$\mathcal{D}$-modules on flag varieties and localization of $\mathfrak{g}$-modules.

José Simental.

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1 (Twisted) Differential operators and $\mathcal{D}$-modules.

1.1 Differential Operators.

Let X be an algebraic variety, and $\mathcal{R}$ a sheaf of rings on X. For $\mathcal{R}$-modules $\mathcal{F}$ and $\mathcal{G}$, define the sheaf $\mathcal{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$ as follows. For any open set $U \subset X$, the sections of $\mathcal{H}om_{\mathcal{R}}(\mathcal{F}, \mathcal{G})$ on U are $\mathrm{Hom}_{\mathcal{R}|_{\mathcal{U}}}(\mathcal{F}|_U, \mathcal{G}|_U)$. For any algebraic variety X, let $\mathbb{C}_X$ be the locally constant sheaf of functions to the complex numbers $\mathbb{C}$.

Now assume X is a smooth algebraic variety. Let $\mathcal{O}_X$ be its sheaf of regular functions and $\mathcal{V}_X$ be its sheaf of vector fields, that is, $\mathcal{V}_X := Der_{\mathbb{C}}(\mathcal{O}_X) := \{\theta \in \mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X) : \theta(fg) = f\theta(g) + \theta(f)g, f, g \in \mathcal{O}_X\}$. The sheaf of differential operators in X, $\mathcal{D}_X$ is, by definition, the subsheaf (of associative rings) of $\mathcal{E}nd_{\mathbb{C}_X}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\mathcal{V}_X$.

Theorem 1.1 *In the setup of the previous paragraph, if $\dim X = n$, then for every point $p \in X$ there exists an affine open neighborhood U of p, $x_1, \dots, x_n \in \mathcal{O}_X(U)$ and $\partial_1, \dots, \partial_n \in \mathcal{V}_X(U)$ such that $[\partial_i, \partial_j] = 0$, $\partial_i(x_j) = \delta_{ij}$ and $\mathcal{V}_X(U)$ is free over $\mathcal{O}_X(U)$ with basis $\partial_1, \dots, \partial_n$. Such a set $\{x_i, \partial_i\}$ is said to be a local coordinate system.*

Proof. Let $\mathfrak{m}_p$ be the maximal ideal of $\mathcal{O}_{X,p}$. Since X is smooth of dimension n, there exist functions $x_1, \dots, x_n$ generating $\mathfrak{m}_p$. Then, $dx_1, \dots, dx_n$ is a basis of $\Omega_{X,p}$, where Ω_X is the cotangent sheaf of X. Take $\partial_1, \dots, \partial_n \in \mathcal{V}_X$ to be a dual basis. $\square$

Remark 1.2 *A local coordinate system does not give an isomorphism from U to an affine subvariety of $\mathbb{C}^n$. We only have an étale morphism $U \to \mathbb{C}^n$.*

Theorem 1.3 *Let U be an affine open subset of X and $\{x_i, \partial_i\}$ a local coordinate system on U. Then, any differential operator of order $\leq k$ on U can be uniquely written in the form*

$$\sum_{k_1 + \cdots + k_n \leq k} f_{k_1, \dots, k_n} \partial_1^{k_1} \cdots \partial_n^{k_n}.$$

Where $f_{k_1, \dots, k_n} \in \mathcal{O}_X(U)$.

Note that $\mathcal{D}_X$ can be described by generators and relations as follows: it is generated by $\mathcal{O}_X$ and $\mathcal{V}_X$ with relations:
- $f_1 \cdot f_2 = f_1 f_2$,
- $f \cdot \xi = f\xi$,
- $\xi \cdot f = \xi(f) + f\xi$,
- $\xi_1 \cdot \xi_2 - \xi_2 \cdot \xi_1 = [\xi_1, \xi_2]$.

Where $f, f_1, f_2 \in \mathcal{O}_X$ and $\xi, \xi_1, \xi_2 \in \mathcal{V}_X$.

1.2 A filtration on $\mathcal{D}_X$.

Let A be a commutative ring and let M, N be A-modules. Define A-modules $Diff_A^{\leq n}(M, N) \subseteq \mathrm{Hom}_{\mathbb{Z}}(M, N)$ inductively by:

1. $Diff_A^{\leq 0}(M, N) := \mathrm{Hom}_A(M, N)$.
2. $Diff_A^{\leq n+1} := \{\theta \in \mathrm{Hom}_{\mathbb{Z}}(M, N) : [f, \theta] \in Diff_A^{\leq n}(M, N) \text{ for every} f \in A\}$.

Set $Diff_A(M, N) = \bigcup_n Diff_A^{\leq n}(M, N)$.

Remark 1.4 *If, moreover, A is a k-algebra, then we take the differential morphisms $Diff_A^{\leq n}(M, N)$ inside* $\mathrm{Hom}_k(M, N)$.

