Lecture 15, 03/03/25 (Rem in Sec 2.2 added 3/17)

1) Kempf-Ness theorem

2) Applications.

Refs: [PV], Sec. 6.12; [CdS], Part VIII; [L].

## 1) Kempf-Ness theorem

### 1.1) Statement

Let $G$ be a reductive group & $V$ be its finite dimensional rational representation. Let $K\subset G$ be a maximal compact subgroup of $G$, it's defined up to $G$-conjugation. There is a $K$-invariant Hermitian scalar product $(\cdot,\cdot)$ on $V$. Note that $(xv,v)\in\sqrt{-1}\mathbb{R}$ $\forall x\in\mathfrak{k}$ & $v\in V$.

Define a map $\mu: V\longrightarrow \mathfrak{k}^*$ by $\langle\mu(v),x\rangle=\sqrt{-1}(xv,v)$

Exercise: $\mu$ is $K$-equivariant.

Remark: Here's how one computes $\langle\mu(v),x\rangle, x\in\mathfrak{k}$. The element $x$ is skew-Hermitian so $V=\bigoplus_{\alpha\in\mathbb{R}} V_\alpha$, where $x$ acts on $V_\alpha$ by $\sqrt{-1}\alpha$.

Write $v=\sum_\alpha v_\alpha$ w. $v_\alpha\in V_\alpha$. Then

$$\langle\mu(v),x\rangle=\sqrt{-1}(xv,v)=[(V_\alpha,V_\beta)=0 \text{ for } \alpha\neq\beta]=-\sum_\alpha \alpha(v_\alpha,v_\alpha)$$

Here's the main result for this lecture.

**Theorem** (Kempf-Ness) Let $v\in V$.

1) $Gv\cap\mu^{-1}(0)\neq\emptyset \Leftrightarrow Gv$ is closed.
2) If $Gv\cap\mu^{-1}(0)\neq\emptyset$, then this intersection is a single $K$-orbit.

## 1.2) Preparation

We need some of the ingredients from Lec 11:

I) Cartan decomposition: we have a maximal torus $T\subset G$ w. $T_K:=K\cap T$ being maximal compact in $T$. Then we have $G=KTK$. Moreover, $T=T_K\times\exp(\mathfrak{h}_{\mathbb{R}})$ (where $\mathfrak{h}=Lie(T)$, $\mathfrak{h}_{\mathbb{R}}=\mathbb{R}\otimes_{\mathbb{Z}}\mathfrak{X}_*(T)$; under identification $T\simeq(\mathbb{C}^\times)^n$, we have $\exp(\mathfrak{h}_{\mathbb{R}})=(\mathbb{R}_{>0})^n$). Then we have

$$G=K\exp(\mathfrak{h}_{\mathbb{R}})K.$$

II) The following was established in proving the Hilbert-Mumford thm: if $u,v\in V$ are s.t. $Gu\subset\overline{Gv}$, then $\exists\, k\in K$ s.t. $\overline{Tkv}\cap Gu\neq\emptyset$ (Sec. 1.2.2 of Lec 11). Moreover, by Sec 1.1.4 in Lec 11, $\exists\,\gamma\in\mathfrak{X}_*(T)$ s.t. $\lim_{t\to 0}\gamma(t)kv$ exists and lies in $Gu$.

## 1.3) Proof of Thm.

Step 1: $Gv$ is closed $\Rightarrow Gv\cap\mu^{-1}(0)\neq\emptyset$: Since $Gv$ is closed $\exists$ point $v_0\in Gv$ with minimal $(v_0,v_0)$. Take $\xi\in\mathfrak{k}$ and consider

$$v_t=\exp(t\sqrt{-1}\xi)v_0,\ t\in\mathbb{R}$$

Note that $\frac{d}{dt}(v_t,v_t)\big|_{t=0}=(\sqrt{-1}\xi v_0,v_0)+(v_0,\sqrt{-1}\xi v_0)=[\sqrt{-1}\xi$

acts by Hermitian operator$] = 2\sqrt{-1}(\xi v_0, v_0)$. But $t=0$ is a point of minimum of $t \mapsto (v_t, v_t)$. Hence $\sqrt{-1}(\xi v_0, v_0) = \mu(v_0) = 0$.

Step 2: $Gv \cap \mu^{-1}(0) \neq \emptyset \Rightarrow Gv$ is closed: Can assume $\mu(v)=0$. By II) in Sec 1.2, can replace $v$ w. $kv$ (this doesn't violate $\mu(v)=0$ by Exercise in Sec. 1.1) and assume $\lim_{t\to 0} \gamma(t)v \in Gu$ (for the unique closed $Gu \subset \overline{Gv}$).

