DEFORMATION TYPES OF MODULI SPACES OF STABLE SHEAVES ON K3 SURFACES

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Abstract. These are notes for the MIT-NEU graduate seminar on Moduli of Sheaves on K3 surfaces. The goal is to understand the deformation type of moduli spaces of stable sheaves on a K3 surface.

Contents

1. Introduction

Let X be a projective K3 surface over $\mathbb{C}$. We will be concerned here with moduli spaces of Gieseker stable sheaves on X. Since the discrete invariants (e.g. Chern character) of a sheaf are constant in flat families, we will consider the moduli space of stable sheaves with fixed discrete invariants. We've already explored one class of examples: Hilbert schemes of n points $\mathrm{Hilb}^n_X = X^{[n]}$ on X. The ideal sheaf of such a length n zero-dimensional subscheme of X is a stable sheaf of Chern character $(\mathrm{rk}, c_1, \mathrm{ch}_2) = (1, 0, -n)$, and all stable sheaves with this Chern character are ideal sheaves of zero-dimensional length n subschemes.

In general, given a Chern character $\mathrm{ch} = (\mathrm{rk}, c_1, \mathrm{ch}_2)$ that could come from a sheaf, e.g. $\mathrm{rk} > 0$ or $\mathrm{rk} = 0$ and c_1 is effective, and an ample class H with which to measure stability, we might first ask:

Is the moduli space of H-stable sheaves of Chern character $(\mathrm{rk}, c_1, \mathrm{ch}_2)$ *nonempty?*

If it is nonempty, we've already seen that it is smooth and computed its dimension to be $\dim T_{[E]}X = \dim \mathrm{Ext}^1(E, E)$. And if H is a "generic" polarization with respect to the invariants of the sheaf, $M_H(v)$ is proper, since the notions of stable and semistable sheaves agree. To recall this, we define a more convenient packaging of the discrete invariants of a sheaf. For a sheaf E, let $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}_X} \in H^{\mathrm{even}}(X, \mathbb{Z})$ be its Mukai vector. As X is a K3 surface, $\mathrm{td}_X = 1 + \omega$ where $\omega \in H^4(X, \mathbb{Z})$ is the fundamental class of X. So $v(E) = (\mathrm{rk}(E), c_1(E), \mathrm{rk}(E) + \mathrm{ch}_2(E))$. We call a Mukai vector $(\mathrm{rk}, c_1, a\omega)$ primitive if $\gcd(\mathrm{rk}, c_1, a) = 1$. See section 1.2 of [9] for the fact the generic polarizations H with respect to primitive Mukai vectors give rise to the same notion of stable and semistable sheaves. Define the Mukai lattice of X to be $H^{\mathrm{even}}(X, \mathbb{Z})$ with pairing

$$\begin{aligned} \langle x, y \rangle &:= -\int_X x^\vee y \\ &= \int_X x_1 y_1 - x_0 y_2 - x_2 y_0, \end{aligned}$$

where x_i is the component of x in degree $2i$. Then the Grothendieck Riemann Roch theorem gives

$$\begin{aligned} \chi(E, F) &:= \mathrm{Ext}^0(E, F) - \mathrm{Ext}^1(E, F) + \mathrm{Ext}^2(E, F) \\ &= -\langle v(E), v(F) \rangle. \end{aligned}$$

Let $M_H(v)$ denote the moduli space of H-stable sheaves of Mukai vector v. The above analysis gives that *if $M_H(v)$ is nonempty*, then it is smooth of dimension

$$\dim M_H(v) = \dim \mathrm{Ext}^1(E, E) = 2 - \chi(E, E) = 2 + \langle v(E)^2 \rangle.$$

However, we still have not established that there exist stable sheaves of a given Mukai vector.

The goal of these notes is to appreciate the following theorem of Yoshioka [9, Theorem 8.1]:

Theorem 1.1. *Let X be a projective K3 surface and let $v = (\mathrm{rk}, c_1, a\omega)$ be a primitive Muaki vector. If $\mathrm{rk}\, v > 0$ or c_1 is ample, then $M_H(v)$ is deformation equivalent to $\mathrm{Hilb}_X^{\langle v^2 \rangle/2+1}$.*

Remark. As a sanity check, the dimensions of both spaces agree! We have

$$\begin{aligned} \dim \mathrm{Hilb}_X^{\langle v^2 \rangle/2+1} &= 2\left(\langle v^2 \rangle/2 + 1\right) \\ &= 2 + \langle v^2 \rangle. \end{aligned}$$

Besides settling the question of existence of stable sheaves of a given Mukai vector, this theorem allows one to reduce questions about deformation-invariant properties to Hilbert schemes of points, e.g. the Betti numbers and Hodge structure. In particular, moduli spaces of stable sheaves with primitive Mukai vector give no new examples of irreducible holomorphic symplectic (IHS) varieties.

The proof of this theorem (in the level of generality achieved by Yoshioka) is rather technical and quite delicate. For that reason, we'll focus here on the two main steps of the argument: reduction to proving the theorem for a single K3 surface, and proof of deformation equivalence in this case. In the first step, we will use elliptic K3 surfaces. These lend themselves nicely to the problem for two reasons: (1) there exist polarized elliptic K3 surfaces of every degree $2d$, and (2) we can use the fibration to leverage results about elliptic curves. For the second step, we we will sketch an earlier argument, due to Bridgeland [1], which proves the desired result in a special case.

1.1. **Outline of Proof.** As indicated above, the first step is to reduce the problem of showing that the moduli of stable sheaves $M_H(v)$ on an *arbitrary* K3 surface X is deformation equivalent to a Hilbert scheme of points, to showing that $M_{H'}(v')$ on some *fixed* K3 surface X' is (for appropriate H' and v'). We can do this for two reasons: (1) the moduli space of polarized K3 surfaces of fixed degree is connected, and (2) we can construct the moduli of stable sheaves in the relative setting.

The second step of the proof is really where the magic happens. We begin with hypothetical stable sheaves E of Mukai vector $v(E) = (\mathrm{rk}(E), c_1(E), \mathrm{rk}(E) + c_1(E))$ on an elliptic K3 surface X, and we want to somehow relate these to ideal sheaves of points on some (possibly different) K3 surface. Throughout, we will assume that the polarization is "suitable", which means a torsion-free sheave is stable if and only if its restriction to a generic fiber is stable. Suitable polarization exist, as observed by Friedman [2]. This is a key point, which we can assume from the techniques of the first step, that allows us to leverage the theory of elliptic curves in our situation.

