. Let D be the dualizing complex in $D^b(\mathrm{Mod}\,R)$, then the Grothendieck-Serre duality yields, for any complexes of coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on Y,

$$R\,\mathrm{Hom}_R(R\,\mathrm{Hom}_Y(\mathcal{F},\mathcal{G}), D) \cong R\,\mathrm{Hom}_Y(\mathcal{G}, \mathcal{F}\otimes K_Y[\dim Y]).$$

Therefore, for any subcategory I of $D^b(\mathrm{Coh}\,\mathcal{A})$, the relative Serre functor S sends $I^{\perp}$ into ${}^{\perp}I$.

Proposition 3.5. *Let Y be a connected smooth quasi-projective variety over k with trivial canonical bundle, and $\mathcal{A}$ an Azumaya algebra on Y. Let $F : D^b(\mathrm{Coh}\,\mathcal{A}) \to D$ be a non-zero triangulated functor what admits a fully-faithful right adjoint F'. Then F is an equivalence of categories.*

Proof. As the Serre functor intertwines left and right perpendiculars of any subcategory, if $K_Y \cong \mathcal{O}_Y$, then the left and right perpendiculars of any subcategory coincide. In particular, under the assumptions we have $D^b(\operatorname{Coh}\mathcal{A}) = D \oplus D^{\perp}$. But we know $D^b(\operatorname{Coh}\mathcal{A})$ is indecomposable since Y is connected, and $D \neq 0$ therefore $D \perp= 0$. □

Corollary 3.6. *Let $X = \operatorname{Hilb}^n(\mathbb{A}^2)^{(1)}$, and let $\mathbb{O}_X$ be the quantization of X constructed in [Vain14], then there is an equivalence of derived categories*

$$D^b(\operatorname{Coh}\mathbb{O}_X) \cong D^b(\operatorname{Mod}\Gamma(\mathbb{O}_X))$$

The only thing need to show is that $\mathbb{O}$ has no higher cohomology. But this follows from the Grauert-Riemenschneider vanishing argument as in [Vain14].

Corollary 3.7. *Let X be $\operatorname{Hilb}^n(\mathbb{A}^2)$ and let $\mathcal{E}$ be any Procesi bundle on X, then $R\operatorname{Hom}_X(\mathcal{E}, -)$ induces an equivalence*

$$D^b(\operatorname{Coh}X) \cong D^b(\operatorname{Coh}_{\mathfrak{S}_n} V).$$

4. Splitting of the Azumaya algebras

4.1. Existence of Procesi bundles in positive characteristic.

Proposition 4.1. *The Azumaya algebra $\mathbb{O}_X$ splits on $X_0^\wedge$.*

We want show that there is an Azumaya algebra $\mathcal{A}$ on $(V/\mathfrak{S}_n)$ such that $Fr_*\mathbb{O}_X$ is Morita equivalent to $\pi^*\mathcal{A}$. Or equivalently, we want to construct a class $\beta \in H^2_{et}((V/\mathfrak{S}_n), \mathbb{G}_m)_{tor}$ such that $[Fr_*\mathbb{O}_X] = \pi^*\beta$. As is explained in Appendix A, there is a norm map $\eta_* : H^2_{et}(V, \mathbb{G}_m)^{\mathfrak{S}_n}_{tor} \to H^2_{et}((V/\mathfrak{S}_n), \mathbb{G}_m)_{tor}$. Then, the class $\beta \in H^2_{et}((V/\mathfrak{S}_n), \mathbb{G}_m)_{tor}$ can be chosen to be the $\eta_*(\mathcal{W})$, where $\mathcal{W}$ is viewed as a $\mathfrak{S}_n$-invariant class on V.

Let us verify that $\pi^*\beta = [Fr_*\mathbb{O}_X]$. As the restriction to any open subset induces an injection on Brauer groups, it suffices to show that after restricted to X^0, we have $\pi^*\beta = [Fr_*\mathbb{O}_X]$. Let V^0 be the open subset of V on which $\mathfrak{S}_n$ acts freely. In particular, η is Galois on V^0. Therefore we have $\eta_*\mathcal{W}|_{V^0} = \mathcal{W}^{\mathfrak{S}_n}|_{V^0/\mathfrak{S}_n}$. Recall that when restricted to X^0 $Fr_*\mathbb{O}_X$ is also isomorphic to $\pi^*\mathcal{W}^{\mathfrak{S}_n}$. Therefore, we have $\pi^*\beta = [Fr_*\mathbb{O}_X]$. In other words, the Azumaya algebra $Fr_*\mathbb{O}_X$ is the pull-back of an Azumaya algebra on $(V/\mathfrak{S}_n)^{(1)}$.

