QUANTUM HAMILTONIAN REDUCTION II: SHEAF LEVEL

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1. Introduction

One of the main highlights of the previous semester was an interplay between the following objects: the nilpotent cone in $\mathfrak{g}$, the cotangent bundle $T^*(G/B)$, the universal enveloping algebra $U(\mathfrak{g})$ (or more precisely, its central reduction $U_\lambda(\mathfrak{g})$) and the sheaf $D^\lambda_{G/B}$ of λ-twisted differential operators on G/B. In our present story, $\mathrm{Sym}_n(\mathbb{C}^2)$ is an analog of the nilpotent cone, $\mathrm{Hilb}_n(\mathbb{C}^2)$ is an analog of $T^*(G/B)$.

Yi has used quantum Hamiltonian reduction to quantize an affine algebraic variety $\mathrm{Sym}_n(\mathbb{C}^2)$ that is obtained by classical Hamiltonian reduction. The result of quantization is a filtered algebra that should be thought as an analog of $U_\lambda(\mathfrak{g})$. What I want to do in this talk is to quantize the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ getting an analog of $D^\lambda_{G/B}$. It is obtained by a GIT Hamiltonian reduction, similarly on one hand to $\mathrm{Hilb}_n(\mathbb{C}^2)$ and on the other hand to the quantum Hamiltonian reductions constructed by Yi. The variety $\mathrm{Hilb}_n(\mathbb{C}^2)$ is not affine and so is not given by a single algebra, rather by a sheaf of algebras. A quantization is therefore should also be a sheaf. We will start by describing this new formalism. First, we will slightly generalize the quantization formalism for graded algebras, instead of $\mathbb{Z}_{\geqslant 0}$-graded algebras we will consider $\mathbb{Z}$-graded ones. Then we will define a quantization of an algebraic symplectic variety (although the formalism makes sense for all Poisson schemes) that will be a sheaf. Next we will see how this formalism is compatible with the usual quantization formalism for affine algebraic varieties (i.e., for algebras). To pass from a quantum algebra to a quantum sheaf we will use a version of the localization known as a *microlocalization*. With this in hand, we will develop the formalism of GIT quantum Hamiltonian reduction needed to quantize $\mathrm{Hilb}_n(\mathbb{C}^2)$.

2. Quantizations of algebras revisited

We start with a $\mathbb{Z}$-graded unital associative commutative algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$ equipped with a Poisson bracket $\{\cdot,\cdot\}$ of degree -1, i.e., $\{A_i, A_j\} \subset A_{i+j-1}$.

Definition 2.1. A (filtered) quantization $\mathcal{A}$ of A is a filtered unital associative algebra $\mathcal{A} = \bigcup_{i\in\mathbb{Z}} \mathcal{A}_{\leqslant i}$ satisfying $[\mathcal{A}_{\leqslant i}, \mathcal{A}_{\leqslant j}] \subset \mathcal{A}_{i+j-1}$ with fixed identification $\mathrm{gr}\,\mathcal{A} \cong A$ of graded Poisson algebras. Finally, we require that $\mathcal{A}$ is complete and separated in the topology defined by the filtration, i.e., the natural homomorphism $\mathcal{A} \to \varprojlim_{i\to-\infty} \mathcal{A}/\mathcal{A}_{\leqslant i}$ is an isomorphism. For those who still remembers Analysis: the last condition simply means that every fundamental (Cauchy) sequence converges to a unique limit.

Notice that all conditions but the last one has appeared before. Let us motivate the condition of being complete and separated.

1) In the $\mathbb{Z}_{\geqslant 0}$-graded/filtered situation it is obviously satisfied.

2) It allows to do "induction by degree". I don't want to explain what this means but rather will give exercises where this is useful starting from the following.

Exercise 2.1. Let $\mathcal{A}, \mathcal{A}'$ be two quantizations of A and let $\varphi : \mathcal{A} \to \mathcal{A}'$ be a filtration preserving homomorphism. If the induced homomorphism $\operatorname{gr}\varphi : A \to A$ is an isomorphism, then so is φ.

