$\mathcal{D}$-modules on flag varieties and localization of $\mathfrak{g}$-modules, II.

José Simental.

October 18, 2013

1 Reminder of last time.

We recall results of last time that we are going to use here.

Recall that we have the Harish-Chandra isomorphism $HC : \mathfrak{z} \to \mathbb{C}[\mathfrak{h}^*]^W$, where $\mathfrak{z}$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Then, every central character (= algebra homomorphism from $\mathfrak{z}$ to $\mathbb{C}$) has the form χ_λ, $\chi_\lambda(z) = HC(z)(\lambda)$, and $\chi_\lambda = \chi_\mu$ if and only if λ and μ are W-conjugate.

For every $\lambda \in \mathfrak{h}^*$, we have a homogeneous twisted sheaf of differential operators $\mathcal{D}_\lambda := \mathcal{U}^\circ/\mathcal{J}_\lambda$, where $\mathcal{J}_\lambda$ is the two-sided ideal generated by elements of the form $\xi - (\lambda+\rho)^\circ(\xi)$, for $\xi \in \mathfrak{b}^\circ$. The morphism $\Psi_\lambda : \mathcal{U}(\mathfrak{g}) \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$ factors through $\mathcal{U}_\lambda := \mathcal{U}(\mathfrak{g})/\operatorname{Ker}(\chi_\lambda)\mathcal{U}(\mathfrak{g})$.

Theorem 1.1 (Beilinson-Bernstein) *The morphism $\Psi_\lambda : \mathcal{U}_\lambda \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$ is an isomorphism.*

Recall that the strategy to prove Theorem 1.1 is to see that its associated graded coincides with the pullback $\gamma^* : \mathbb{C}[\mathcal{N}] \to \Gamma(T^*\mathcal{B}, \mathcal{O}_{T^*\mathcal{B}})$ of the Springer resolution $\gamma : T^*\mathcal{B} \to \mathcal{N}$.

2 Cohomology of $\mathcal{D}_\lambda$-modules.

We have two induced functors. The first functor is the *global sections functor*:

$$\begin{gathered}\operatorname{Mod}_{qc}(\mathcal{D}_\lambda) \to \operatorname{Mod}\text{-}\mathcal{U}(\mathfrak{g}) \\ \mathcal{M} \mapsto \Gamma(\mathcal{B}, \mathcal{M}).\end{gathered}$$

Note that $\Gamma(\mathcal{B}, \mathcal{M}) = \operatorname{Hom}_{\mathcal{O}_\mathcal{B}}(\mathcal{O}_\mathcal{B}, \mathcal{M})$. Also, note that $\operatorname{Hom}_{\mathcal{O}_\mathcal{B}}(\mathcal{O}_\mathcal{B}, \mathcal{M}) = \operatorname{Hom}_{\mathcal{D}_\lambda}(\mathcal{D}_\lambda, \mathcal{M})$: if $\mathcal{M}$ has a $\mathcal{D}_\lambda$-module structure, then any $\mathcal{O}_\mathcal{B}$-homomorphism $\mathcal{O}_\mathcal{B} \to \mathcal{M}$ admits a unique extension to a $\mathcal{D}_\lambda$-linear homomorphism $\mathcal{D}_\lambda \to \mathcal{M}$. The next functor is the *localization functor*:

$$\begin{gathered}\operatorname{Mod}\text{-}\mathcal{U}(\mathfrak{g}) \to \operatorname{Mod}_{qc}(\mathcal{D}_\lambda) \\ M \mapsto \mathcal{D}_\lambda \otimes_{\mathcal{U}(\mathfrak{g})} M.\end{gathered}$$

Note that the global sections functor is right adjoint to the localization functor. Our next goal is to study these functors.

2.1 Abelian Beilinson-Bernstein theorem.

The goal of this subsection is to state and prove two fundamental theorems of Beilinson-Bernstein on the cohomology of $\mathcal{O}_\mathcal{B}$-coherent $\mathcal{D}_\lambda$-modules. The first one, Theorem 2.2, concerns the vanishing of the higher cohomology of modules. The second Theorem 2.6, tells us when every $\mathcal{O}_\mathcal{B}$-coherent $\mathcal{D}_\lambda$-module is generated by its global sections. A strategy to do this is to realize every $\mathcal{O}_\mathcal{B}$-coherent submodule of such a module as a direct summand in a sheaf without higher cohomology. To do this, we will use the Borel-Weil-Bott theorem, which tells us that the sheaf $\mathcal{L}(\lambda)$ is ample whenever $\lambda \in P$ is antidominant and regular.