Definition 1.5 *Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{O}_X$ modules. Define $\mathcal{D}_X(\mathcal{M}, \mathcal{N})$ by gluing $Diff_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ on affine open subsets, that is, for any affine open subset $U \subseteq X$,*

$$\Gamma(U, \mathcal{D}_X(\mathcal{M}, \mathcal{N})) := Diff_{\mathcal{O}(U)}(\mathcal{M}(U), \mathcal{N}(U)).$$

Note that $\mathcal{D}_X = \mathcal{D}_X(\mathcal{O}_X, \mathcal{O}_X)$. It follows that $\mathcal{D}_X$ is a filtered sheaf. In local coordinates $(U, \{x_i, \partial_i\})$, $F_i\,\mathcal{D}_X(U) = \sum \mathcal{O}_U \partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_n^{\alpha_n}, \sum \alpha_j \leq i$, that is, $F_i\,\mathcal{D}_X(U)$ is given by differential operators of order $\leq i$. Note that if $P \in F_l\,\mathcal{D}_X$, $Q \in F_m\,\mathcal{D}_X$, then $[P, Q] \in F_{l+m-1}\,\mathcal{D}_X$. Then, $\operatorname{gr}\mathcal{D}_X$ is a sheaf of commutative algebras. Let us look closer at $\operatorname{gr}\mathcal{D}_X$. Take a local coordinate system $(U, \{x_i, \partial_i\})$. Set $\xi_i = \bar{\partial}_i \in F_1\,\mathcal{D}_X(U)/F_0\,\mathcal{D}_X(U)$. Then, $\operatorname{gr}\mathcal{D}_X(U) = \mathcal{O}_U[\xi_1, \xi_2, \dots, \xi_n]$. Let $\pi : T^*X \to X$ be the projection. Regard $\xi_1, \dots, \xi_n$ as the local coordinate system of the cotangent space $\bigoplus \mathbb{C}dx_i$. Then, $\mathcal{O}_U[\xi_1, \dots, \xi_n]$ is identified with $\pi_*\mathcal{O}_{T^*X}(U)$. These identifications are natural and they can be glued together, so that we have an isomorphism,

$$\operatorname{gr}\mathcal{D}_X \cong \pi_*\mathcal{O}_{T^*X}.$$

A natural isomorphism can be constructed as follows. First of all, note that $F_0\,\mathcal{D}_X = \mathcal{O}_X$. This follows easily by the description of $\mathcal{D}_X$ in local coordinates. Alternatively, we have a monomorphism $\mathcal{O}_X \hookrightarrow F_0\,\mathcal{D}_X$. An inverse map is given by $F_0\,\mathcal{D}_X \ni P \mapsto P(1)$. Now we see that $F_1\,\mathcal{D}_X = \mathcal{O}_X \oplus \mathcal{V}_X$. Note that, if $P \in F_1\,\mathcal{D}_X$, then $f \mapsto [P, f]$ is a derivation on $\mathcal{O}_X$. Then, we get a morphism $F_1\,\mathcal{D}_X \to \mathcal{O}_X \oplus \mathcal{V}_X$, $P \mapsto (P(1), [P, \bullet])$. This is an isomorphism.

So we have isomorphisms $\mathcal{O}_X = \operatorname{gr}_0\mathcal{D}_X$, $\mathcal{V}_X = \operatorname{gr}_1\mathcal{D}_X$. These give rise to a map $\pi_*\mathcal{O}_{T^*X} \to \operatorname{gr}\mathcal{D}_X$. By the above considerations in local coordinates, this is an isomorphism.

1.3 Twisted differential operators.

Definition 1.6 *Let $\mathcal{D}$ be a sheaf of rings on X that admits an inclusion $\iota : \mathcal{O}_X \hookrightarrow \mathcal{D}$. We say that $\mathcal{D}$ is a sheaf of twisted differential operators (TDO) if the embedding $\mathcal{O}_X \overset{\iota}{\hookrightarrow} \mathcal{D}$ is locally isomorphic to the standard embedding $\mathcal{O}_X \overset{\iota_X}{\hookrightarrow} \mathcal{D}_X$.*

For any line bundle $\mathcal{L}$ on X, $\mathcal{D}_X^{\mathcal{L}} = \mathcal{D}_X(\mathcal{L}, \mathcal{L})$ is a TDO, as $\mathcal{L}$ is locally isomorphic to $\mathcal{O}_X$.

One can also get a sheaf of TDO from a closed 1-cocycle $\alpha \in \Omega^1_{cl}$ as follows. Consider an open cover $X = \bigcup U_i$ and a 1-cocycle $\alpha = (\alpha_{ij}) \in \Omega^1_{cl}$. Then, $\mathcal{D}(U_i) := \mathcal{D}_X(U_i)$, and the transition function from U_j to U_i maps a vector field ξ to $\xi + \langle \xi, \alpha_{ij} \rangle$. In fact, by this procedure we can get all TDO.

Proposition 1.7 *Let $\varphi : \mathcal{D}_X \to \mathcal{D}_X$ be an endomorphism such that $\varphi|_{\mathcal{O}_X} = \mathrm{id}$. Then, there exists $\omega \in \Omega^1_{cl}$ such that $\varphi(\theta) = \theta - \omega(\theta)$ for any vector field $\theta \in \mathcal{V}_X$. Moreover, φ is completely determined by ω and it is an automorphism of $\mathcal{D}_X$.*

Proof. Let $f \in \mathcal{O}_X$, $\theta \in \mathcal{V}_X$. Then, $[\varphi(\theta), f] = \varphi([\theta, f]) = \varphi(\theta(f)) = \theta(f)$. Then,$[\varphi(\theta), f](1) = \theta(f)(1)$, so $\varphi(\theta)(f) = \theta(f) + f\varphi(\theta)(1)$. Set $\omega(\theta) = -\varphi(\theta)(1)$. This is a 1-form. Note that $\omega([\theta, \eta]) = -\varphi([\theta, \eta])(1) = -[\varphi(\theta), \varphi(\eta)](1) = \varphi(\theta)(\omega(\eta)) - \varphi(\eta)(\omega(\theta)) = \theta(\omega(\eta)) - \eta(\omega(\theta))$. It follows that ω is closed. The last statement of the Proposition is clear. □

The following is then an exercise in Čech cohomology.