Let $\pi\colon X \to C$ be the elliptic fibration. Then we can construct a relative moduli space $\mathcal{M}$ of stable sheaves on the fibers of π, which is itself an elliptic K3 surfaces fibered over C. Letting P be the pushforward of the Poincare bundle on $X \times_C \mathcal{M}$, we have a Fourier-Mukai transform

$$\Psi_P\colon D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(\mathcal{M})).$$

It turns out that this is an equivalence of categories, which, in our case, will take stable sheaves E to (possibly shifted) sheaves (e.g. complexes concentrated in one degree). The new Mukai vector satisfies

$$v(\Psi(E)) = \Psi^H(v)$$

where Ψ^H is the Fourier-Mukai transform on cohomology. We will show that this gives a birational map

$$M_H(X, v) \dashrightarrow M_{H'}(\mathcal{M}, \Psi^H(v)).$$

By appropriately choosing the numerics of the moduli space $\mathcal{M}$ we can arrange that

$$\Psi^H(v) = (1, 0, -\langle v^2\rangle/2).$$

More precisely in section 3 we will prove

Theorem 1.2 ([1])**.** *Let X be an elliptic K3 surface, f the numerical class of a fiber, and $v = r + c_1 + a\omega$ a primitive Mukai vector with $r > 1$ and $(c_1 \cdot f, r) = 1$. If H is a suitable polarization, then $M_H(X, v)$ is birational to $\mathcal{M}^{[\langle v^2\rangle/2+1]}$, where $\mathcal{M}$ is another elliptic $K3$ surface.*

To finish the argument, we need to show that if $M_H(v)$ is birational to a Hilbert scheme of points on a K3 surface, then it is in face deformation equivalent. This is guaranteed by the following two results:

Proposition 1.3. *Let X be a projective symplectic variety and Y an irreducible holomorphic symplectic variety. If X is birational to Y, then X is irreducible holomorphic symplectic as well.*

The proof of this result is given in Cor 6.2.7 of [5], where they prove that $M_H(v)$ is irreducible holomorphic symplectic.

Proposition 1.4 ([4, Theorem 4.6])**.** *Two irreducible holomorphic symplectic varieties which are birational are deformation equivalent.*

2. Reduction to Elliptic K3 Surfaces

The first major step in the proof of this theorem is to reduce to understanding the deformation type of the moduli space of stable sheaves on a particular K3 surface. As indicated above, we will specialize to elliptic K3 surfaces. For this, we will need to understand how to recognize an elliptic K3 surface from its Picard lattice.

2.1. **Torelli Theorem for K3 Surfaces.** Recall that for curves, the Torelli Theorem states that a curve C is determined by its Jacobian, equivalently by its weight 1 Hodge structure on $H^1(C,\mathbb{Z})$. Hence the Torelli map from M_g to A_g is an injection. However, a simple dimension count shows that not all abelian varieties are Jacobians of curves, so this map cannot be surjective. The magical fact for (polarized) K3 surfaces is that they are also determined by their (polarized) weight 2 Hodge structure on $H^2(X,\mathbb{Z})$; but even better the Torelli map is surjective, so given a polarized weight 2 Hodge structure which is of the type of a K3 surface, it is possible to construct the corresponding K3 surface. We will discuss this now, and give some examples.

Recall that for any K3 surface X, the second cohomology $H^2(X,\mathbb{Z})$ together with its intersection pairing is an even, unimodular lattice of signature $(3,19)$. In fact, this uniquely determines the lattice to be

$$\Lambda_{K3} = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3},$$

which we call the K3 lattice.

We will say that a weight 2 polarized Hodge structure V is of K3 type if $V \otimes \mathbb{C} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$ with $\dim V^{2,0} = \dim V^{0,2} = 1$ and $\dim V^{1,1} = 19$. The primitive second cohomology (orthogonal complement of the polarization) of an algebraic K3 surface is then a Hodge structure of K3 type.

Theorem 2.1 (Surjectivity of the Period Map). *Let $v_d \in \Lambda_{K3}$ be such that $\langle v_d, v_d\rangle = d > 0$. Any polarized Hodge structure V of K3 type on $v_d^\perp$ comes from a polarized K3 surface (X, L), where L is big and nef on X, and $V = c_1(L)^\perp \subset H^2(X,\mathbb{Z})$.*

Proof. This stronger version of the surjectivity of the period map (e.g. [3, Theorem 4.1]) follows from a more general theorem which identifies all possible ample cones, see **Surjectivity Theorem** [7, Section 12, pg. 76]. In that notation, we take R to be the union of all e such that $e^2 = -2$ and $v_d \cdot e = 0$. □

Recall that by the Lefschetz (1,1) Theorem we have that $\mathrm{Pic}(X) = \mathrm{NS}(X) \simeq H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$. So this theorem allows one "exhibit" a K3 surface with a desired Picard lattice.

Definition 2.2. An embedding of lattices $M \hookrightarrow \Lambda$ is called primitive if Λ/M is free. Two primitive embeddings $M \hookrightarrow \Lambda$ and $M \hookrightarrow \Lambda'$ are isomorphic if there is an isometry $\Lambda \xrightarrow{\simeq} \Lambda'$ which induces the identity on M.

The main lattice-theoretic result with interesting geometric consequences is:

Proposition 2.3 ([8, Corollary 1.12.3]). *Any even lattice S of signature $(1, \rho-1)$ for $\rho \leqslant 10$ admits a unique primitive embedding $S \hookrightarrow \Lambda_{K3}$. Hence there exists an algebraic K3 surface X and an isometry $\mathrm{NS}(X) \simeq S$.*

Example 2.4. Let U be the hyperbolic lattice $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $U \hookrightarrow \Lambda_{K_3}$ is an even lattice of signature $(1,1)$, there exist K3 surfaces X with $\mathrm{Pic}(X) \supset U$. The claim is that such an X

is an elliptic K3 surface. To see this, first call the two basis classes F and C. Then we can specify $2F+C$ to be ample (since $(2F+C)^2 = 4$). Hence by the Nakai-Moishezon criterion, $-F$ is not effective. Thus by Serre duality $h^2(X, F) = h^0(X, -F) = 0$. The Riemann-Roch theorem gives

$$\chi(F) = h^0(F) - h^1(F) = \frac{F^2}{2} + 2 = 2.$$

Thus $h^0(F) \geqslant 2$ and F is effective and moves in at least a pencil. In fact, $|F|$ is a base-point free linear series. Indeed, any base point would have to be on some effective curve F in the series. By twisting the subscheme sequence defining F by $\mathcal{O}_X(F)$, we have

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(F) \to \mathcal{O}_F(F) \simeq N_{F/X} \to 0.$$

The corresponding long exact sequence in cohomology is

$$0 \to H^0(\mathcal{O}_X) \to H^0(F) \to H^0(N_{F/X}) \to H^1(\mathcal{O}_X) = 0.$$

Hence if F has a base-point, then so does $N_{F/X}$. But by the adjunction formula, $N_{F/X} \simeq \omega_F$, the dualizing sheaf. As the Reiamnn-Roch theorem implies that ω_F is base-point free, we have that the linear series $|F|$ on X is as well.