Let $x = \sqrt{-1}d_1\gamma(1) \in \sqrt{-1}\mathfrak{h}_{\mathbb{R}}^{k} = Lie(T_k)$. We have decomposition

$$V = \bigoplus_{n\in\mathbb{Z}} V_n, \quad V_n = \{v \in V \mid \gamma(t)v = t^n v \Leftrightarrow xv = \sqrt{-1}nv\}$$

Then existence of $\lim_{t\to 0}\gamma(t)v$ is equivalent to $v_i = 0$ for $i<0$. On the other hand, by Remark in Sec 1.1,

$$\langle \mu(v), x\rangle = -\sum_{n\in\mathbb{Z}} n(v_n, v_n).$$

Combining $\langle\mu(v), x\rangle = 0$ w. $v_i = 0\ \forall i<0$, we arrive at $v = v_0 \Rightarrow \lim_{t\to 0}\gamma(t)v = v \Rightarrow Gu = Gv \Rightarrow Gv$ is closed.

Step 3: We prove 2). It remains to prove that if $v\in\mu^{-1}(0)$ & $g\in G$ are s.t. $\mu(gv)=0$, then $gv\in Kv$. Write $g$ as $k_1\exp(x)k_2$ for $k_i\in K$, $x\in\mathfrak{h}_{\mathbb{R}}$, see I) in Sec. 1.2. Replacing $v$ w. $k_2v$ we can assume $\mu(\exp(x)v)=0$. Write $v$ as $\sum_{a\in\mathbb{R}} v_a$ w. $x.v_a = av_a \Rightarrow \exp(x)v_a = e^a v_a$. Then

$$\mu(v) = 0 \Leftrightarrow \sum a(v_a, v_a) = 0 \Leftrightarrow$$

$$(1) \qquad \sum_{a>0} a(v_a, v_a) = \sum_{b<0}(-b)(v_b, v_b)$$

$$\mu(\exp(x)v) = 0 \Leftrightarrow \sum a e^{2a}(v_a, v_a) = 0 \Leftrightarrow$$

(2) $$\sum_{a>0} a\underbrace{e^{2a}}_{>1}(v_a,v_a) = \sum_{b<0}(-b)\underbrace{e^{2b}}_{<1}(v_b,v_b)$$

It follows that l.h.s of (2) $\geqslant$ l.h.s. of (1) w. equality iff $(v_a,v_a)=0$ $\forall a>0$, while r.h.s. of (2) $\leq$ r.h.s. of (1) w. equality iff $(v_b,v_b)=0$ $\forall b<0$. It follows that $\mu(\exp(x)v)=0 \Rightarrow \exp(x)v=v$ finishing the proof. $\square$

## 2) Applications

Our main goal here is to prove the following result already mentioned in Lec 14, due to Matsushima & Onishchik:

**Thm**: Let $G$ be a reductive group & $H\subset G$ an algebraic subgroup. If $G/H$ is affine, then $H$ is reductive.

The proof is based on the Kempf-Ness theorem as well as understanding the nature of the map $\mu$: it's a moment map.

### 2.1) Moment maps.

Let $M$ be a $C^\infty$-manifold. Recall that by a symplectic form on $M$ one means a closed & non-degenerate 2-form $\omega$. A basic example is a non-degenerate skew-symmetric $\mathbb{R}$-bilinear form on a real vector space. In particular, if $V$ is a $\mathbb{C}$-vector space w.

Hermitian scalar product $(\cdot;\cdot)$, then $\omega = -2\cdot \mathrm{Im}(\cdot;\cdot)$ is a symplectic form (on the real vector space $V$).

Now suppose $K$ is a Lie group acting on $M$ preserving $\omega$. The action induces a Lie algebra homomorphism $\mathfrak{k} \longrightarrow \mathrm{Vect}(M)$, $x \mapsto x_M$. Note that $x_M$ is a symplectic vector field $\Leftrightarrow$ the contraction $\iota_{x_M}\omega$ is a closed 1-form. If it's exact, then $\exists\, H_x \in C^\infty(M)$ w. $\iota_{x_M}\omega = dH_x$. We want to be able to choose $H_x$'s in a nice way:

Definition: By a Hamiltonian $K$-action on $M$ we mean an action as above together w. a $K$-equivariant linear map (comoment map) $x \mapsto H_x : \mathfrak{k} \longrightarrow C^\infty(M)$ s.t.

(3): $\iota_{x_M}\omega = dH_x \quad \forall\, x \in \mathfrak{k}$

We define the moment map $\mu : M \to \mathfrak{k}^*$ by
$$\langle \mu(m), x \rangle := H_x(m).$$
It is $K$-equivariant.