Assume $\mu \in P$ is antidominant. By the Borel-Weil-Bott theorem (from last time, Theorem 2.5 1)), $\mathcal{L}(\mu)$ is generated by its global sections. We know that the global sections of $\mathcal{L}(\mu)$ are $L^-(\mu)$, the simple module with lowest weight μ. Then, we have $p_\mu : \mathcal{O}_\mathcal{B} \otimes_\mathbb{C} L^-(\mu) \twoheadrightarrow \mathcal{L}(\mu)$. Taking the dual of p_μ, we get a morphism $\mathcal{H}om_{\mathcal{O}_\mathcal{B}}(\mathcal{L}(\mu), \mathcal{O}_\mathcal{B}) \to \mathcal{O}_\mathcal{B} \otimes_\mathbb{C} \operatorname{Hom}_\mathbb{C}(L^-(\mu), \mathbb{C})$. This is injective. Rewriting, we have an injective morphism $\mathcal{L}(-\mu) \to \mathcal{O}_\mathcal{B} \otimes_\mathbb{C} L^+(-\mu)$. If we tensor with the locally free module $\mathcal{L}(\mu)$, we get an injective morphism,

$$i_\mu : \mathcal{O}_\mathcal{B} \to \mathcal{L}(\mu) \otimes_\mathbb{C} L^+(-\mu).$$

Tensoring with a $\mathcal{O}_\mathcal{B}$-coherent $\mathcal{D}_\lambda$ module we get,

$$i_{\mu,\mathcal{M}} : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(\mu) \otimes_\mathbb{C} L^+(-\mu).$$

Note that $i_{\mu,\mathcal{M}}$ is always injective because i_μ locally splits. We want to show that, if λ is antidominant, $i_{\mu,\mathcal{M}}$ splits as morphism of sheaves of vector spaces. Note that this splitting will be constructed using differential operators, so it is not a splitting of $\mathcal{O}_\mathcal{B}$-modules. To do so, we will realize the image of $i_{\mu,\mathcal{M}}$ as a generalized eigensheaf for the action of the center $\mathfrak{z}$ of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{M} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(\mu) \otimes_\mathbb{C} L^+(-\mu)$.

An essential ingredient will be the following construction. Let F be a finite dimensional $\mathfrak{g}$-module. Recall that F has a filtration by $\mathfrak{b}$-submodules $0 = F_0 \subset F_1 \subset \cdots \subset F_m = F$, where $\dim F_i = i$, $\mathfrak{n} F_i \subseteq F_{i-1}$ and $\mathfrak{h}$ acts on the 1-dimensional quotient F_i/F_{i-1} by an integral character ν_i. The ν_i's are just the weights of F. Consider the trivial vector bundle $\mathcal{B} \times F \twoheadrightarrow \mathcal{B}$. Its sheaf of sections is $\mathcal{F} := \mathcal{O}_\mathcal{B} \otimes_\mathbb{C} F$. Note that $\mathcal{B} \times F$ has a filtration $0 = U_0 \subset U_1 \subset \cdots \subset U_m$, where

$$U_i := \{(gB, v) \in \mathcal{B} \times F : v \in g(F_i)\}.$$

This defines a filtration on $\mathcal{F}$, $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathcal{F}$. These are G-equivariant coherent sheaves on $\mathcal{B}$. Recall that we have an equivalence between the category of G-equivariant coherent sheaves on $\mathcal{B}$ and the category of representations of the Borel subgroup B. Under this equivalence, $\mathcal{F}_i$ corresponds to F_i. It then follows that $\mathcal{F}_i \,/\, \mathcal{F}_{i-1} = \mathcal{L}(\nu_i)$.

It follows that, more generally, for any quasi-coherent $\mathcal{O}_\mathcal{B}$-module $\mathcal{M}$, $\mathcal{M} \otimes_\mathbb{C} F$ has a filtration with succesive quotients being $\mathcal{M} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(\nu_i)$. Now assume $\mathcal{M}$ is a $\mathcal{D}_\lambda$-module. Then, $\mathcal{M}$ is a $\mathfrak{g}^\circ = \mathcal{O}_\mathcal{B} \otimes_\mathbb{C} \mathfrak{g}^\circ$-module such that the subbundle of Borel subalgebras $\mathfrak{b}^\circ$ acts with character $(\lambda + \rho)^\circ$. Similarly, $\mathfrak{b}^\circ$ acts with character ν_i° on $\mathcal{L}(\nu_i)$. It follows that $\mathcal{M} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(\nu_i)$ is a $\mathfrak{g}^\circ$-module and the $\mathfrak{b}^\circ$ acts on it with character $(\lambda + \nu_i + \rho)^\circ$. In other words, the action of $\mathcal{U}^\circ$ on $\mathcal{M} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(\nu_i)$ factors through the quotient $\mathcal{D}_{\lambda+\nu_i}$. By Theorem 1.1, the center $\mathfrak{z}$ of $\mathcal{U}(\mathfrak{g})$ acts on $\mathcal{M} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(\nu_i)$ with character $\chi_{\lambda+\nu_i}$. It follows that