And so the above exact sequence gives that $h^0(F) = 2$ and we have a morphism

$$\pi \colon X \xrightarrow{|F|} \mathbb{P}^1.$$

By the adjunction formula, if $p_a(F)$ denotes the arithmetic genus of F, then

$$2p_a(F) - 2 = F^2 = 0, \qquad \Rightarrow \qquad p_a(F) = 1.$$

And so the above map π is an elliptic fibration. Note that it also has a section, which is in the class $C - F$. A similar argument to the one above shows that it is effective.

2.2. **Deforming to Elliptic K3 Surfaces.** Recall that the goal is to show that for any K3 surface X and primitive Mukai vector v with $\operatorname{rk} v > 0$, the moduli space of stable sheaves $M_H(X, v)$ is deformation equivalent to some moduli space of stable sheaves $M_{H'}(Y, w)$, where Y is an elliptic $K3$ surface. The key technical lemma is the following proposition, which makes the moduli of sheaves construction in families:

Lemma 2.5 ([9, Prop 5.1])**.** *Let T be a connected curve, $(\mathcal{X}, \mathcal{L}) \xrightarrow{p} T$ a smooth family of polarized K3 surfaces. Assume that there exists $t_1 \in T$ such that $\rho(\mathcal{X}_{t_1}) = 1$. Let $v = r + d\mathcal{L} + a\omega \in Rp_*\mathbb{Z}$ be a primitive Mukai vector. Then there exists an algebraic space $\mathcal{M}(v) \to T$, smooth and proper, such that $\mathcal{M}(v)_t = M_{H_t}(v_t)$, where H_t is a general ample class except perhaps at finitely many places $t \in T$.*

As a result, all of the moduli spaces of sheaves which are fibers of the morphism $\mathcal{M}(v) \to T$ are deformation equivalent. We'll use this to prove:

Theorem 2.6. *Let X_1 and X_2 be K3 surfaces, $v_1 = r + \xi_1 + a_1\omega$ and $v_2 = r + \xi_2 + a_2\omega$ primitive Mukai vectors such that*

(1) $r > 0$
(2) $\ell(v_1) = \ell(v_2) = \ell$
(3) $\langle v_1^2 \rangle = \langle v_2^2 \rangle = 2s$
(4) $a_1 \equiv a_2 \pmod{\ell}$,

and H_1 and H_2 generic polarizations. Then $M_{H_1}(v_1)$ and $M_{H_2}(v_2)$ are deformation equivalent.

Proof Sketch. First, we may assume that the ξ_i are ample by twisting up by a sufficiently multiply of H_i. The operation $E \mapsto E \otimes H_i^{\otimes n_i}$ does not affect stability with respect to H_i. On Mukai vectors this corresponds to $v_i \mapsto v_i \operatorname{ch}(H_i^{\otimes n_i})$.

Second, we may assume that the X_i are elliptic K3 surfaces of degree $2\xi_i^2$. This follows almost directly from Lemma 2.5, since the moduli of polarized $K3$ surfaces of degree $2\xi_i^2$ is connected. Hence we can find curves T_i with the desired properties joining (X_i, ξ_i) to elliptic K3 surfaces. In fact, we can assume that the surfaces $X_1 = X_2$ since elliptic K3 surfaces have polarizations of every degree; namely, $\sigma + n_i f$, if σ is the class of a section and f is the class of a fiber.

So we may assume that $X_1 = X_2 = X$is an elliptic K3 surface and that $\xi_i/\ell = \sigma + n_i f$. Because $\langle v_1^2 \rangle = \langle v_2^2 \rangle$, we have

$$\begin{aligned} \xi_1^2 - 2ra_1 &= \xi_2^2 - 2ra_2 \\ \ell^2(2n_1 - 2) - 2ra_1 &= \ell^2(2n_2 - 2) - 2ra_2 \\ \ell^2 n_1 - ra_1 &= \ell^2 n_2 - ra_2 \\ r(a_2 - a_1) &= \ell^2(n_2 - n_1). \end{aligned}$$

Now the difference

$$\begin{aligned} v_2 - v_1 &= \xi_2 - \xi_1 + (a_2 - a_1)\omega \\ &= (n_2 - n_1)\ell f + (a_2 - a_1)\omega \\ &= \frac{r(a_2 - a_1)f}{\ell} + (a_2 - a_1)\omega \\ &= (r + (\sigma + n_1 f)\ell + a_1\omega) \cdot \left(\frac{(a_2 - a_1)f}{\ell}\right) \\ &= v_1 \cdot \left(\frac{(a_2 - a_1)f}{\ell}\right). \end{aligned}$$

Hence $v_2 = v_1 \cdot \exp\left(\frac{(a_2-a_1)f}{\ell}\right)$. And $\exp\left(\frac{(a_2-a_1)f}{\ell}\right)$ is the Chern character of a line bundle. This corresponds to the operation of twisting the sheaves by the line bundle $\frac{a_2-a_1}{\ell} \cdot f$. Since this preserves stability [9, Lemma 1.1], we may reduce to the case $v_1 = v_2$. We reduce to $H_1 = H_2$ by again applying Lemma 2.5 and passing through Picard rank 1. □

2.3. **Producing the Elliptic K3 Surface.** Let X be our projective K3 surface, H generic polarization, and v primitive Mukai vector with $\operatorname{rk} v > 0$, Given the result of Theorem 2.6, our goal here is to produce an elliptic K3 surface Y with Mukai vector w such that the hypothesis of Theorem 2.6 are satisfies for (X, v) and (Y, w). Our main tool with be the Torelli theorem!

The choice Yoshioka gives is as follows. Let $v = \ell(r + c_1) + a\omega)$ and choose $k > 0$ such that $n := r \cdot k - c_1^2 > 0$. Let b be such that $b + \ell r = a - \ell k$. We may assume by altering k that b is relatively prime to r.