Example: Let $V$ be a vector space and $\omega \in \Lambda^2 V^*$, non-degenerate. Then we claim that we can take $H_x(v) = \frac{1}{2}\omega(xv, v)$. Indeed, the map is $K$-equivariant: $[kH_x](v) = \frac{1}{2}\omega(xk^{-1}v, k^{-1}v) = \frac{1}{2}\omega(\mathrm{Ad}(k)x\,v, v) = H_{\mathrm{Ad}(k)x}(v)$ & $[d_v H_x](u) = \frac{1}{2}(\omega(xv,u) + \omega(v, xu)) = [x \text{ is symplectic}]$

$= \omega(xv,u) = [\iota_{x_V}\omega](u)$, which checks (3). In particular, for $\omega = -2\cdot \mathrm{Im}(\cdot,\cdot)$ we recover $\mu(v): x \mapsto \sqrt{-1}\,(xv,v)$

In our proof of Theorem we will need the following general property of a moment map.

Lemma: Let $\mu$ be a moment map for a Hamiltonian action of $K$ on $(M,\omega)$. Then $\forall\ x,y\in\mathfrak{k}$ & $m\in M$ we have

$$\omega_m(x_{M,m}, y_{M,m}) = \langle \mu(m), [x,y]\rangle.$$

Proof:

$$\omega_m(x_{M,m}, y_{M,m}) = [\iota_{x_M}\omega]_m(y_{M,m}) = [(3)] = d_m H_x(y_{M,m}) =$$

$\langle d_m\mu(y_{M,m}), x\rangle =$ [$\mu$ is $K$-equivariant, hence $\mathfrak{k}$-equivariant: intertwines $y_M$ w. $y_{\mathfrak{k}^*}$] $= \langle \underbrace{y_{\mathfrak{k}^*,\mu(m)}}_{\in\mathfrak{k}^*}, x\rangle =$ [def'n of coadjoint action] $= -\langle\mu(m), [y,x]\rangle$. $\square$

## 2.2) Proof of Theorem.

Since $G/H$ is affine, it embeds as a closed $G$-stable subvariety into some $G$-rep'n $V$. Let $v$ be the image of $H$. Take $(\cdot,\cdot)$, $\mu$ as in Sec. 1. We can replace $v$ w. a $G$-conjugate & achieve $\mu(v)=0$, thx to Kempf-Ness thm.

Since $\mathfrak{g} = \mathbb{C}\otimes_{\mathbb{R}}\mathfrak{k}$, we have $\mathbb{C}\otimes_{\mathbb{R}}(\mathfrak{k}\cap\mathfrak{h}) \hookrightarrow \mathfrak{h} \Rightarrow$

(4) $\dim_{\mathbb{R}}(\mathfrak{k}\cap\mathfrak{h}) \leq \dim_{\mathbb{C}}\mathfrak{h}$.

Note that $H$ is reductive $\Leftrightarrow H^\circ$ is. To prove that $H^\circ$ is reductive, it suffices to show that $H^\circ \cap K$ is Zariski dense in $H^\circ$, which would follow if we show (4) is an equality, cf. Lemma in Sec. 1.3 of Lec 2. So it remains to show

(5) $\dim_{\mathbb{R}}(\mathfrak{k}\cap\mathfrak{h}) \geqslant \dim_{\mathbb{C}}\mathfrak{h} \Leftrightarrow \dim_{\mathbb{R}}\mathfrak{k}.v \leqslant \frac{1}{2}\dim_{\mathbb{R}}\mathfrak{g}.v$

Note that:

(i) $\mathfrak{g}.v$ is a complex subspace, hence $\omega = -2\,\mathrm{Im}(\cdot,\cdot)$ is non-degenerate on $\mathfrak{g}.v$.

(ii) By Lemma in Sec. 2.1, $\omega_v(x.v, y.v) = \langle \mu(v), [x,y]\rangle = 0$ $\forall$ $x,y\in\mathfrak{k} \Rightarrow \mathfrak{k}.v$ is an isotropic subspace of $\mathfrak{g}.v$. This yields the equivalent formulation of (5) & finishes the proof. $\square$

Remark (added 3/17) The argument of the proof implies that for $v\in\mu^{-1}(0)$, $\mathrm{Lie}(\mathrm{Stab}_K(v))$ is a real form of $\mathrm{Lie}(\mathrm{Stab}_G(v))$. In particular, if $\mathrm{Stab}_K(v)$ is finite, then so is $\mathrm{Stab}_G(v)$.

### 2.3) Luna's closedness criterion.

Thm (Luna): Let $G$ be a reductive group acting on an affine variety $X$. Let $H\subset G$ be reductive & $x\in X^H$. If $N_G(H)x$ is closed, then so is $Gx$.