$$\prod_i (z - \chi_{\lambda+\nu_i}(z))$$

annihilates $\mathcal{M} \otimes_\mathbb{C} F$ for every $z \in \mathfrak{z}$. Then, the action of $\mathfrak{z}$ on $\mathcal{M} \otimes_\mathbb{C} F$ is locally finite, and $\mathcal{M} \otimes_\mathbb{C} F$ decomposes into the direct sum of its generalized $\mathfrak{z}$-eigensheaves.

For a $\mathcal{U}^\circ$-module $\mathcal{M}$ and $\lambda \in \mathfrak{h}^*$, denote by $\mathcal{M}_{[\lambda]}$ the generalized $\mathfrak{z}$-eigensheaf of $\mathcal{M}$ with eigencharacter χ_λ. Note that $\mathcal{M}_{[\lambda]} = \mathcal{M}_{[\mu]}$ whenever λ, μ belong to the same W-orbit.

Lemma 2.1 *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then, for every $\mathcal{O}_\mathcal{B}$-quasi-coherent $\mathcal{D}_\lambda$-module $\mathcal{M}$, and every antidominant integral weight μ, $i_{\mu,\mathcal{M}}$ splits. In particular, $\mathcal{M} \cong [\mathcal{M} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(\mu) \otimes_\mathbb{C} L^+(-\mu)]_{[\lambda]}$.*

Proof. We know that the eigencharacters of $\mathcal{M} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(\mu) \otimes_\mathbb{C} L^+(-\mu)$ are of the form $\chi_{\lambda+\mu+\nu_i}$ where ν_i is a weight of $L^+(-\mu)$. Assume $\chi_{\lambda+\mu+\nu_i} = \chi_\lambda$ for some weight ν_i of $L^+(-\mu)$. Then, for some $w \in W$, $w(\lambda) = \lambda + \mu + \nu_i$, so $(-\mu - \nu_i) + w(\lambda) - \lambda = 0$. But λ is antidominant, so $w(\lambda) - \lambda$ is positive, that is, it is a non-negative linear combination of simple roots. Since $L^+(-\mu)$ is the irreducible module with highest weight $-\mu$, $-\mu - \nu_i$ is also positive. It follows that $w(\lambda) = \lambda$ and $\mu = -\nu_i$. Then, the generalized eigensheaf with eigencharacter χ_λ is $\mathcal{M} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(\mu) \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(-\mu) = \mathcal{M}$. □

Theorem 2.2 (Beilinson-Bernstein) *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then, $H^i(\mathcal{B}, \mathcal{M}) = 0$ for every quasi-coherent $\mathcal{D}_\lambda$-module and $i > 0$. In particular, the global sections functor $\Gamma(\mathcal{B}, \bullet) : \mathrm{Mod}_{qc}(\mathcal{D}_\lambda) \to \mathrm{Mod}\text{-}\mathcal{U}_\lambda$ is exact.*

Proof. [1] Let $\mathcal{W}$ be an $\mathcal{O}_\mathcal{B}$-coherent submodule of $\mathcal{M}$. By Borel-Weil-Bott (more precisely, Theorem 2.5 2) of last time) we can find an antidominant weight μ such that $H^i(\mathcal{B}, \mathcal{W} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(\mu)) = 0$ for $i > 0$. Then, $H^i(\mathcal{B}, \mathcal{W} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{L}(\mu) \otimes_\mathbb{C} L^+(-\mu)) = 0$. Now consider the following commutative diagram:

[1] Note that the argument I gave on the October 18 talk is incorrect: the morphism $i_{\mu,\mathcal{W}}$ does not necessarily split.

$$\begin{tikzcd}
H^i(\mathcal{B},\mathcal{W}) \arrow[r] \arrow[d] & H^i(\mathcal{B},\mathcal{M}) \arrow[d] \\
0 = H^i(\mathcal{B},\mathcal{W}\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{L}(\mu)\otimes_\mathbb{C} L^+(-\mu)) \arrow[r] & H^i(\mathcal{B},\mathcal{M}\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{L}(\mu)\otimes_\mathbb{C} L^+(-\mu))
\end{tikzcd}.$$