Proposition 1.8 *TDO on a smooth variety X are classified by the first cohomology $H^1_{Zar}(X, \Omega^1_{cl})$.*

To prove Proposition 1.8, one uses a covering U_i of X by open affine subsets such that the sheaf $\mathcal{D}$ is locally isomorphic to $\mathcal{D}_X$ in each U_i, and the transition morphisms in each intersection $U_i \cap U_j$.

Remark 1.9 *If $\mathcal{L}$ is a line bundle, then we know that $\mathcal{D}_X^{\mathcal{L}}$ is a TDO. The Picard group is naturally isomorphic to $H^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ is the subsheaf of invertible elements in $\mathcal{O}_X$. There exists a homomorphism $\mathcal{O}_X^* \to \Omega^1_{cl}$ given by taking the logarithmic derivative, $f \mapsto dlog(f) = f^{-1}df$, which induces morphisms $H^p(dlog) : H^p(X, \mathcal{O}_X^*) \to H^p(X, \Omega^1_{cl})$. It is an exercise to show that the 1-cocycle corresponding to $\mathcal{D}_X^{\mathcal{L}}$ is $H^1(dlog)(\mathcal{L})$.*

Note that it follows that any TDO $\mathcal{D}$ is filtered, and moreover, $\operatorname{gr}\mathcal{D} = \pi_*\mathcal{O}_{T^*X}$.

1.4 Homogeneous TDO.

In this subsection, we assume X is a homogeneous G-variety, where G is a semisimple algebraic group. Later, we will specialize the results of this subsection to the case $X = G/B$, where B is a Borel subgroup of G. Let $\mathfrak{g} := \mathrm{Lie}(G)$. Recall that the *universal enveloping algebra* of $\mathfrak{g}$ is $\mathcal{U}(\mathfrak{g}) = T\,\mathfrak{g}\,/(u \otimes v - v \otimes u - [u,v], u, v \in \mathfrak{g})$.

Differentiating the action of G on $\mathcal{O}_X$, we get a G-equivariant Lie algebra homomorphism $\tau : \mathfrak{g} \to \Gamma(X, \mathcal{V}_X)$, that can be extented to a G-equivariant algebra homomorphism $\tau : \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X)$. More generally, let $\mathcal{D}$ be a TDO on X with an algebraic action γ of G on $\mathcal{D}$ and a G-equivariant algebra homomorphism $\alpha : \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D})$ satisfying the following conditions:

1. The multiplication in $\mathcal{D}$ is G-equivariant.
2. For $\xi \in \mathfrak{g}$, we have an equality $\xi_X = [\alpha(\xi), \bullet]$, where ξ_X is the the derivation induced by differentiating the G-action on $\mathcal{D}$.

In this case, we say that α is a *quantum comoment map* and we call $(\mathcal{D}, \gamma, \alpha)$ a *homogeneous sheaf of twisted differential operators (HTDO)*. For example, $(\mathcal{D}_X, \gamma_X, \tau)$ is an HTDO, where γ_X is the natural action of G on $\mathcal{D}_X$.

Our next goal is to classify HTDO. To do this, we study $(\mathcal{D}_X, \gamma_X, \tau)$ more closely.

Consider the trivial bundle $\mathfrak{g} \times X \twoheadrightarrow X$, and its sheaf of sections $\mathfrak{g}^\circ := \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$. We can define a bracket on $\mathfrak{g}^\circ$ by

$$[f \otimes \xi, g \otimes \eta] := f\tau(\xi)g \otimes \eta - g\tau(\eta)f \otimes \xi + fg \otimes [\xi, \eta].$$

This makes $\mathfrak{g}^\circ$ a sheaf of Lie algebras. It is also an $\mathcal{O}_X$-bimodule. Note, however, that the Lie bracket is not $\mathcal{O}_X$-bilinear. We can extend $\tau : \mathfrak{g} \to \Gamma(X, \mathcal{V}_X)$ to $\tau^\circ : \mathfrak{g}^\circ \to \mathcal{V}_X$ by $\tau^\circ(f \otimes \xi) = f\tau(\xi)$. Note that this satisfies the following identity:

$$[f \otimes \xi, h(g \otimes \eta)] = \tau^\circ(f \otimes \xi)(h)(g \otimes \eta) + h[f \otimes \xi, g \otimes \eta]. \tag{1}$$

where $f, g, h \in \mathcal{O}_X$, $\xi, \eta \in \mathfrak{g}$.

Let $\mathcal{U}^\circ$ be the sheaf of algebras generated by $\mathfrak{g}^\circ$ and $\mathcal{O}_X$ subject to the following relations. Denote by ι both maps $\mathfrak{g}^\circ \to \mathcal{U}^\circ$, $\mathcal{O}_X \to \mathcal{U}^\circ$ taking a section to the corresponding generator.

(a) $\iota(fg) = \iota(f) \cdot \iota(g)$.

(b) $\iota([a, b]) = \iota(a) \cdot \iota(b) - \iota(b) \cdot \iota(a)$.

(c) $\iota(fa) = \iota(f) \cdot \iota(a)$.

(d) $\iota(a) \cdot \iota(f) - \iota(f) \cdot \iota(a) = \iota(\tau^\circ(a)(f))$.

Where $f, g \in \mathcal{O}_X$, $a, b \in \mathfrak{g}^\circ$.