Let L be the even rank 3 lattice with intersection matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2n \end{pmatrix}.$$

Note the the signature of L is $(1,2)$, so by Proposition 2.3 L admits a primitive embedding into the $K3$ lattice. Also, since L contains a copy of U, the K3 surface Y produced by the Torelli theorem is an elliptic K3 with a section. Call σ the class of a section and f the class of a fiber. Let ζ_n be the last basis class, with $\zeta_n \cdot f = \zeta_n \cdot \sigma = 0$ and $\zeta_n^2 = -2n$. Note that if $n > 1$, then every fiber of π is irreducible. So we may assume this by making k larger.

Now we set $w = \ell(r + (-\zeta_n + f)) + (b + \ell r)\omega$. Then we have

$$\begin{aligned} \langle w^2 \rangle &= \ell^2(-\zeta_n + f)^2 - 2\ell r(b + \ell r) \\ &= \ell^2(-2n) - 2\ell r(b + \ell r) \\ &= \ell^2(-2rk + c_1^2) - 2\ell r(a - \ell k) \\ &= \ell^2 c_1^2 - 2\ell r a \\ &= \langle v^2 \rangle. \end{aligned}$$

In addition, it can be easily seen that the other assumptions of Theorem 2.6 hold.

3. Fourier-Mukai Transforms on Elliptic K3 Surfaces

Recall from the introduction that the theorem we are aiming for is:

Theorem 3.1 ([1, Theorem 1.1]). *Let X be an elliptic K3 surface, f the numerical class of a fiber, and $v = r + c_1 + a\omega$ a primitive Mukai vector with $r > 1$ and $(c_1 \cdot f, r) = 1$. If H is a suitable polarization, then $M_H(X, v)$ is birational to $\mathcal{M}^{[\langle v^2 \rangle/2+1]}$, where $\mathcal{M}$ is another elliptic $K3$ surface.*

We begin by recalling the definition and some properties of Fourier-Mukai transforms in general. Then we will discuss Fourier-Mukai transforms on elliptic curves and their relative versions fro elliptic fibrations. Finally we will put this all together to define the above map and prove that it is birational.

3.1. **Fourier-Mukai Tranforms.** Let $D(X) := D^b(\mathrm{Coh}(X))$ be the bounded derived category of coherent sheaves on X. Recall that the objects of $D(X)$ are complexes of coherent sheaves on X. Let $[i]$ be the shift functor that shifts complexes i places to the left. For a complex A, let $\mathscr{H}^i(A)$ denote the ith cohomology sheaf of A, e.g. $\ker(A_i \to A_{i+1})/\mathrm{im}(A_{i-1} \to A_i)$. We say that $E \in D(X)$ is a sheaf if $\mathscr{H}^i(E) = 0$ for $i \neq 0$.

On a product of varieties $X \times Y$, let π_X and π_Y denote the projections onto each factor

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

Given $P \in D(X \times Y)$ define the Fourier-Mukai transform with kernel P to be the map

$$\Phi^P \colon D(Y) \to D(X),$$
$$\Phi^P(y) = R\pi_{X,*}(P \otimes^L \pi_Y^*(y)).$$

If P is a sheaf on $X \times Y$, flat over Y, then the above tensor product is exact. When the kernel is obvious from context, we will omit it and simply write $\Phi = \Phi^P$.

We can also define a Fourier-Mukai functor in the other direction, which in good circumstances will be an inverse of Φ. Let $P^\vee := \mathrm{RHom}(P, \mathcal{O}_{X \times Y})$, and define

$$Q := P^\vee[\dim X + \dim Y - \dim P].$$

We then set

$$\Psi = \Psi^Q : D(X) \to D(Y),$$

to be the corresponding Fourier-Mukai functor. The reason for the shift is to give Q the best chance of being a sheaf. For example, if P is a torsion sheaf (e.g. $\dim P \neq \dim X + \dim Y$), then $\mathrm{Hom}(P, \mathcal{O}_{X\times Y}) = 0$, but higher Ext groups need not vanish. If P is a vector bundle, then it is clear that Q is as well.

Grothendieck-Verdier duality gives that $\Psi[\dim P - \dim Y]$ is a left adjoint of Φ, so if Φ is fully faithful, then

$$\Psi \circ \Phi \simeq \mathrm{Id}_{D(Y)}[\dim Y - \dim P]. \tag{1}$$

By Grothendeick Riemann Roch, there is a cohomological Fourier-Mukai functor defined such that the following diagram

$$\begin{tikzcd} D(Y) \arrow[r, "\Phi"] \arrow[d, "\mathrm{ch}"] & D(X) \arrow[d, "\mathrm{ch}"] \\ H^{\mathrm{even}}(Y, \mathbb{Z}) \arrow[r, "\Phi^H"] & H^{\mathrm{even}}(X, \mathbb{Z}) \end{tikzcd}$$

commutes.

As indicated in the introduction, we will use Fourier-Mukai transforms to turn stable sheaves of arbitrary Mukai vector into ideal sheaves of points on a (possibly different) K3 surface. First we must understand when the Fourier-Mukai transform of a sheaf is again a sheaf.

Definition 3.2. A sheaf $E \in \mathrm{Coh}(Y)$ is $\Phi - \mathsf{WIT}_i$ (weak index theorem) if

$$\Phi^j(E) := \mathscr{H}^j(\Phi(E)) = 0$$

for $i \neq j$.

Equivalently, E is $\Phi - \mathrm{WIT}_i$ if $\Phi(E)[i]$ is a sheaf. In this case, we will call the sheaf $\Phi^i(E)$ the transform of E, and denote it by $\hat{E}$.

If Φ is fully faithful, we can relate the Ext groups of transforms of WIT sheaves to the Ext groups of the original sheaves. If A is $\Phi - \mathrm{WIT}_a$ and B is $\Phi - \mathrm{WIT}_b$, then

$$\begin{aligned} \mathrm{Ext}^i_Y(A, B) &= \mathrm{Hom}_{D(Y)}(A, B[i]) \\ &= \mathrm{Hom}_{D(X)}(\Phi(A), \Phi(B)[i]) \\ &= \mathrm{Hom}_{D(X)}(\hat{A}[-a], \hat{B}[-b+i]) \\ &= \mathrm{Ext}^{i+a-b}_X(\hat{A}, \hat{B}). \end{aligned}$$

The relation

$$\mathrm{Ext}^i_Y(A, B) = \mathrm{Ext}^{i+a-b}_X(\hat{A}, \hat{B}) \tag{2}$$

is referred to as the Parseval theorem.