Since the diagram commutes and, by the previous lemma, $H^i(\mathcal{B},\mathcal{M}) \to H^i(\mathcal{B},\mathcal{M}\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{L}(\mu)\otimes_\mathbb{C} L^+(-\mu))$ is injective, we get that $H^i(\mathcal{B},\mathcal{W}) \to H^i(\mathcal{B},\mathcal{M})$ is the zero map. Since $\mathcal{M}$ is the direct limit of its $\mathcal{O}_\mathcal{B}$-coherent submodules and cohomology commutes with direct limits, we get that $H^i(\mathcal{B},\mathcal{M}) = 0$. □

Corollary 2.3 *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then, for every $\mathcal{U}_\lambda$-module V, the natural map φ_V of V to $\Gamma(\mathcal{B},\mathcal{O}_\mathcal{B}\otimes_{\mathcal{U}(\mathfrak{g})}V)$ is an isomorphism of $\mathfrak{g}$-modules.*

Proof. By the previous theorem, the global sections functor Γ is exact. Then, the functor $\Gamma(\mathcal{B},\mathcal{O}_\mathcal{B}\otimes_{\mathcal{U}(\mathfrak{g})}\bullet)$ is right exact. Now let $V \in \mathcal{U}_\lambda$. There exists an exact sequence $(\mathcal{U}_\lambda)^{\oplus I} \to (\mathcal{U}_\lambda)^{\oplus J} \to V \to 0$. Then, we get a commutative diagram,

$$\begin{tikzcd}
(\mathcal{U}_\lambda)^{\oplus I} \arrow[r] \arrow[d] & (\mathcal{U}_\lambda)^{\oplus J} \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\
\Gamma(\mathcal{B},\mathcal{D}_\lambda)^{\oplus I} \arrow[r] & \Gamma(\mathcal{B},\mathcal{D}_\lambda)^{\oplus J} \arrow[r] & \Gamma(\mathcal{B},\mathcal{O}_\mathcal{B}\otimes_{\mathcal{U}(\mathfrak{g})}V) \arrow[r] & 0.
\end{tikzcd}$$

The first two vertical maps are isomorphisms. Then, the third vertical map is also an isomorphism. □

Denote by $\mathrm{Qmod}_{qc}(\mathcal{D}_\lambda)$ the quotient category of $\mathrm{Mod}_{qc}(\mathcal{D}_\lambda)$ modulo the full subcategory formed by quasi-coherent $\mathcal{D}_\lambda$ modules without global sections.

Corollary 2.4 *Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then, the localization functor induces an equivalence from* $\mathrm{Mod}\text{-}\mathcal{U}_\lambda$ *to* $\mathrm{Qmod}_{qc}(\mathcal{D}_\lambda)$.

Proof. Let $\mathcal{M} \in \mathrm{Qmod}_{qc}(\mathcal{D}_\lambda)$. By adjointness, we have a natural morphism $\psi_\mathcal{M} : \mathcal{D}_\lambda\otimes_{\mathcal{U}_\lambda}\Gamma(\mathcal{B},\mathcal{M}) \to \mathcal{M}$. Let $\mathcal{K}'$ and $\mathcal{K}''$ be the kernel and cokernel of this morphism, respectively. Then, we get an exact sequence $0 \to \mathcal{K}' \to \mathcal{D}_\lambda\otimes_{\mathcal{U}_\lambda}\Gamma(\mathcal{B},\mathcal{M}) \to \mathcal{M} \to \mathcal{K}'' \to 0$. Applying the global sections functor we find that $\Gamma(\mathcal{B},\mathcal{K}') = 0$, $\Gamma(\mathcal{B},\mathcal{K}'') = 0$. The result follows. □

Now we show another result due to Beilinson-Bernstein, that says that when $\lambda \in \mathfrak{h}^*$ is antidominant and regular, every quasi-coherent $\mathcal{D}_\lambda$ module $\mathcal{M}$ is generated by its global sections. The strategy is similar to that of the proof of Theorem 2.2 but somewhat easier. Recall that for any integral antidominant weight we μ we have a surjective morphism $p_\mu : \mathcal{O}_\mathcal{B}\otimes_\mathbb{C} L^-(\mu) \twoheadrightarrow \mathcal{L}(\mu)$. Note that this morphism locally splits. Then, for every quasi-coherent module $\mathcal{M}$ we get an epimorphism $p_{\mu,\mathcal{M}} : \mathcal{M}\otimes_\mathbb{C} L^-(\mu) \to \mathcal{M}\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{L}(\mu)$. The following is an analog of Lemma 2.1.