Remark 1.10 *In the formalism of Lie algebroids, $\mathfrak{g}^\circ$ is a Lie algebroid with $\tau^\circ : \mathfrak{g}^\circ \to \mathcal{V}_X$ the anchor map (this follows from Equation (1)), and $\mathcal{U}^\circ$ is the universal enveloping algebra of $\mathfrak{g}^\circ$.*

Note that τ° induces a map $\tau^\circ : \mathcal{U}^\circ := \mathcal{U}(\mathfrak{g}^\circ) \to \mathcal{D}_X$. This map is an epimorphism. This follows from the fact that $\mathcal{D}_X$ is generated by $\mathcal{V}_X$ and $\mathcal{O}_X$, and the following Proposition.

Proposition 1.11 *The morphism $\tau^\circ : \mathfrak{g}^\circ \to \mathcal{V}_X$ is an epimorphism.*

Proof. Since $\mathfrak{g}^\circ$ and $\mathcal{V}_X$ are locally free, it suffices to show that the induced map on the geometric fibers of $\mathfrak{g}^\circ$ and $\mathcal{V}_X$ is surjective. But this is clear. □

So $\mathcal{D}_X = \mathcal{U}^\circ / \mathrm{Ker}(\tau^\circ)$. We will see that any HTDO admits a similar description. To do that, we find $\mathrm{Ker}(\tau^\circ)$. Of course, first we need an explicit characterization of $\mathcal{U}^\circ$.

Proposition 1.12 *As a sheaf, $\mathcal{U}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$. The product is given by*

$$(f \otimes \xi)(g \otimes \eta) = f\tau(\xi)g \otimes \eta + fg \otimes \xi\eta.$$

Where $f, g \in \mathcal{O}_X$, $\xi \in \mathfrak{g}$, $\eta \in \mathcal{U}(\mathfrak{g})$.

Proof. Follows from the relations defining $\mathcal{U}^\circ$. $\square$

Note that it also follows that $\mathcal{U}^\circ$ is a filtered algebra, by setting $F_p\mathcal{U}^\circ := \mathcal{O}_X \otimes_{\mathbb{C}} F_p\mathcal{U}(\mathfrak{g})$, where $F_p\mathcal{U}(\mathfrak{g})$ is the standard filtration on $\mathcal{U}(\mathfrak{g})$. Note that τ° is a filtered morphism, that $F_0\mathcal{U}^\circ = \mathcal{O}_X$, $F_1\mathcal{U}^\circ = \mathcal{O}_X \oplus \mathfrak{g}^\circ$ and that $\mathcal{U}^\circ$ is generated by $F_1\mathcal{U}^\circ$ as a sheaf of algebras.

Let $\mathfrak{b}^\circ = \operatorname{Ker}(\tau^\circ : \mathfrak{g}^\circ \to \mathcal{V}_X)$, so that $\mathfrak{b}^\circ$ consists of those sections $\sum f_i \otimes \xi_i$ such that $\sum f_i \tau(\xi_i) = 0 \in \mathcal{V}_X$. Note that $\mathcal{J}_0 := \mathfrak{b}^\circ\mathcal{U}^\circ$ is a sheaf of two-sided ideals in $\mathcal{U}^\circ$: it is easy to see that, if $\sum f_i \otimes \xi_i in\, \mathfrak{b}^\circ$ and $\eta \in \mathfrak{g}$, then $[1 \otimes \eta, \sum f_i \otimes \xi_i] \in \mathfrak{b}^\circ$. Similarly, if $g \in \mathcal{O}_X$, then

$$\left[\sum f_i \otimes \xi_i, g \otimes 1\right] = \sum f_i \tau(\xi_i) g \otimes 1 = 0.$$

Alternatively, one can see that $\mathcal{J}_0$ is a sheaf of two-sided ideals from the following Proposition.

Proposition 1.13 $\operatorname{Ker}(\tau^\circ : \mathcal{U}^\circ \to \mathcal{D}_X) = \mathcal{J}_0$.

Proof. It is clear that $\mathcal{J}_0$ is contained in $\operatorname{Ker}(\tau)$. Moreover, for any $x \in X$, the geometric fiber $T_x(\mathcal{J}_0) = \mathfrak{b}_x^\circ\mathcal{U}(\mathfrak{g})$ is the kernel of the induced map from the geometric fiber $T_x(\mathcal{U}^\circ) = \mathcal{U}(\mathfrak{g})$ to the geometric fiber of $\mathcal{D}_X$ at x. The result follows. $\square$

Then, we have that $\mathcal{D}_X = \mathcal{U}^\circ/\mathcal{J}_0$.

Now we show that any HTDO admits a similar description. Let (D, γ, α) be an HTDO. Then, for every $\xi \in \mathfrak{g}$ and $f \in \mathcal{O}_{\mathrm{X}}$, $[\alpha(\xi), f] = \tau(\xi)f$. It follows that $\alpha : \mathfrak{g} \to F_1\mathcal{D}$. Note that we can also extend α to an algebra homomorphism $\alpha^\circ : \mathcal{U}^\circ \to \mathcal{D}$, $f \otimes \xi \mapsto f\alpha(\xi)$. Moreover, α° is filtered and $\operatorname{gr}\alpha^\circ = \operatorname{gr}\tau^\circ$, so $\alpha^\circ(\mathfrak{b}^\circ) \subseteq F_0\mathcal{D} = \mathcal{O}_{\mathrm{X}}$. Then, an HTDO determines a G-equivariant morphism from the G-homogeneous $\mathcal{O}_{\mathrm{X}}$-module $\mathfrak{b}^\circ$ to $\mathcal{O}_{\mathrm{X}}$.