It follows from the Hensel's Lemma that on the formal neighborhood of the origin, any Azumaya algebra on $(V/\mathfrak{S}_n)$ splits. Our $Fr_*\mathbb{O}_X|_{X_0^\wedge}$ is the pull-back of a splitting Azumaya algebra, hence also splits.

Corollary 4.2. *There is an equivalence of derived categories*

$$D^b(\operatorname{Coh}X_0^\wedge) \cong D^b(\operatorname{Mod}\mathcal{W}^{\mathfrak{S}_n\wedge}).$$

Theorem 4.3. *There exists a Procesi bundle on X.*

Proof. We have

$$D^b(\operatorname{Coh}X_0^\wedge) \cong D^b(\operatorname{Coh}\mathbb{O}|_{X_0^\wedge}) \cong D^b(\operatorname{Mod}\mathcal{W}^{\mathfrak{S}_n\wedge}) \cong D^b(\operatorname{Coh}_{\mathfrak{S}_n}\mathcal{W}^\wedge) \cong D^b(\operatorname{Coh}_{\mathfrak{S}_n}V_0^\wedge).$$

Let $\hat{\mathcal{E}}$ be the complex of coherent sheaves on $X_0^\wedge$ corresponding to $\mathcal{O}_{V_0^\wedge}$. We would like to show that $\hat{\mathcal{E}}$ is actually a vector bundle, with no higher self-extensions, whose endomorphism ring is isomorphic to $k[\![V]\!]\#\mathfrak{S}_n$.

Let $\mathcal{S}$ be the splitting bundle of $\mathcal{O}$ on $X_0^\wedge$, and let S be the equivariant splitting bundle of $\mathcal{W}^\wedge$. Tracing back all the equivalences above, one can see that $\mathcal{S}^\vee$ is send to $S^\vee$. ALso $S^\vee$ is the bimodule induces the equivalence between $\operatorname{Mod}\mathcal{W}^{\mathfrak{S}_n\wedge}$ and $\operatorname{Coh}_{\mathfrak{S}_n} V_0^\wedge$. By standard Morita theory, $S^\vee$ is a projective generator in $\operatorname{Coh}_{\mathfrak{S}_n} V_0^\wedge$. In particular, every indecomposable projective objet is a direct summand of $S^\vee$. Note that $k[\![V]\!]\#\mathfrak{S}_n$ is a direct sum of indecomposable projective objects. Therefore, every indecomposable direct summand is send to a direct summand of $\mathcal{S}^\vee$ via the derived equivalence, which is a vector bundle. So, $k[\![V]\!]\#\mathfrak{S}_n$ itself is send to a vector bundle on $X_0^\wedge$.

The other claims follow from the fact that $R\operatorname{Hom}(\hat{\mathcal{E}},\hat{\mathcal{E}}) \cong k[\![V]\!]\#\mathfrak{S}_n$. □

Corollary 4.4. *There is an equivalence of derived categories*

$$D^b(\operatorname{Coh} X) \cong D^b(\operatorname{Coh}_{\mathfrak{S}_n} V).$$

4.2. **Lifting to characteristic zero.**

Proposition 4.5.
(1) *Let R be a local Noetherian ring with maximal ideal m and residue field k. Let X_R be a variety smooth over R, whose fiber over k is denoted by X_k. Then every almost exceptional object $\mathcal{E}$ on X_k extends uniquely to an almost exceptional object $\hat{\mathcal{E}}$ on $\mathfrak{X}$, the formal completion of X_R at X_k.*
(2) *If moreover R is complete with respect to the m-adic topology, and X_R is equipped with a positive weight $\mathbb{G}_m$-action such that $\mathcal{E}$ is equivariant, then $\mathcal{E}$ extends uniquely to a $\mathbb{G}_m$-equivariant vector bundle $\mathcal{E}_R$ on X_R.*

Proof of (1). Denote $R/(m^n)$ by R_n, and denote the base change of X_R to R_n by X_n. Iteratively, we need to extend the vector bundle $\mathcal{E}_n$ on X_n to a vector bundle $\mathcal{E}_{n+1}$ on X_{n+1}. By standard deformation theory, the obstruction of the extension lies in $\operatorname{Ext}^2(\mathcal{E},\mathcal{E})\otimes(m^n/m^{n+1})$, and the set of equivalence classes of extensions is a torsor under $\operatorname{Ext}^1(\mathcal{E},\mathcal{E})\otimes(m^n/m^{n+1})$. By assumption, both groups are trivial. Therefore, the extension in each step exists and is unique.