Since our goal is to use the Fourier-Mukai functor to turn the universal family over $M_H(v)$ into a family of ideal sheaves of zero-dimensional subschemes, we need to know that the Fourier-Mukai functor behaves well with families of WIT sheaves. This is provided by the following Lemma, which loosely says that Φ takes families of $\Phi - \mathrm{WIT}$ sheaves to families of sheaves.

Lemma 3.3 ([1, 2.4]). *Let S be a scheme and $\mathscr{E}$ a sheaf on $Y \times S$, flat over S. Further suppose that the Fourier-Mukai kernel P is a sheaf on $X \times Y$. Then*

$$U := \{s \in S : \mathscr{E}_s \textit{ is } \Phi - WIT_i\}$$

is open in S. In addition there exists a sheave $\mathscr{F}$ on $X \times U$, flat over U, such that for $s \in U$, $\mathscr{F}_s = \Phi^i(\mathscr{E}_s)$.

3.2. **Fourier-Mukai Transforms on Elliptic Curves and Elliptic Surfaces.**

3.2.1. *Elliptic Curves.* We consider now the case that X and Y are both elliptic curves. Let a, b be coprime integers with $a > 0$. Let Y be the moduli space of stable bundles with Chern class $c(E) = (r(E), d(E)) = (a, b)$, where $r(E)$ is the rank of E and $d(E)$ is its degree. In fact, $Y \simeq X$, but we will continue with the differentiated notation to reduce confusion.

Let P be the Poincaré bundle on $X \times Y$, satisfying the property that for all $y \in Y$, $P|_{X\times\{y\}}$ is the bundle parameterized by the point y. Let $\Phi = \Phi^P : D(Y) \to D(X)$. Since P is a vector bundle, $Q = P^\vee$ is again a vector bundle. Let $\Psi = \Phi^Q : D(X) \to D(Y)$. As noted above, $\Psi[1]$ is a left adjoint to Φ.

Proposition 3.4 ([1, Prop 3.1]). *The functor Φ is an equivalence of categories.*

In particular, $\Psi \circ \Phi = \mathrm{Id}_{D(Y)}[1]$. Thus the cohomological Fourier-Mukai functors satisfy

$$\Psi^H \circ \Phi^H = -\mathrm{Id}_{H^{\mathrm{even}}(Y,\mathbb{Z})}\,. \tag{3}$$

In fact, we can understand the cohomological Fourier-Mukai functor Φ^H very concretely. If X and Y are elliptic curves, then $H^{\mathrm{even}}(Y,\mathbb{Z})$ and $H^{\mathrm{even}}(X,\mathbb{Z})$ are both rank 2 free $\mathbb{Z}$-modules. So Φ^H is an invertible 2×2 integer matrix. But we can say even more!

For $y \in Y$, we have that

$$\begin{aligned}\Phi(\mathcal{O}_y) &= R\pi_{X,*}(P \otimes \pi_Y^*(\mathcal{O}_y)),\\ &= R\pi_{X,*}(P \otimes \pi_Y^*(\mathcal{O}_{X\times\{y\}})),\\ &= R\pi_{X,*}(P|_{X\times\{y\}})),\\ &= P|_{X\times\{y\}},\end{aligned}$$

as π_X is an isomorphism restricted to $X \times \{y\}$. So, as $\mathcal{O}_y$ has Chern class $(0, 1)$ and $P|_{X\times\{y\}}$ has Chern class (a, b), we have that

$$\Phi^H \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

So the matrix corresponding to Φ^H is of the form $\begin{pmatrix} c & a \\ d & b \end{pmatrix}$ with $bc - ad = \pm 1$. We can in fact say that it is $+1$. By (3) we have that

$$\Psi^H = \pm \begin{pmatrix} -b & d \\ a & -c \end{pmatrix}.$$

But as observed above, $a = r(Q|_{\{x\}\times Y}) > 0$ as Q is a sheaf.

The c and d in the matrix corresponding to Φ^H in $\mathrm{SL}_2(\mathbb{Z})$ are only defined modulo the addition of a fixed multiple n of a to c, and b to d. This corresponds to twisting the kernel P by a line bundle of degree n on Y.

3.2.2. *Elliptic Surfaces.* Our goal now is the make the entire picture above relative! Let $\pi\colon X \to C$ be an elliptic K3 surface with a section. Let f be the numerical class of a fiber of π. For any sheaf E on X, let $d(E) = c_1(E) \cdot f$ denote the fiber degree of E. Further let $\mu(E) := \frac{d(E)}{r(E)}$ denote the fiber slope of E. We will say that E is a fiber sheaf if $r(E) = d(E) = 0$; equivalently if the support of E is contained in the union of finitely many fibers. We will construct relative Fourier-Mukai functors which agree with the functors above when restricted to the fibers of π.

Let $\mathcal{M}(X/C)$ be the moduli space of stable, pure-dimension 1 sheaves supported on fibers of π.

Definition 3.5. For coprime integers a and b, let $\hat{\pi}\colon J_X(a,b) \to C$ be the union of components of $\mathcal{M}(X/C)$ with a point corresponding to a sheaf of rank a and degree b supported on a nonsingular fiber.

The coprimality assumptions guarantee that $J_X(a,b)$ is a projective, fine moduli space whose points correspond to strictly stable sheaves. For convenience we will refer to $J_X(a,b)$ as Y. Mukai showed that there is a Poincare sheaf P on $X \times_C Y$, with the property that for all $y \in Y$, $P|_{\pi^{-1}(\hat{\pi}(y)) \times \{y\}}$ is the sheaf parameterized by the point y.

Extension by 0 (e.g. pushforward) defines an isomorphism

$$Y \xrightarrow{\sim} M_H((0, af, -b)),$$

since any stable torsion sheaf E with $c_1(E) = af$ is in fact supported on a fiber (since its support must be contained in fibers and also connected). And from Mukai's analysis of moduli spaces with $\langle v^2 \rangle = 2$, we know that both sides are smooth projective K3 surfaces.