Fix a point $x_0 \in X$. Let $B_0 = \operatorname{Stab}_X(x_0)$, and let $\mathfrak{b}_0$ be the fiber of $\mathfrak{b}^\circ$ at x_0. Then, B_0 acts on $\mathfrak{b}_0^*$. Let $I(\mathfrak{b}_0^*)$ be the space of B_0-invariants. There is a natural linear isomorphism between $I(\mathfrak{b}_0^*)$ and the space of G-equivariant morphisms σ of $\mathfrak{b}^\circ$ to $\mathcal{O}_{\mathrm{X}}$. Then, for each $\lambda \in I(\mathfrak{b}_0^*)$, let σ_λ be the associated G-equivariant morphism $\sigma_\lambda : \mathfrak{b}^\circ \to \mathcal{O}_{\mathrm{X}}$. Let $\varphi_\lambda : \mathfrak{b}^\circ \to \mathcal{U}^\circ$ be given by $\varphi_\lambda(s) = s - \sigma_\lambda(s)$. Let $\mathcal{J}_\lambda$ be the sheaf of two-sided ideals of $\mathcal{U}^\circ$ generated by the image of φ_λ. Finally, set $\mathcal{D}_{X,\lambda} := \mathcal{U}^\circ/\mathcal{J}_\lambda$.

Theorem 1.14 *$\mathcal{D}_{X,\lambda}$ is an HTDO. Moreover, the map $\lambda \mapsto \mathcal{D}_{X,\lambda}$ is an isomorphism between $I(\mathfrak{b}_0^*)$ and the set of isoclasses of HTDO on X.*

For a proof of Theorem 1.14 see, for example, [3, Section 1.2].

1.5 $\mathcal{D}$-modules.

By $\mathcal{D}$-module, we mean a left module over the sheaf $\mathcal{D}_X$ of differential operators. Clearly, every $\mathcal{D}$-module is also an $\mathcal{O}_X$-module. On the other hand, given an $\mathcal{O}_X$-module $\mathcal{M}$, giving a $\mathcal{D}$-module structure to $\mathcal{M}$ is equivalent to giving a $\mathbb{C}$-linear morphism $\nabla : \mathcal{V}_X \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{M}), \theta \mapsto \nabla_\theta$ satisfying the following conditions:

1. $\nabla_{f\theta}(s) = f\nabla_\theta(s)$.
2. $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s)$.
3. $\nabla_{[\theta_1,\theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$.

For $f \in \mathcal{O}_X$, $s \in \mathcal{M}$, $\theta, \theta_1, \theta_2 \in \mathcal{V}_X$.

Note that if $\mathcal{V}$ is a vector bundle, then ∇ defines a connection on $\mathcal{V}$, and condition 3. amounts to saying that this connection is flat.

A $\mathcal{D}$-module is called quasi-coherent if it is quasi-coherent as an $\mathcal{O}_X$-module. Denote by $\operatorname{Mod}_{qc}(\mathcal{D}_X)$ the category of quasi-coherent $\mathcal{D}$-modules.

An algebraic variety X is said to be $\mathcal{D}$-affine if the global sections functor $\Gamma : \operatorname{Mod}_{qc}(\mathcal{D}_X) \to \operatorname{Mod}(\Gamma(X, \mathcal{D}_X))$, $\mathcal{M} \mapsto \Gamma(X, \mathcal{M})$ is exact and if every module in $\operatorname{Mod}_{qc}(\mathcal{D}_X)$ is generated by its global sections. Clearly, every affine algebraic variety is $\mathcal{D}$-affine. We'll see later that, for a semisimple algebraic group G and a Borel subgroup B, the corresponding flag variety G/B is $\mathcal{D}$-affine.

2 $\mathcal{D}$-modules on G/B.

2.1 Universal enveloping algebras and the Harish-Chandra isomorphism.

Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Let Φ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$, and Φ^+ a choice of positive roots. Also, let $\{\alpha_1, \ldots, \alpha_l\} \subseteq \Phi^+$ be a choice of simple roots. For each $i = 1, \ldots, l$, let $\alpha_i^\vee \in \mathfrak{h}$ be the coroot of α_i, and let $\pi_i \in \mathfrak{h}^*$ be the fundamental weight corresponding to α_i, that is $\langle \pi_i, \alpha_j^\vee \rangle = \delta_{ij}$. Denote by $Q = \mathbb{Z}\Phi$ the root lattice, Q^+ its positive part, and $P = \sum \mathbb{Z}\pi_i$ the weight lattice. Finally, let $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^l \pi_i$ be the Weyl vector.

In this subsection, we want to study the center $\mathfrak{z} = Z(\mathcal{U}(\mathfrak{g}))$ of the universal enveloping algebra of $\mathfrak{g}$. Since, in particular, $\mathfrak{z}$ commutes with $\mathfrak{h}$, any element $z \in \mathfrak{z}$ acts as a scalar on any highest weight module $M = \mathcal{U}(\mathfrak{g})v_\lambda$ with highest weight $\lambda \in \mathfrak{h}^*$. Since every such module is a quotient of the Verma module Δ_λ, this scalar only depends on λ. Then, for any $z \in \mathfrak{z}$, we get a function $\Xi_z : \mathfrak{h}^* \to \mathbb{C}$.

We show that Ξ_z is polynomial. Indeed, it follows by the PBW theorem that $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \bigoplus (\mathfrak{n}_- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\,\mathfrak{n}_+)$. Since $\mathfrak{h}$ is abelian, $\mathcal{U}(\mathfrak{h}) = S\,\mathfrak{h}$, the symmetric algebra. Consider then the projection $\mathrm{pr} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$. It follows that, for any highest weight module $M = \mathcal{U}(\mathfrak{g})v_\lambda$, and any $u \in \mathcal{U}(\mathfrak{g})$, $uv_\lambda = \mathrm{pr}(u)(\lambda)v_\lambda +$ terms of lower weight. Then, $\Xi_z(\lambda) = \mathrm{pr}(z)(\lambda)$, so Ξ_z is indeed polynomial. We get a map $\Xi : \mathfrak{z} \to \mathcal{U}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$, which is clearly an algebra morphism.