As $\operatorname{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ for $i > 0$, we also have $\operatorname{Ext}^i(\hat{\mathcal{E}},\hat{\mathcal{E}}) = 0$ for $i > 0$. The fact that $\operatorname{End}(\hat{\mathcal{E}})$ has finite homological dimension can be checked by calculating the Ext groups using the spectral sequence from the m-adic filtration. □

We need to recall the formal function theorem of Grothendieck. Let $p : X \to Y$ be a proper morphism. Let $Y_y^\wedge$ be the formal neighborhood of $y \in Y$, and let $X_y^\wedge$ be the preimage of $Y_y^\wedge$ in X. Let $\mathcal{E}$ be a vector bundle on $X_y^\wedge$. Then $(R^i p_*\mathcal{E})_y^\wedge \cong R^i p|_{X_y^\wedge *}(\mathcal{E}|_{X_y^\wedge})$.

Proof of (2). [1] Denote $R/(m^n)$ by R_n. Note that both X_n and $\mathcal{E}_n$ are flat over R_n. As a projective variety over an affine R_n-variety, X_n can be written as $\operatorname{Proj}\mathcal{B}_n^\bullet$ for

[1] The proof will be skipped in the talk

some graded ring $\mathcal{B}_n^\bullet$. The vector bundle $\mathcal{E}_n$ corresponds to a graded module $\mathcal{M}_n^\bullet$ over $\mathcal{B}_n^\bullet$, flat a module over R_n. The system $\mathcal{M}_n^\bullet$ for $n \geq 0$ satisfies the property that $\mathcal{M}_n^\bullet/(m^{n-1}) \cong \mathcal{M}_{n-1}^\bullet$. The fact that X_n carries a contracting action of $\mathbb{G}_m(R_n)$, with respect to which $\mathcal{E}_n$ is equivariant, decomposes $\mathcal{B}_n^\bullet$ into weight spaces $\mathcal{B}_n^\bullet \cong \oplus_{i=0}^\infty \mathcal{B}_n^\bullet[i]$, and $\mathcal{M}_n^\bullet \cong \oplus \mathcal{M}_n^\bullet[i]$. This weight space decomposition is compatible with respect to the grading. Each $\mathcal{M}_n^k[i]$ and $\mathcal{B}_n^k[i]$ are finitely generated flat (hence free) modules over R_n.

The formal scheme $\mathcal{X}_R$ is $\varinjlim \operatorname{Proj} \mathcal{B}_n^\bullet$. Please note that $\varprojlim \mathcal{B}_n^\bullet$ is different from $\oplus_i \varprojlim \mathcal{B}_n^\bullet[i]$; but the space of finite vectors with respect to $\varprojlim \mathbb{G}_m(R_n)$ in the former is the later. Denote $\oplus_i \varprojlim \mathcal{B}_n^\bullet[i]$ by $\mathcal{B}^\bullet$. We have $\operatorname{Proj} \mathcal{B}^\bullet \cong X_R$. Similarly, define $\mathcal{M}^\bullet := \oplus_i \varprojlim \mathcal{M}_n^\bullet[i]$. We want to show that $\mathcal{M}^\bullet$ is a finitely generated graded $\mathcal{B}^\bullet$-module, and corresponds to a vector bundle $\mathcal{E}_R$ on X_R with the property that $\mathcal{E}_R|_{\mathcal{X}_R} \cong \hat{\mathcal{E}}$ (or equivalently $\mathcal{E}_R|_{X_n} \cong \mathcal{E}_n$). Let $\{b_s^n \mid s \in I_B\}$ be a finite set of homogeneous generators of $\mathcal{B}_n^\bullet$ as R_n-algebra, and let $\{f_h^n \mid h \in I_M\}$ be a finite set of homogeneous generators of $\mathcal{M}_n^\bullet$ as $\mathcal{B}_n^\bullet$-module. In particular, monomials in b_s^n and f_h^n of degree i generates $\mathcal{M}_n^\bullet[i]$ as R_n-module. Take any homogeneous lifting of b_s^n in $\mathcal{B}_{n+1}^\bullet$, called b_s^{n+1}, and homogeneous liftings of f_h^n called f_h^{n+1}. Then the Nakayama's Lemma implies that monomials of b_s^n and f_h^{n+1} of degree i generates $\mathcal{M}_{n+1}^\bullet$ as $\mathcal{B}_{n+1}^\bullet$-module. Iterate this process we get elements $b_s \in \mathcal{B}^\bullet$ and $f_h \in \mathcal{M}^\bullet$ whose monomials of degree i generates $\mathcal{M}^\bullet[i]$. This shows the finite-generation of $\mathcal{M}^\bullet$. By construction each $\mathcal{M}^k[i]$ is free as module over R, therefore $\mathcal{E}$ is flat over R.