Proof: $\exists$ $G$-equivariant closed embedding of $X$ into a rational representation $V$ so we can assume $X=V$. We can choose maximal compact subgroups $K_H \subset K_N \subset K$ in $H \subset N := N_G(H) \subset G$. Choose a $K$-invariant Hermitian scalar product $(\cdot,\cdot)$ on $V$ & form the moment maps $\mu_N, \mu$ for $K_N, K$. They are related as follows: if $p: \mathfrak{k}^* \twoheadrightarrow \mathfrak{k}_N^*$ is the restriction map, then $\mu_N = p \circ \mu$. Note that $x$ is fixed by $K_H$, hence $\mu(x) \in (\mathfrak{k}^*)^{K_H}$. Also

$$Z_K(K_H) \subset K_N \Rightarrow \mathfrak{k}^{K_H} \subset \mathfrak{k}_N \Rightarrow [(\mathfrak{k}/\mathfrak{k}_N)^*]^{K_H} = 0 \Rightarrow p|_{(\mathfrak{k}^*)^{K_H}} \text{ is injective.}$$

Since $\mu(x) \in (\mathfrak{k}^*)^{K_H}$, we have $\mu_N(x) = 0 \Rightarrow \mu(x) = 0$.

Now we are done by the Kempf-Ness theorem $\square$

Corollary: If $N_G(H)/H$ is finite (e.g. $H$ is a maximal torus), then $Gx$ is closed.

Remark: In fact, the opposite inclusion in Thm is true as well: if $Gx$ is closed, then $N_G(H)x$ is closed. This can be deduced from Vinberg's lemma (Sec 3.1 of Lec 6) and is left as an exercise.

## 2.4) Characteristic of a nilpotent element in $\mathfrak{g}$.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ & $(\cdot,\cdot)_{ad}$ denote the Killing form. Let $e\in\mathfrak{g}$ be a nonzero nilpotent element. As was mentioned in Sec. 2.3.1 of Lec 10, we can include $e$ into an $\mathfrak{sl}_2$-triple $(e,h,f)$.

**Proposition:** $h$ is a characteristic of $e$ (w.r.t. $(\cdot,\cdot)_{ad}$) in the sense of Sec 2 of Lec 12.

Proof: Recall the Levi subgroup $G_0(h)=Z_G(h)$ & its normal subgroup $\underline{G}_0(h)$: the connected subgroup w. Lie algebra $\underline{\mathfrak{g}}_0(h)=\{x\in\mathfrak{g}_0(h)| (h,x)_{ad}=0\}$. As we've mentioned in Example in Sec. 1.1 of Lec 13, it's enough to show that $\underline{G}_0(h)e$ is closed.

Let $S$ denote the connected subgroup of $G$ w. Lie algebra $\mathrm{Span}_{\mathbb{C}}(e,h,f)\simeq\mathfrak{sl}_2$ (so that $S\simeq SL_2$ or $SO_3$). Let $K_S$ denote the image of $SU_2$ under $SL_2\twoheadrightarrow S$, it's a maximal compact subgroup. One can find $K$ w. $K_S\subset K$.

Let $\tau$ be the anti-linear automorphism of $\mathfrak{g}$ that is the identity on $\mathfrak{k}$. It's restriction to $\mathfrak{s}=\mathrm{Span}_{\mathbb{C}}(e,h,f)\simeq\mathfrak{sl}_2$ fixes $\mathfrak{su}_2$, hence is given by $x\mapsto -\bar{x}^t$. Therefore

(*) $\quad\tau(e)=-f,\ \tau(f)=-e,\ \tau(h)=-h.$

On the other hand, $(\cdot,\cdot)_{ad}$ is negative definite on $\mathfrak{k}$ (b/c $\mathfrak{k}$ acts by skew-Hermitian operators on any rational representation of $G$). Hence $(\cdot,\cdot)_H:=-(x,\tau(y))_{ad}$ is a Hermitian scalar product on $\mathfrak{g}$

invariant w.r.t. $K$. Observe that $\forall\, x \in \mathfrak{g}_0(h)$:

$$(**) \qquad ([x,e],e)_H = [(*)] = ([x,e],f)_{ad} = [\text{invariance}] = (x,h) = 0$$

Note that $K_e(h) = \mathrm{Stab}_K(\sqrt{-1}h)$ is a maximal compact subgroup of $G_0(h)$ (the latter is connected & $\mathrm{Lie}(K_0(h))$ is a real form of $\mathfrak{g}_0(h)$). Also $\underline{\mathfrak{k}}_0(h) = \mathfrak{k}_0(h) \cap \underline{\mathfrak{g}}_0(h)$ is a real form of $\underline{\mathfrak{g}}_0(h)$ so the Lie algebra of a maximal compact subgroup $\underline{K}_0(h)$ of $\underline{G}_0(h)$. Let $\mu$ denote the moment map from $\underline{K}_0(h) \curvearrowright \mathfrak{g}$. $(**)$ says precisely that $\mu(e)=0$. So $\underline{G}_0(h)e$ is closed in $\mathfrak{g}$ (& hence in $\mathfrak{g}_2(h)$) by the Kempf-Ness theorem $\square$