The kernel of our relative Fouier-Mukai transform will be the extension of P by 0 to $X \times Y$, which we will also call P. Note that since this sheaf is supported on $X \times_C Y$, we have that $\dim P = 3$, not 4. So we set $Q = P^\vee[1]$. By [1, Lemma 5.1] Q is a sheaf. Define the relative Fourier-Mukai functors

$$\Phi = \Phi^P\colon D(Y) \to D(X)$$
$$\Psi = \Phi^Q\colon D(X) \to D(Y).$$

By similar arguments to the case of Fourier-Mukai transforms for elliptic curves, Bridgeland [1] shows that these relative functors are equivalences, so in particular

$$\Psi \circ \Phi \simeq \mathrm{Id}_{D(Y)}[-1]. \tag{4}$$

Let $p \in C$, and define $X_p := \pi^{-1}(p)$ and $Y_p := \hat{\pi}^{-1}(p)$ be nonsingular fibers. Note that by construction Y_p is the moduli space of stable bundles of rank a and degree b on X_p. Let $i_p\colon X_p \hookrightarrow X$ and $j_p\colon Y_p \hookrightarrow Y$ be the inclusions of the fibers. Finally let P_p be the restriction of P to $X_p \times Y_p$ and let

$$\Phi_p = \Phi_p^{P_p}\colon D(Y_p) \to D(X_p),$$

be the restriction of Φ to the fiber over p.

Proposition 3.6. *Φ and Ψ restrict to the elliptic curve Fourier-Mukai functors with parameters (a,b) on the fibers of $\hat{\pi}\colon Y \to C$ and $\pi\colon X \to C$, e.g.*

$$Li_p^* \circ \Phi \simeq \Phi_p \circ Lj_p^*,$$
$$Lj_p^* \circ \Psi \simeq \Psi_p \circ Li_p^*,$$

Proof. The proofs of both results are identical, so we show the first. We have the following commutative diagram, the bottom and top "squares" of which are cartesian:

$$\begin{tikzcd} & X_p \arrow[r, hook, "i_p"] & X & \\ X_p \times Y_p \arrow[ur, "\pi_{X_p}"] \arrow[dr, "\pi_{Y_p}"'] \arrow[rrr, "k_p"] & & & X \times Y \arrow[ul, "\pi_X"'] \arrow[dl, "\pi_Y"] \\ & Y_p \arrow[r, hook, "j_p"] & Y & \end{tikzcd}$$

Using the base change theorem for derived functors we have

$$\begin{aligned} Li_p^*\left(\pi_{X,*}\left(P \otimes \pi_Y^*(-)\right)\right) &\simeq \pi_{X_p,*}\left(Lk_p^*\left(P \otimes \pi_Y^*(-)\right)\right) \\ &\simeq \pi_{X_p,*}\left(P_p \otimes Lk_p^* \pi_Y^*(-)\right) \\ &\simeq \pi_{X_p,*}\left(P_p \otimes \pi_{Y_p}^* Lj_p^*(-)\right). \end{aligned}$$

□

By restricting to the fibers, we have

Corollary 3.7. *If E is a sheaf on X of rank $r(E)$ and fiber degree $d(E)$, then*

$$\begin{pmatrix} r(\Psi(E)) \\ d(\Psi(E)) \end{pmatrix} = \begin{pmatrix} -b & a \\ d & -c \end{pmatrix} \begin{pmatrix} r(E) \\ d(E) \end{pmatrix},$$

for some $c, d \in \mathbb{Z}$ such that $bc - ad = 1$.

Recall that our goal is to set the numerical parameters (a, b, c, d) appropriately such that for E on X with given Mukai vector v, $\Psi(E)$ is (1) a sheaf, and (2) more particularly an ideal sheaf of a zero-dimensional subscheme of Y. We'll begin with (1), understanding when Ψ and Φ bring (families of) sheaves into (families of) sheaves.

3.3. **Weak Index Theorem Sheaves.**

Recall (4) that

$$\Psi \circ \Phi \simeq \mathrm{Id}_{D(Y)}[-1], \qquad \Phi \circ \Psi \simeq \mathrm{Id}_{D(X)}[-1].$$

First note that this implies that if E is $\Psi - \mathrm{WIT}_i$, then $\hat{E}$ is $\Phi - \mathrm{WIT}_{1-i}$ and visa versa; indeed, $\Phi^j(\Psi(E)) = 0$, unless $j = 1$, in which case it is E.

More generally, isomorphism of functors feeds nicely into the Grothendieck composition of derived functor spectral sequence! Taking the second isomorphism of functors, the E_2 page of this spectral sequence is:

$$\begin{array}{|lll} \vdots & \iddots & \\ \Phi^0(\Psi^2(E)) & \Phi^1(\Psi^2(E)) & \cdots \\ \Phi^0(\Psi^1(E)) & \Phi^1(\Psi^1(E)) & \cdots \\ \Phi^0(\Psi^0(E)) & \Phi^1(\Psi^0(E)) & \cdots \\ \hline \end{array}$$

This spectral sequence abuts to the rightward shift of E, so

$$E_2^{p,q} = \Phi^p(\Psi^q(E)) \qquad \Rightarrow \qquad \begin{cases} E & : \ p+q=1 \\ 0 & : \ \text{else.} \end{cases}$$

Using Proposition 3.6, $\Phi^i(\Psi^j(E)) = 0$ for i or j greater than 1 (or less than 0), since it restricts to 0 on a generic fiber. So in fact the E_2 page is concentrated in the square box where $0 \leqslant p, q \leqslant 1$.

$$\begin{array}{|ccc}
0 & 0 & \iddots \\
0 & 0 & 0 \\
\Phi^0(\Psi^1(E)) & \Phi^1(\Psi^1(E)) & 0 \\
\Phi^0(\Psi^0(E)) & \Phi^1(\Psi^0(E)) & 0 \\
\hline
\end{array}$$

All differentials pass out of this box, so the spectral sequence must degenerate on the E_2 page. Hence we further have that $\Phi^0(\Psi^0(E)) = \Phi^1(\Psi^1(E)) = 0$, since the sequence abuts to the shift of E. To summarize, we have three pieces of information

(1) $\Phi^0(\Psi^0(E)) = 0$, so $\Psi^0(E)$ is $\Phi - \mathrm{WIT}_1$,
(2) $\Phi^1(\Psi^1(E)) = 0$, so $\Psi^1(E)$ is $\Phi - \mathrm{WIT}_0$,
(3) We have an exact sequence $0 \to \Phi^1(\Psi^0(E)) \to E \to \Phi^0(\Psi^1(E)) \to 0$. By our first observation, $\Phi^1(\Psi^0(E))$ is $\Psi - \mathrm{WIT}_0$ and $\Phi^1(\Psi^0(E))$ and $\Psi - \mathrm{WIT}_1$.

Similarly, we have another spectral sequence with the roles of Φ and Ψ reversed, which gives the reversed information.

Lemma 3.8. *For any sheaf E on X, there is a unique short exact sequence*

$$0 \to A \to E \to B \to 0,$$

where A is $\Psi - WIT_0$ and B is $\Psi - WIT_1$.