3) Even if we remove the condition, $\varprojlim_{i\to-\infty} \mathcal{A}/\mathcal{A}_{\leqslant i}$ is always a quantization.

4) Last not least, our notion of a filtered quantization is compatible with that of a formal quantization that was introduced in Yi's lecture. Namely, let $\mathcal{A}_\hbar$ be a formal quantization of A (in particular, it is complete and separated in the $\hbar$-adic topology). Assume in addition that $\mathcal{A}_\hbar$ is equipped with a $\mathbb{C}^\times$-action by algebra automorphisms such that $\hbar$ has degree 1 and the isomorphism $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \xrightarrow{\sim} A$ (which is a part of a quantization data) is $\mathbb{C}^\times$-equivariant.

Exercise 2.2. $\mathcal{A}_{\hbar,fin}/(\hbar-1)\mathcal{A}_{\hbar,fin}$ is a filtered quantization of A.

Conversely, the $\hbar$-adic completion $\widehat{R}(\mathcal{A})$ of the Rees algebra of $\mathcal{A}$ is a formal quantization equipped with a $\mathbb{C}^\times$-action. The correspondences $\mathcal{A}_\hbar \mapsto \mathcal{A}_{\hbar,fin}/(\hbar-1)\mathcal{A}_{\hbar,fin}$ and $\mathcal{A} \mapsto \widehat{R}(\mathcal{A})$ are mutually inverse.

3. Quantizations of varieties

Let X be a smooth variety equipped with a symplectic form ω and a $\mathbb{C}^\times$-action such that $t.\omega = t\omega$. We want to quantize (more precisely, to produce a filtered deformation) of the structure sheaf $\mathcal{O}_X$.

We need to deal with the fact that not all sections of $\mathcal{O}_X$ are graded. Basically, the algebra $\Gamma(U, \mathcal{O}_X)$ is naturally graded if and only if U is $\mathbb{C}^\times$-stable. We know how to quantize $\Gamma(U, \mathcal{O}_X)$ when U is affine and $\mathbb{C}^\times$-stable (let us note that the latter algebra is only $\mathbb{Z}$-graded, in general, that is why we developed the formalism in the previous section). As the following result shows, there are sufficiently many $\mathbb{C}^\times$-stable open affine subsets.

Theorem 3.1 (Sumihiro). *Every $x \in X$ has a $\mathbb{C}^\times$-stable open affine neighborhood.*

By a conical topology, we mean the topology where "open" means Zariski open+ $\mathbb{C}^\times$-stable. This topology is rich enough to work with algebraic varieties, as the theorem above shows. And we can view $\mathcal{O}_X$ as a sheaf of graded algebras in the conical topology.

Definition 3.2. A quantization of $\mathcal{O}_X$ is a sheaf (in conical topology) $\mathfrak{A} = \bigcup_{i\in\mathbb{Z}} \mathfrak{A}_{\leqslant i}$ of filtered algebras (meaning in particular that the filtrations glue together) such that $\operatorname{gr}\mathfrak{A} = \mathcal{O}_X$ and $\mathfrak{A} = \varprojlim_{i\to-\infty} \mathfrak{A}/\mathfrak{A}_{\leqslant i}$.

Recall that it is enough to define the sections of a sheaf on a base of topology to recover the whole sheaf. In our case, it is enough to define the sections of $\mathfrak{A}$ on open $\mathbb{C}^\times$-stable affine subsets U. Those are filtered quantizations of $\Gamma(U, \mathcal{O}_X)$.

4. Microlocalization

Now we have two notions of quantizations of an affine variety X: by a sheaf $\mathfrak{A}$ and by an algebra $\mathcal{A}$. We need to introduce ways to pass from one formalism to another. Passing from sheaves to algebras is straightforward.

Exercise 4.1. If $\mathfrak{A}$ is a quantization of $\mathcal{O}_X$, then $\Gamma(\mathfrak{A})$ is a quantization of $\mathbb{C}[X]$ (yet again, completeness is important here).