Remark 2.1 *Even though the restriction of* $\mathrm{pr} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ *to* $\mathfrak{z}$ *is an algebra homomorphism, the map* pr *is not.*

Now consider the automorphism $t_{-\rho}$ of $\mathbb{C}[\mathfrak{h}^*]$, $t_{-\rho} : f(\lambda) \mapsto f(\lambda - \rho)$, and let $HC : \mathfrak{z} \to S(\mathfrak{h})$ be $HC = t_{-\rho}\Xi$, so that any $z \in \mathfrak{z}$ acts on any highest weight representation with highest weight λ by $HC(z)(\lambda + \rho)$. The reason for twisting the morphism Ξ is that HC has its image in $S(\mathfrak{h})^W$. Indeed, it is known that, for any integral dominant weight λ and any $w \in W$, there exists a nonzero morphism $\Delta_{w*\lambda} \to \Delta_\lambda$, where $w * \lambda = w(\lambda + \rho) - \rho$ is called the *ρ-shifted action* of W on $\mathfrak{h}$. It follows that $\Xi_z(\lambda) = \Xi_z(w * \lambda)$, or, equivalently, that $HC(z)(\lambda) = HC(z)(w\lambda)$. Since the lattice of integral dominant weights is dense in Zariski topology, it follows that $HC(z) \in S(\mathfrak{h})^W$.

Theorem 2.2 *The morphism* $HC : \mathfrak{z} \to S(\mathfrak{h})^W$ *is an isomorphism. It is called the Harish-Chandra isomorphism.*

A strategy to prove Theorem 2.2 is to compare HC with the Chevalley isomorphism. Recall that this is an isomorphism $\mathbf{res} : S(\mathfrak{g})^G \to S(\mathfrak{h})^W$ that is given by restriction of a polynomial map in $\mathfrak{g}^*$ to $\mathfrak{h}^*$. On the other hand, we have an isomorphism of $\mathfrak{g}$-modules $\mathbf{sym} : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, given by

$$\mathbf{sym} : x_1 \ldots x_n \mapsto \frac{1}{n!} \sum_{s \in \mathfrak{S}_n} x_{s(1)} x_{s(2)} \ldots x_{s(n)}.$$

This map induces an isomorphism of vector spaces $\mathbf{sym} : S(\mathfrak{g})^G \to \mathfrak{z}$ whenever G is connected simply-connected. We have the following diagram,

$$\begin{tikzcd} & S(\mathfrak{g})^G \arrow[dl, "\mathbf{sym}"'] \arrow[dr, "\mathbf{res}"] & \\ \mathfrak{z} \arrow[rr, "HC"] & & S(\mathfrak{h})^W. \end{tikzcd}$$

This diagram is not commutative. However, it is commutative 'up to lower degree terms', as follows. Recall that each of the vector spaces $\mathfrak{z}$, $S(\mathfrak{g})^G$ and $S(\mathfrak{h})^W$ are filtered. Then, we have that, for any $p \in (S^n(\mathfrak{g}))^G$, $HC(\mathbf{sym}(p)) \equiv \mathbf{res}(p)$ mod $S^{n-1}(\mathfrak{h})^W$. Note that it suffices to show that $\mathrm{pr}(\mathbf{sym}(p)) \equiv \mathbf{res}(p)$ mod $S^{n-1}(\mathfrak{h})^W$. In fact, a more general statement holds.

Lemma 2.3 *For any* $p \in S^n(\mathfrak{g})$, $\mathrm{pr}(\mathbf{sym}(p)) \equiv \mathbf{res}(p) \mod S^{n-1}(\mathfrak{h})$.

Proof. Let $p = (\prod_{\alpha \in \Phi^+} f_\alpha^{k_\alpha})(\prod_i h_i^{n_i})(\prod_{\alpha \in \Phi^+} e_\alpha^{m_\alpha}) \in S(\mathfrak{g})^n$. Then,

$$\mathrm{pr}(\mathbf{sym}(p)) \equiv \begin{cases} \prod_i h_i^{n_i}, & \text{if } m_\alpha, k_\alpha = 0. \\ 0 & \text{otherwise} \end{cases} \equiv \mathbf{res}(p) \mod S^{n-1}(\mathfrak{h}).$$

The result follows. $\square$

An algebra homomorphism $\mathfrak{z} \to \mathbb{C}$ is called a *central character*. For any $\lambda \in \mathfrak{h}^*$, define a central character $\chi_\lambda : \mathfrak{z} \to \mathbb{C}$ by defining $\chi_\lambda(z) = HC(z)(\lambda)$. Theorem 2.2 has the following easy consequence.