Proof. Existence follows from observation (3) above: $A = \Phi^1(\Psi^0(E))$ and $B = \Phi^0(\Psi^1(E))$. Suppose that there existed another sequence $0 \to A' \to E \to B' \to 0$. The composition $b \circ a'$,

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "a'"] \arrow[drr, dashed] & E \arrow[r, "b'"] \arrow[d, equal, "b\circ a'"] & B' \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "a"] & E \arrow[r, "b"] & B \arrow[r] & 0
\end{tikzcd}$$

lives in the group $\mathrm{Hom}(A', B)$. By the Parseval Theorem (2),

$$\begin{aligned} \mathrm{Hom}_X(A', B) &= \mathrm{Ext}^0_X(A', B) \\ &= \mathrm{Ext}^{-1}_Y(\hat{A}', \hat{B}) \\ &= 0. \end{aligned}$$

So the map is necessarily 0, and so factors through A. Considering $\mathrm{Hom}(A, B')$, we see that the same is true in the other direction, and so $A = A'$ and $B = B'$. $\square$

Lemma 3.9. *Let E be a torsion-free sheaf on X. If E is $\Psi - WIT_0$, then the fiber slope $\mu(E) \geqslant b/a$. If E is $\Psi - WIT_1$, then $\mu(E) \leqslant b/a$.*

Proof. Recall that by Corollary 3.7, Ψ^H to the matrix $\begin{pmatrix} -b & a \\ d & -c \end{pmatrix}$. In particular $r(\Psi(E)) = -br(E) + ad(E)$.

- If E is $\Psi - \mathrm{WIT}_0$, then $\Psi(E)$ is a sheaf, so $r(\Psi(E)) \geqslant 0$. Hence $\mu(E) \geqslant b/a$.
- If E is $\Psi - \mathrm{WIT}_1$, then $\Psi(E)[1]$ is a sheaf, so $r(\Psi(E)) \leqslant 0$. Hence $\mu(E) \leqslant b/a$. □

Lemma 3.10. *Let T be a torsion $\Psi - WIT_1$ sheaf on X. Then T is a fiber sheaf.*

Proof. Since T is a torsion sheaf, $r(T) = 0$ and so it suffices to show that $d(T) = 0$. As above we have $r(\Psi(T)) = -br(T) + ad(T) = ad(T) \leqslant 0$, since T is $\Psi - \mathrm{WIT}_1$. Thus $d(T) = c_1(T) \cdot f \leqslant 0$, since $a \geqslant 0$. But as $c_1(T)$ is effective and f is nef, $d(T) = 0$. □

Recall that our goal is to use the Fourier-Mukai functor to transforms *stable* sheaves with Mukai vector v to ideal sheaves of points. We now see that stability (with respect to a suitable polarization) is exactly what will guarantee for us that $\Psi(E)$ is again a sheaf.

Lemma 3.11. *Let E be a torsion-free sheaf on X, such that the restriction of E to a generic fiber is stable. Then if $\mu(E) < b/a$, E is $\Psi - WIT_1$.*

Proof. Consider the unique short exact sequence

$$0 \to A \to E \to B \to 0,$$

where A is $\Psi - \mathrm{WIT}_0$ and B is $\Psi - \mathrm{WIT}_1$. We want to show that $A = 0$. If not, then A is torsion-free because E is. Lemma 3.9 implies that the fiber slope

$$\mu(A) \geqslant b/a > \mu(E),$$

which contradicts our assumption that the restriction of E to a generic fiber is stable. □

Finally, we have that the opposite holds: the Fourier-Mukai transform of a sheaf whose restriction to the general fiber is simple again has this property. Recall that a sheaf E is called simple if $\mathrm{Hom}(E, E) = k$.

Lemma 3.12. *Let E be $\Phi - WIT$ sheaf on Y whose restriction to a general fiber of $\hat{\pi}$ is simple. Then the restriction of $\hat{E}$ to a general fiber of π is simple.*

Proof. Because of Prop 3.6, this follows directly from the Parseval theorem (2) for Fourier-Mukai transforms on elliptic curves:

$$\mathrm{Hom}(E, E) = \mathrm{Ext}^0(E, E) = \mathrm{Ext}^0(\hat{E}, \hat{E}). \qquad \square$$

Remark. For elliptic curves, a simple sheaf is either a stable vector bundle or the skyscrapper sheaf of a point (see Remark 3.4 of [**?**]). Hence, because we will always work with a suitable polarization, this result shows that the (torsion-free) transform of a stable sheaf is stable.

3.4. **Proof of the Main Theorem.** We will now carry out the process that we have been alluding to for the past seven pages!

Let $\pi\colon X \to C$ be an elliptic K3 surface with a section and let f be the numerical class of a smooth fiber. Fix $v = (r, c_1, r + \mathrm{ch}_2)$ a primitive Mukai vector with $r > 1$ and $d = c_1 \cdot f$ relatively prime to r. Let H be a suitable polarization on X, e.g. one for which a torsion-free sheaf E on X is stable if and only if its restriction to a generic fiber of π is stable.

Let a, b be the unique integers such that

$$br - ad = 1, \qquad 0 < a < r.$$

Let $Y = J_X(a, b) \simeq M_H((0, af, -b))$. Let H' be a suitable polarization on Y. Let P be the Poincare sheaf on $X \times Y$, and let Φ and Ψ be the Fourier-Mukai functors defined in Section 3.2.2. So we have that Φ corresponds to the matrix $\begin{pmatrix} r & a \\ d & b \end{pmatrix}$ and Ψ corresponds to the matrix $\begin{pmatrix} -b & a \\ d & -r \end{pmatrix}$ on the fibers. Hence if E is a stable sheaf on X of rank and fiber degree $(r(E), d(E)) = (r, d)$, then the complex $\Psi(E)$ has rank and fiber degree

$$\begin{pmatrix} r(\Psi(E)) \\ d(\Psi(E)) \end{pmatrix} = \begin{pmatrix} -b & a \\ d & -r \end{pmatrix} \begin{pmatrix} r \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Furthermore, since $br - ad = 1$, we have that

$$\frac{d}{r} < \frac{b}{a}.$$

So by Lemma 3.11, E is $\Psi - \mathrm{WIT}_1$ (e.g. a sheaf concentrated in degree 1.) So by the above calculation $\hat{E}$ is a rank 1 sheaf of fiber degree 0. However, our goal was to produce a sheaf of rank one with trivial first Chern class. We can arrange that by modifying the relative Fourier-Mukai functor P. Let $L = c_1(\hat{E})$ be the fiber degree 0 line bundle on Y, and replace P by $P \otimes \pi_Y^*(L^{-1})$. Since the fiber degree of L (and hence L^{-1}) is 0 this does not alter the cohomological Fourier-Mukai transform on the fibers. But by the push-pull formula:

$$\pi_{Y*}(P \otimes \pi_X^*(E) \otimes \pi_Y^*(L^{-1})) = \pi_{Y*}(P \otimes \pi_X^*(E)) \otimes L^{-1}.$$

Hence, making such an alteration, we may assume that $c_1(\hat{E}) = 0$. Thus the Mukai vector of the transform

$$v(\hat{E}) = (1, 0, -\langle v^2 \rangle / 2) = v',$$

since $\langle v(\hat{E})^2 \rangle = \langle v(E)^2 \rangle$ as the Fourier-Mukai transform is an isometry of Mukai lattices. So as desired, the transform of E is an ideal sheaf of points on Y, another elliptic K3 surface.