To pass from an algebra to a sheaf (in conical topology) one needs some version of the localization. First of all, let us notice that the principal open subsets X_f with homogeneous f form a base of topology and, moreover, $X_f \cap X_g = X_{fg}$. So to define a pre-sheaf $\mathfrak{A}$ it is enough to define the sections $\mathfrak{A}(X_f)$ together with homomorphisms $\mathfrak{A}(X_f) \to \mathfrak{A}(X_{fg})$. Then one still needs to check that the result is a sheaf.

The sections $\mathfrak{A}(X_f)$ will be constructed via so called *microlocalization*. Namely, consider the Rees algebra $R(\mathcal{A})$ and its quotient $R_k(\mathcal{A}) := R(\mathcal{A})/(\hbar^k)$ so that $\widehat{R}(\mathcal{A}) := \varprojlim_{k\to+\infty} R_k(\mathcal{A})$. Let S_k be the preimage of $\{f^n, n \geqslant 0\} \subset A = R_1(\mathcal{A})$ in $R_k(\mathcal{A})$. We claim that there is a localization $R_k(\mathcal{A})[S_k^{-1}]$, i.e., an algebra $R_k(\mathcal{A})(S_k^{-1})$ together with homomorphism $\iota : R_k(\mathcal{A}) \to R_k(\mathcal{A})[S_k^{-1}]$ such that $\iota(S_k)$ is invertible and $(R_k(\mathcal{A})[S_k^{-1}], \iota)$ is universal with this property.

There is one problem with noncommutative localization: in order to do algebra operations we need every left fraction $s^{-1}a$ to be also a right fraction bt^{-1} and vice versa. Unlike in the commutative case, this is not automatic. The coincidence of the left fractions and the right fractions is known as the Ore condition.

However, in our case, this is not a problem. Indeed, every element $s \in S_k$ has the property that $\mathrm{ad}^k(s) = 0$ (here we use that A is commutative).

Exercise 4.2. Suppose that B is an algebra and S a multiplicative set such that $\mathrm{ad}^k(s) = 0$ for any s. Then S satisfies the Ore condition.

From the universality, we have a natural (surjective) homomorphism $R_{k+1}(\mathcal{A})[S_{k+1}^{-1}] \to R_k(\mathcal{A})[S_k^{-1}]$. So we can form the inverse limit $\widehat{R}(\mathcal{A})[f^{-1}] := \varprojlim_{k\to+\infty} R_k(\mathcal{A})[S_k^{-1}]$. By the construction, this is a formal quantization of $A[f^{-1}]$. Also it comes equipped with a $\mathbb{C}^\times$-action (it is present on every step of our construction). So we can pass to a filtered quantization as was described above, it will be denoted by $\mathcal{A}[f^{-1}]$. There is a natural homomorphism $\mathcal{A} \to \mathcal{A}[f^{-1}]$ of filtered algebras. Moreover, we have $\mathcal{A}[(fg)^{-1}] = \mathcal{A}[f^{-1}][g^{-1}] = \mathcal{A}[g^{-1}][f^{-1}]$. It is then straightforward to check that the sections $\mathfrak{A}(X_f) = \mathcal{A}[f^{-1}]$ do constitute a pre-sheaf. With a bit more care one checks that this pre-sheaf is a sheaf. Moreover, the correspondences $\mathcal{A} \mapsto \mathfrak{A}, \mathfrak{A} \mapsto \mathcal{A}$ are inverse to each other.

Exercise 4.3. Prove these boring facts.

5. Quantum Hamiltonian reduction

Our setting here is as follows. We start with a vector space U. Form $V := U \oplus U^*$, it is a symplectic vector space, let ω denote the form. Let G be a reductive subgroup of $\mathrm{Sp}(V)$. Let $\mu : V \to \mathfrak{g}^*$ denote the standard moment map. Let θ be a character such that G acts on $\mu^{-1}(0)^{\theta-ss}$ freely. We can form the Hamiltonian reduction $V///^\theta G := \mu^{-1}(0)//^\theta G$ and it is a smooth symplectic variety. Moreover, it comes equipped with a torus action like in Section 3. In more detail, $\mathbb{C}^\times$ acts on V by $t.(u, u^*) = (u, t^{-1}u^*)$. The action commutes with G and μ has degree 1. So it restricts to $\mu^{-1}(0)$. The set $V^{\theta-ss}$ is also $\mathbb{C}^\times$-stable. So $\mathbb{C}^\times$ acts on $\mu^{-1}(0)^{\theta-ss}$ and commutes with G so descends to $V///^\theta G$ rescaling the symplectic form.