Corollary 2.4 *Any central character coincides with χ_λ for some $\lambda \in \mathfrak{h}^*$. Moreover, $\chi_\lambda = \chi_\mu$ if and only if λ and μ lie in the same W-orbit.*

2.2 Equivariant vector bundles on the flag variety.

Let $T \subseteq B$ be a maximal torus with $\mathrm{Lie}(T) = \mathfrak{h}$, and N be the unipotent radical of B so that $B = NT$. Note that the fiber at $B \in \mathcal{B}$ of any G-equivariant vector bundle is a representation of B. Conversely, given a representation U of B, consider the trivial vector bundle $U \times G$ on G. This descends to a G-equivariant vector bundle on $\mathcal{B}$ whose fiber at B is precisely U. In other words, G-equivariant vector bundles of $\mathcal{B}$ are in one-to-one correspondence with representations of B. In particular, G-equivariant line bundles on $\mathcal{B}$ are in one-to-one correspondence with 1-dimensional representations of B. These, in turn, are in one-to-one correspondence with characters of T, as N acts trivially on any 1-dimensional representation of B. Then, for any character $\lambda \in P = \mathrm{Hom}(T, \mathbb{C}^\times) \subseteq \mathfrak{h}^*$, we get a G-equivariant line bundle $\mathcal{L}(\lambda)$. Note that, since $\mathcal{L}(\lambda)$ is G-equivariant, the sheaf of differential operators $\mathcal{D}^{\mathcal{L}(\lambda)}_{\mathcal{O}_\mathcal{B}}$ is an HTDO.

Theorem 2.5 (Borel-Weil-Bott) *Let $\lambda \in P$. Then, we have,*

1. *If λ is antidominant, then the line bundle $\mathcal{L}(\lambda)$ is generated by its global sections.*
2. *The line bundle $\mathcal{L}(\lambda)$ is ample if and only if λ is antidominant and regular.*
3. *If $\lambda - \rho$ is not regular, then $H^i(\mathcal{B}, \mathcal{L}(\lambda)) = 0$ for $i \geq 0$.*
4. *If $\lambda - \rho$ is regular, then there exists $w \in W$ such that $w \cdot \lambda := w(\lambda - \rho) + \rho$ is antidominant, and*

$$H^i(X, \mathcal{L}(\lambda)) = \begin{cases} L^-(w \cdot \lambda) & \text{if } i = l(w), \\ 0 & \text{otherwise.} \end{cases}$$

Where $L^-(w \cdot \lambda)$ is the irreducible module with lowest weight $w \cdot \lambda$.

Note that it follows that if λ is antidominant, then $\Gamma(X, \mathcal{L}(\lambda)) = L^-(\lambda)$, and $H^i(X, \mathcal{L}(\lambda)) = 0$ for $i > 0$.

2.3 HTDO on the flag variety.

Let G be a semisimple algebraic group with Lie algebra $\mathfrak{g}$, and let B be a Borel subgroup, with $\mathcal{B} = G/B$ the corresponding flag variety. Then, $\mathcal{B}$ is an homogeneous G-variety, so we can apply the results of Subsection 1.4 to $\mathcal{B}$.

Recall the sheaf $\mathfrak{g}^\circ = \mathcal{O}_\mathcal{B} \otimes_\mathbb{C} \mathfrak{g}$, which is the sheaf of sections of the trivial bundle $\mathcal{B} \times \mathfrak{g}$. Inside this bundle, we have the homogeneous bundle of Borel subalgebras, $\mathcal{F}$, whose fiber at a point $x \in \mathcal{B}$ is the Borel subalgebra $\mathfrak{b}_x$ corresponding to x. Let $\mathfrak{b}^\circ$ be the sheaf of sections of this bundle. Also, we have the homogeneous bundle whose fiber at each point x is $[\mathfrak{b}_x, \mathfrak{b}_x]$. Let $\mathfrak{n}^\circ$ be the sheaf of sections of this bundle. Clearly, we have $\mathfrak{n}^\circ \hookrightarrow \mathfrak{b}^\circ \hookrightarrow \mathfrak{g}^\circ$.

Recall the epimorphism $\tau^\circ : \mathfrak{g}^\circ \to \mathcal{V}_\mathcal{B}$. Its kernel is precisely $\mathfrak{b}^\circ$. Pick as a basepoint $x_0 = B \in \mathcal{B}$. Then, we have $\mathfrak{b}_0 = \mathfrak{b}$. An element $\lambda \in \mathfrak{h}^*$ determines a B-invariant function $\mathfrak{b} \to \mathbb{C}$, and therefore a G-equivariant map $\lambda^\circ : \mathfrak{b}^\circ \to \mathcal{O}_\mathcal{B}$. Let $\mathcal{J}_\lambda$ be the ideal in $\mathcal{U}^\circ$ generated by elements of the form $\xi - (\lambda + \rho)^\circ(\xi)$ for $\xi \in \mathfrak{b}^\circ$, and let $\mathcal{D}_\lambda := \mathcal{U}^\circ / \mathcal{J}_\lambda$. The following is then a consequence of Theorem 1.14.

Proposition 2.6 *$\mathcal{D}_\lambda$ is an HTDO.*

We remark that $\mathcal{D}_{-\rho} = \mathcal{D}_\mathcal{B}$, and that for an integral weight λ, $\mathcal{D}_\lambda$ is the sheaf of differential operators on the line bundle $\mathcal{L}(\lambda + \rho)$.

By definition of an HTDO, we have a G-equivariant algebra homomorphism $\Psi_\lambda : \mathcal{U}(\mathfrak{g}) \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$. Recall the central character $\chi_\lambda : \mathfrak{z} \to \mathbb{C}$. Let J_λ be the ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\mathrm{Ker}(\chi_\lambda) \subseteq \mathfrak{z}$. Let $\mathcal{U}_\lambda := \mathcal{U}(\mathfrak{g})/J_\lambda$.

