Problem set (#1 on Lec 1, #2 on Lec 3, #3 on Lec 4)

Problem 1: We want a matrix interpretation of quadratic forms. Let $q(x_1,\dots x_n)=\sum_{i\leq j} q_{ij}x_ix_j$ with $q_{ij}\in\mathbb{F}$. Define its matrix, $B=(b_{ij})_{i,j=1}^{n}$ by

$$b_{ij}=\begin{cases} q_{ii} & \text{if } i=j\\ \frac{1}{2}q_{ij} & \text{if } i<j\\ \frac{1}{2}q_{ji} & \text{if } i>j\end{cases}$$

Note that, by the construction, B coincides with its transpose, B^T, such matrices are called symmetric. Let $\vec{x}$ denote the row vector $(x_1,\dots x_n)$ so that $\vec{x}^T$ is the column vector $\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$.

1) Check that $q(\vec{x})=\vec{x}B\vec{x}^T\ \forall\vec{x}$

2) Show that 1) recovers B uniquely (assuming it's symmetric).

3) Let q,q' be quadratic forms & B,B' be their matrices. Show that $q\sim q'\Leftrightarrow\exists$ invertible matrix $A\,|\,B'=ABA^T$

4) Let $q=[q_1,\dots q_n]$ & $q'=[q_1',\dots,q_n']$ be two equivalent diagonal forms. Show that

a) The number of 0's among $q_1,\dots,q_n$ is equal to the number of 0's among $q_1',\dots,q_n'$.

6) $\exists\, x\in \mathbb{F}\setminus\{0\} \mid \prod_{i=1}^n q_i' = x^2 \prod_{i=1}^n q_i$.

Problem 2: This problem is about classifying elements of $\mathbb{Q}_2\setminus\{0\}$ up to multiplication by a complete square.

1) For $\alpha,\beta\in\mathbb{Q}_p\setminus\{0\}$, we have α/β is a complete square $\Leftrightarrow$ $\mathrm{ord}(\alpha)\equiv \mathrm{ord}(\beta) \bmod 2$ & $p^{-\mathrm{ord}(\alpha)}\alpha \equiv p^{-\mathrm{ord}(\beta)}\beta \bmod 8$.

2) $\forall\, \beta\in\mathbb{Q}_2\setminus\{0\}$ there's unique pair $(i,j)\in\{-3,-1,1,3\}\times\{0,1\}$ such that $\beta/(i2^j)$ is a complete square.

Problem 3*: This problem proves that every quadratic form over $\mathbb{Q}_2$ in $n\geqslant 5$ variables represents 0. As in Lec 4, it's enough to assume that q is diagonal, $q=[q_1,\dots q_n]$ (and that $n=5$). Also it's enough to assume $q_i\neq 0$ $\forall i$.

1) Show that if q_i/q_j is not a complete square for $i\neq j$, then $[q_1,\dots,q_5]$ represents 0 (hint: look at Corollary 2(2)).

2) Show that $[1,1]\sim[2,2]\sim[-3,-3]\sim[-6,-6]$ (hint: use 4) of Problem 1).

3) Show that $[1,1,2,-3,-6]$ represents 0.

4) Show that $[1,1,1,1]$ represents every nonzero element in $\mathbb{Q}_2$.

5) Deduce that q represents 0.