Such an $\mathcal{E}$ is automatically locally free. Being locally free is equivalent to being projective on every members of an affine cover of X_R. Let U_R be an affine subset, flat over R. We want to show that $\operatorname{Ext}^i_{U_R}(\mathcal{E}|_{U_R}, M) = 0$ for any coherent sheaf M on U_R and any $i > 0$. For this it suffices to show that $\operatorname{Ext}^i_{U_R}(\mathcal{E}|_{U_R}, M)/m = 0$. By the formal function theorem, $\operatorname{Ext}^i_{U_R}(\mathcal{E}|_{U_R}, M)^\wedge_0 \cong \varprojlim \operatorname{Ext}^i_{U_n}(\mathcal{E}|_{U_n}, M|_{U_n})$. But $\mathcal{E}|_{U_n}$ is isomorphic to $\mathcal{E}_n$ restricted to an affine open set, hence is projective. Each $\operatorname{Ext}^i_{U_n}(\mathcal{E}|_{U_n}, M|_{U_n})$ vanishes. □

Remark 4.6. Note that by the formal function theorem, $H^i(X_R, \mathscr{E}\mathrm{nd}(\mathcal{E}))^\wedge_m \cong \varprojlim H^i(X_n, \mathscr{E}\mathrm{nd}(\mathcal{E}_n))$.

Let R be a finitely generated $\mathbb{Z}$-algebra on which X is well-defined, whose field of fractions is denoted by K. Let k be a geometric point in $\operatorname{Spec} R$ with large enough characteristic, and $\tilde{R}$ the formal completion of R at k, whose field of fractions is denoted by $\tilde{K}$. On X_k there is a Procesi bundle by Theorem 4.3. Then the Proposition implies that $\mathcal{E}_{\tilde{K}}$ exists and satisfies the axioms of a Procesi bundle. Let R' be a finitely generated K-subalgebra in $\tilde{K}$, so that $\mathcal{E}_{R'}$ is well-defined and there is an isomorphism $\operatorname{End}_{X_{R'}}(\mathcal{E}_{R'}) \cong R'[V]\#\mathfrak{S}_n$ as $\mathbb{G}_m$-equivariant $R'[V]^{\mathfrak{S}_n}$-algebras. Then the field of fractions K' of R' is a field of characteristic zero on which a Procesi bundle exists.

Appendix A. The Brauer groups

Let us recall some facts about Brauer groups. For any scheme Y, the set of Azumaya algebras, up to Morita equivalence, is called the Brauer group of this scheme, denoted by $Br(Y)$. The group structure is given by tensor of Azumaya algebras. Clearly it is an abelian group with identity given by the structure sheaf.

We list some basic properties without proof that are used in the body of the notes. All of them can be found in [M80].

Proposition A.1. (1) *For any Zarisky open subset U of Y, restricting to U induces an injection on Brauer groups $Br(Y) \hookrightarrow Br(U)$.*

(2) *For any affine scheme Y, let $H^2_{et}(Y, \mathbb{G}_m)_{tor}$ be the torsion subgroup of the second étale cohomology. We have an isomorphism $Br(Y) \cong H^2_{et}(Y, \mathbb{G}_m)_{tor}$.*

(3) *Let B be a complete local ring with separably closed residue field, and let B' be a finite étale (flat unramified) extension of B. Then $B' = \prod^n B$ for some n. In particular, any connected finite étale map $Y \to \operatorname{Spec} B$ is an isomorphism, hence any étale cohomology of $\operatorname{Spec} B$ vanishes.*

(4) *Let B be a complete local ring with separably closed residue field, and let B' be a finite extension of B, then any étale cohomology of $\operatorname{Spec} B'$ vanishes.*

For any finite group Γ acting on an affine variety S over a separably closed field, the norm morphism $\eta_* : H^i(S, \mathbb{G}_m) \to H^i(S/\Gamma, \mathbb{G}_m)$ is well-defined. In general for any geometric point $\bar{x}$ in S/Γ, the stalk

$$(R^i \eta_* \mathbb{G}_m)_{\bar{x}} \cong H^i(S \times_{S/\Gamma} \operatorname{Spec} \mathcal{O}_{S/\Gamma, \bar{x}}, \mathbb{G}_m).$$

The later vanishes since $S \times_{S/\Gamma} \operatorname{Spec} \mathcal{O}_{S/\Gamma, \bar{x}}$, as a finite extension of a complete local ring (over a separably closed field), has no étale cohomology. Therefore, $R^i \eta_* \mathbb{G}_m = 0$ for $i > 0$. Then, $H^2_{et}(S/\Gamma, \eta_* \mathbb{G}_m) \cong H^2_{et}(S, \mathbb{G}_m)$ by the Leray spectral sequence. Therefore, the norm map $\eta_* : \mathbb{G}_{mS} \to \mathbb{G}_{mS/\Gamma}$ induces a morphism on the level of étale cohomology.