And this construction works in families. Define

$$U = \{E \in M_H(v) : \hat{E} \text{ is torsion-free}\},$$

and

$$V = \{F \in M_{H'}(v') : F \text{ is } \Phi - \mathrm{WIT}_0\}.$$

Recall that Lemma 3.11 guarantees that all $E \in M_H(v)$ are already $\Psi - \mathrm{WIT}_1$. By Lemma 3.3 and the fact that stability is an open condition in flat families [5, Prop 2.3.1.], U and V are open subsets of $M_H(v)$ and $M_{H'}(v')$ respectively. In what follows we will show that U and V are nonempty and in fact isomorphic (via the Fourier-Mukai functor Φ). After arguing that these moduli spaces are irreducible, this will establish their birationality, completing the proof.

Lemma 3.13. *The Fourier-Mukai transform Φ is an isomorphism between U and V.*

Proof. Since $\Phi \circ \Psi \simeq \mathrm{Id}_{D(X)}[-1]$ and Lemma 3.3 guarantees that Φ behaves well in families, it suffices to show that Φ and Ψ bring U and V into eachother.

Let E be in U. Then since E is $\Psi - \mathrm{WIT}_1$, we have that $\hat{E}$ is $\Phi - \mathrm{WIT}_0$. Thus $\hat{E}$ is in V, since a rank 1 sheaf is stable if and only if it is torsion-free. In the other direction, given $F \in V$, set $E = \hat{F} = \Phi(F)$. We want to show that E is in U, e.g. that it is stable.

Since the restriction of F to a generic fiber is stable, hence simple, the restriction of E to a generic fiber is simple, hence stable. As H is a suitable polarization, it suffices to show that E is torsion-free.

Suppose that E had a torsion subsheaf T. Consider the exact sequence

$$0 \to T \to E \to E/T \to 0.$$

Since Ψ is left exact, and E is $\Psi - \mathrm{WIT}_1$, T is also $\Psi - \mathrm{WIT}_1$. Hence applying Ψ we get an exact sequence

$$0 \to \Psi^0(E/T) \to \hat{T} \to F \to \Psi^1(E/T) \to 0.$$

By Lemma 3.10 T is a fiber sheaf and so $\hat{T}$ is as well. In particular, it is a torsion sheaf; but as F is torsion-free, the map

$$\Psi^0(E/T) \to \hat{T}$$

must be an isomorphism. But $\Psi^0(E/T)$ is $\Phi - \mathrm{WIT}_1$ and $\Psi^1(T) = \hat{T}$ is $\Phi - \mathrm{WIT}_0$. Thus both sheaves are in fact 0 and $T = 0$ and E is torsion-free. $\square$

To show that they are nonempty, we will show that certain ideal sheaves of length $\langle v^2\rangle/2+1 = -\operatorname{ch}_2(v')$ are $\Phi - \mathrm{WIT}_0$.

Lemma 3.14. *Let $F = L \otimes \mathcal{I}_Z$ where $L \in \operatorname{Pic}^0(Y)$ and Z is a set of $\langle v^2\rangle/2+1$ points lying on distinct fibers of $\hat{\pi}\colon Y \to C$. Then F is in V.*

To prove this we need the following Lemma. Recall that $Q = P^\vee[1]$.

Lemma 3.15. *A sheaf F on Y is $\Phi - WIT_0$ if and only if*

$$\operatorname{Hom}_Y(F, Q|_{\{x\}\times Y}) = 0,$$

for all $x \in X$.

Proof. For simplicity we will write Q_x for $Q|_{\{x\}\times Y}$. We have that $Q_x = \Psi(\mathcal{O}_x)$, and is hence $\Phi - \mathrm{WIT}_1$. Thus if F is $\Phi - \mathrm{WIT}_0$, the Parseval Theorem (2) implies that

$$\operatorname{Hom}(F, Q_x) = \operatorname{Ext}^{-1}(\hat{F}, \mathcal{O}_x) = 0.$$

Conversely, if F not $\Phi - \mathrm{WIT}_0$, then as in Lemma 3.8, there is a surjection $F \to B$, where B is a nonzero $\Phi - \mathrm{WIT}_1$ sheaf. Again by the Parseval theorem

$$\operatorname{Hom}(B, Q_x) = \operatorname{Hom}(\hat{B}, \mathcal{O}_x).$$

If x is in the support of $\hat{B}$, then there is a nonzero map $\hat{B} \to \mathcal{O}_x$, and hence from $B \to Q_x$. Composing we have $F \to B \to Q_x$, a nonzero map. $\square$

Proof of Lemma 3.14. By the above Lemma it suffices to show that there are no nonzero maps from $F \to Q_x$ for any $x \in X$. As Q_x is supported on the fiber $Y_{\pi(x)}$ over $\pi(x)$, any such map factors through $F|_{Y_{\pi(x)}}$. The (push-forward to Y of this) restriction is a stable, pure-dimension 1 sheaf on Y with Chern class $(0, f, s)$, where s is the number of points of Z in the fiber $Y_{\pi(x)}$. Hence $s = 1$ for finitely many values of $\pi(x)$, and is otherwise 0. Q_x is also stable of pure dimension 1 with Chern class $(0, af, r)$. But then there cannot be any nonzero maps $F \to Q_x$ since $r/a > 1 \geqslant s$ and Q_x is stable. $\square$

Finally to complete the proof that $M_H(v)$ and $M_{H'}(v')$ are birational, we need that these nonempty open set U and V are dense in their respective moduli spaces. In other words, we need that these spaces are irreducible, e.g. have only one connected component. This general result for moduli of (semi)stable sheaves on a K3 surface with respect to a generic polarization is Theorem 4.1 of [6]. This completes the proof.

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