Lemma 2.7 *For any $\lambda \in \mathfrak{h}^*$, the morphism $\Psi_\lambda : \mathcal{U}(\mathfrak{g}) \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$ factors through $\mathcal{U}_\lambda$.*

Proof. Since the map $\Psi_\lambda : \mathcal{U}(\mathfrak{g}) \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$ is G-equivariant, it is enough to show that the induced map on fibers maps $\mathrm{Ker}(\chi_\lambda)$ to 0. Note that the fibers of $\mathcal{D}_\lambda$ have the form $\mathcal{U}(\mathfrak{g}) / \sum_{x \in \mathfrak{b}} (x - \langle \lambda + \rho, x \rangle) \mathcal{U}(\mathfrak{g})$. Then, for $z \in \mathfrak{z}$, by the PBW theorem, $z \in \mathfrak{n}\mathcal{U}(\mathfrak{g}) + f$ for a unique $f \in \mathcal{U}(\mathfrak{h})$, so that $\chi_\lambda(z) = f(\lambda + \rho)$. It follows that if $f(\lambda + \rho) = 0$, then $z \in \sum_{x \in \mathfrak{b}} (x - \langle \lambda + \rho, x \rangle) \mathcal{U}(\mathfrak{g})$. □

Abusing notation, we denote by Ψ_λ the induced morphism $\Psi_\lambda : \mathcal{U}_\lambda \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$.

Theorem 2.8 (Beilinson-Bernstein) *For any $\lambda \in \mathfrak{h}^*$, the morphism $\Psi_\lambda : \mathcal{U}_\lambda \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$ is an isomorphism.*

The key step to prove Theorem 2.8 is to prove its quasiclassical version, that is, to prove that its associated graded morphism is an isomorphism. In fact, we are going to relate this isomorphism to the Springer resolution.

By the PBW theorem, we know that $\operatorname{gr} \mathcal{U}(\mathfrak{g}) = S(\mathfrak{g})$. We canonically identify $\mathfrak{g} \cong \mathfrak{g}^*$ via the Killing form, so $\operatorname{gr} \mathcal{U}(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}]$. The next Lemma characterizes the ideal of the nilpotent cone $\mathcal{N}$ in $S(\mathfrak{g})$.

Lemma 2.9 *The ideal in $S(\mathfrak{g})$ defining $\mathcal{N}$ is generated by $S(\mathfrak{g})^G_+ = S(\mathfrak{g})^G \bigcap (\bigoplus_{p>0} S(\mathfrak{g})_p)$.*

Proof. Recall the Chevalley isomorphism, $\mathbf{res} : S(\mathfrak{g})^G \to S(\mathfrak{h})^W$. This maps the ideal $S(\mathfrak{g})^G_+$ to $S(\mathfrak{h})^W_+$, which is the ideal of 0 in $\mathfrak{h}/W$. Then, the ideal generated by $S(\mathfrak{g})^G_+$ defines the elements $x \in \mathfrak{g}^*$ such that $\overline{Gx} \cap \mathfrak{h} = 0$, that is, elements whose Jordan decomposition doesn't have a semisimple part. This is precisely $\mathcal{N}$. By results of Kostant, $\mathcal{N}$ is a normal variety and a complete intersection in $\mathfrak{g}$. The result now follows. □

Note that, by lifting elements of $S(\mathfrak{g})^G_+$ to $\mathcal{U}(\mathfrak{g})$, it follows that we have an epimorphism $\mathbb{C}[\mathcal{N}] \twoheadrightarrow \operatorname{gr} \mathcal{U}_\lambda$. We then have the following commutative diagram.

$$\begin{tikzcd}
S(\mathfrak{g}) \arrow[rr] \arrow[dr] & & \operatorname{gr} \Gamma(\mathcal{B}, \mathcal{D}_\lambda) \\
& \operatorname{gr} \mathcal{U}_\lambda \arrow[ur] & \\
& \mathbb{C}[\mathcal{N}] \arrow[u, two heads] &
\end{tikzcd}$$

Recall the Springer resolution $\gamma : T^* \mathcal{B} \to \mathcal{N}$, and its pullback $\gamma^* : \mathbb{C}[\mathcal{N}] \to \Gamma(T^* \mathcal{B}, \mathcal{O}_{\mathrm{T}^* \mathcal{B}})$. Since γ is a resolution of singularities and $\mathcal{N}$ is normal, γ^* is actually an isomorphism. Moreover, the following diagram commutes.

$$\begin{tikzcd}
S(\mathfrak{g}) \arrow[rr] \arrow[dr] & & \operatorname{gr} \Gamma(\mathcal{B}, \mathcal{D}_\lambda) \arrow[d] \\
& \operatorname{gr} \mathcal{U}_\lambda \arrow[ur] & \Gamma(\mathcal{B}, \operatorname{gr} \mathcal{D}_\lambda) \arrow[d, equal] \\
& \mathbb{C}[\mathcal{N}] \arrow[u, two heads] \arrow[r, "\gamma^*"] & \Gamma(T^* \mathcal{B}, \mathcal{O}_{\mathrm{T}^* \mathcal{B}})
\end{tikzcd}$$

Since $\operatorname{gr} \Gamma(\mathcal{B}, \mathcal{D}_\lambda) \to \Gamma(\mathcal{B}, \operatorname{gr} \mathcal{D}_\lambda)$ is injective, it follows that $\mathbb{C}[\mathcal{N}] \twoheadrightarrow \operatorname{gr} \mathcal{U}_\lambda$, $\operatorname{gr} \mathcal{U}_\lambda \to \operatorname{gr} \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$ and $\operatorname{gr} \Gamma(\mathcal{B}, \mathcal{D}_\lambda) \to \Gamma(\mathcal{B}, \operatorname{gr} \mathcal{D}_\lambda)$ are all isomorphisms. Then, $\operatorname{gr} \Psi_\lambda$ is an isomorphism. Since all the algebras we're working with are positively graded, it follows that Ψ_λ is an isomorphism.

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