Appendix B. Uniqueness of the $\mathbb{G}_m$-equivariant structure of the Procesi bundle

For any irreducible representation N of $\mathfrak{S}_n$, the bundle $\hat{\mathcal{E}}^N := \operatorname{Hom}_{\mathfrak{S}_n}(N, \hat{\mathcal{E}})$ is indecomposable. Indeed, let $e_N \in k\mathfrak{S}_n$ be the indecomposable idempotent corresponding to N. Then $\operatorname{End}(\hat{\mathcal{E}}^N) \cong e_N(k[\![V]\!] \# \mathfrak{S}_n) e_N$, Were $\hat{\mathcal{E}}^N$ decomposable, the algebra $e_N(k[\![V]\!] \# \mathfrak{S}_n) e_N$ would have more than one non-trivial idempotents. This is a contradiction. So $\hat{\mathcal{E}}^N$ is indecomposable.

Lemma B.1. *Up to a twist, $H^0(X_0^\wedge, \hat{\mathcal{E}})$ and $k[\![V]\!]$ are isomorphic as $\mathbb{G}_m$-equivariant $k[\![V]\!]^{\mathfrak{S}_n}$-modules.*

Proof. For any irreducible $\mathfrak{S}_n$-representation N, let $k[\![V]\!]^N := \operatorname{Hom}_{\mathfrak{S}_n}(N, k[\![V]\!])$. We only need to show that $k[\![V]\!]^N$ and $\Gamma(\hat{\mathcal{E}}^N)$ are isomorphic as $\mathbb{G}_m$-equivariant $k[\![V]\!]^{\mathfrak{S}_n}$-modules. In turn, it suffices to show that the $\mathbb{G}_m$-equivariant structure on $k[\![V]\!]^N$ is unique.

Let $\hat{V}^0$ be $V^0 \cap V_0^\wedge$, where V^0 is the open subscheme on which $\mathfrak{S}_n$-acts freely. Note that $k[\![V]\!]^N|_{\hat{V}^0/\mathfrak{S}_n}$ is a vector bundle. We only need to show that this bundle is indecomposable and admits a unique $\mathbb{G}_m$-equivariant structure up to twisting. This will follow if we show $\eta^*(k[\![V]\!]^N|_{\hat{V}^0/\mathfrak{S}_n})$ is indecomposable and admits a unique $\mathbb{G}_m$-equivariant structure up to twisting. But

$$\eta^*(k[\![V]\!]^N|_{\hat{V}^0/\mathfrak{S}_n}) \cong k[\![V]\!]^N \otimes_{\hat{V}^0/\mathfrak{S}_n} \mathcal{O}_{\hat{V}_0} \cong \mathcal{O}_{\hat{V}_0} \otimes_k N.$$

The global section of $\mathcal{O}_{\hat{V}_0} \otimes_k N$ is $k[\![V]\!] \otimes N$, which is clearly indecomposable as $k[\![V]\!]\#\mathfrak{S}_n$-module and admits a unique $\mathbb{G}_m$-equivariant structure up to a grading. As its restriction to an open subset, $\mathcal{O}_{\hat{V}_0} \otimes_k N$ also admits a unique $\mathbb{G}_m$-equivariant structure. Therefore, so does $k[\![V]\!]^N$. □

This Lemma implies that we can endow $\hat{\mathcal{E}}$ with suitable $\mathbb{G}_m$-equivariant structure, so that the isomorphism $\Gamma(\hat{\mathcal{E}}) \cong k[\![V]\!]$ is $\mathbb{G}_m$-equivariant. Then using the fact that the natural morphism $k[\![V]\!]\#\mathfrak{S}_n \to \mathrm{End}_{k[\![V]\!]^{\mathfrak{S}_n}}(k[\![V]\!])$ is a $\mathbb{G}_m$-equivariant isomorphism, we conclude that $k[\![V]\!]\#\mathfrak{S}_n$ and $\mathrm{End}(\hat{\mathcal{E}})$ are isomorphic as $\mathbb{G}_m$-equivariant algebras.

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