Now our goal is to quantize $V///^\theta G$. The idea is the same as with $V///G$: use the quantum Hamiltonian reduction for the action of G on $D(U)$, that is a quantization of $\mathbb{C}[T^*U]$.

Again, $V///^\theta f$ is covered by the affine open subsets of the form $V_f///G$ for θ^n-semiinvariant homogeneous f. We want a sheaf $\mathfrak{A}_\lambda$ depending on $\lambda \in \mathfrak{g}^{*G}$. What we need to do is to define the sections $\mathfrak{A}_\lambda(V_f///G)$ (filtered algebras) together with homomorphisms $\iota_{f,g}:\mathfrak{A}_\lambda(V_f///G)\to\mathfrak{A}_\lambda(V_{fg}///G)$.

We set $\mathfrak{A}_\lambda(V_f///G):=D(U)[f^{-1}]///_\lambda G$, we we write $///_\lambda$ for the quantum Hamiltonian reduction. In more detail:

$$D(U)[f^{-1}]///_\lambda G:=(D(U)[f^{-1}]/D(U)[f^{-1}]\{x_U-\langle\lambda,x\rangle\})^G.$$

We remark that G does act on $D(U)[f^{-1}]$ because f is G-semi-invariant and that $x\mapsto x_U$ is still a quantum comoment map. The map $\mathfrak{A}_\lambda(V_f///G)\to\mathfrak{A}_\lambda(V_{fg}///G)$ is just induced by $D(U)[f^{-1}]\to D(U)[(fg)^{-1}]$, this makes sense because the latter is equivariant.

Exercise 5.1. Check that $\mathfrak{A}_\lambda$ is indeed a quantization. Well, I understand this is not pleasant...

The homomorphisms $D(U)///_\lambda G\to D(U)[f^{-1}]///_\lambda G$ give rise to a homomorphism $D(U)///_\lambda G\to\Gamma(\mathfrak{A}_\lambda)$.

Now consider the case of most interest for us: $U=\operatorname{End}(\mathbb{C}^n)\oplus\mathbb{C}^n, G=\operatorname{GL}_n(\mathbb{C})$. For brevity, put $X:=\operatorname{Hilb}_n(\mathbb{C}^2), X_0:=\operatorname{Sym}_n(\mathbb{C}^2)$. We claim that $H^i(X,\mathfrak{A}_\lambda)=0$ for $i>0$, while $H^0(X,\mathfrak{A}_\lambda)=D(U)///_\lambda G$. Consider the Hilbert-Chow morphism. It is a resolution of singularities onto a normal variety, and the source variety is symplectic. It follows that $\mathbb{C}[X]=\mathbb{C}[X_0]$ and $H^i(X,\mathcal{O}_X)=0$ for $i>0$ (the latter is a special case of the Grauert-Riemenschneider theorem). We claim that the latter implies $\operatorname{gr}\Gamma(\mathfrak{A}_\lambda)=\mathbb{C}[X], H^i(X,\mathfrak{A}_\lambda)=0$ for $i>0$. The proof is an exercise (and once again, a hint: use a descending induction on the filtration degree together with completeness). Now consider the natural morphism $D(U)///_\lambda G\to\Gamma(\mathfrak{A}_\lambda)$. This homomorphism preserves the filtration by the construction, while its associated graded is the natural homomorphism $\mathbb{C}[X_0]\to\mathbb{C}[X]$. Since the latter is an isomorphism, we see that $D(U)///_\lambda G\to\Gamma(\mathfrak{A}_\lambda)$.