Lemma 2.5 *Assume λ is antidominant and regular. Then, for every quasi-coherent $\mathcal{D}_\lambda$ module, and every antidominant integral weight μ, the epimorphism $p_{\mu,\mathcal{M}}$ splits. In fact, the generalized $\chi_{\lambda+\mu}$-eigensheaf of $\mathcal{M}\otimes_\mathbb{C} L^-(\mu)$ is $\mathcal{M}\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{L}(\mu)$.*

Proof. We argue similarly to Lemma 2.1. Assume there exists a weight ν_i of $L^-(\mu)$ and $w \in W$ such that $w(\lambda+\nu_i) = \lambda+\mu$. Then, $(w(\lambda)-\lambda)+(w(\nu_i)-\mu) = 0$. Similarly to Lemma 2.1, it follows that $\nu_i = \mu$. The result follows. □

Theorem 2.6 (Beilinson-Bernstein) *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Then, for any quasi-coherent $\mathcal{D}_\lambda$-module $\mathcal{M}$, the morphism $\mathcal{D}_\lambda\otimes_{\mathcal{U}(\mathfrak{g})}\Gamma(\mathcal{B},\mathcal{M}) \to \mathcal{M}$ is surjective. In other words, every quasi-coherent $\mathcal{D}_\lambda$ module is generated by its global sections.*

Proof. Since λ is antidominant, $\Gamma(\mathcal{B},\bullet)$ is exact. Hence, it suffices to show that $\Gamma(\mathcal{B},\mathcal{M}) \neq 0$ for $\mathcal{M} \neq 0$. We can assume that $\mathcal{M}$ is coherent. By Borel-Weil-Bott, we can find a regular antidominant weight ν such that $\Gamma(\mathcal{B},\mathcal{M}\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{L}(\nu)) \neq 0$. Since ν is regular, Lemma 2.5 implies that $L^-(\nu)\otimes\Gamma(\mathcal{B},\mathcal{M}) \neq 0$. We're done. □

Corollary 2.7 *Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Then, the global sections functor is an equivalence of categories* $\mathrm{Mod}_{qc}(\mathcal{D}_\lambda) \to \mathrm{Mod}\text{-}\mathcal{U}_\lambda$. *Its inverse is the localization functor.*

Proof. Follows from Corollary 2.4 and Theorem 2.6. □

As an application of Theorems 2.2, 2.6, we show that the homological dimension of $\mathcal{U}_\lambda$ is finite whenever λ is regular. It is known that if this is not the case then the homological dimension of $\mathcal{U}_\lambda$ is infinite.

Proposition 2.8 *Let $\lambda' \in \mathfrak{h}^*$ be regular. Then the homological dimension of $\mathcal{U}_{\lambda'}$ is finite.*

Proof. Since $\mathcal{U}_{\lambda'} = \mathcal{U}_{w\lambda}$ for any $w \in W$, we can replace λ' by $\lambda \in W\lambda'$ antidominant (and, by hypothesis, regular). Since $\mathcal{D}_\lambda$ is a TDO, the homological dimension of each stalk $\mathcal{D}_{\lambda,x}$ is finite, as this is a filtered algebra whose associated graded algebra has finite homological dimension. Moreover, the homological dimension $\operatorname{hd}\mathcal{D}_{\lambda,x} \leq \dim \mathcal{B}$, so that these homological dimensions are uniformly bounded. It is known that, for any $x \in \mathcal{B}$, $i \in \mathbb{Z}_{>0}$ $\mathcal{E}xt^i_{\mathcal{D}_\lambda}(\mathcal{M}, \mathcal{W})_x = \operatorname{Ext}^i_{\mathcal{D}_{\lambda,x}}(\mathcal{M}_x, \mathcal{W}_x)$, for an $\mathcal{O}_\mathcal{B}$-coherent $\mathcal{D}_\lambda$-module $\mathcal{M}$ and a quasi-coherent $\mathcal{D}_\lambda$-module $\mathcal{W}$. Then, $\mathcal{E}xt^i_{\mathcal{D}_\lambda}(\mathcal{M}, \mathcal{W}) = 0$ for $i > \dim \mathcal{B}$.

On the other hand, we have the Grothendieck spectral sequence $H^p(\mathcal{B}, \mathcal{E}xt^q_{\mathcal{D}_\lambda}(\mathcal{M}, \mathcal{W})) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{D}_\lambda}(\mathcal{M}, \mathcal{W})$. It follows that $\operatorname{Ext}^i_{\mathcal{D}_\lambda}(\mathcal{M}, \mathcal{W}) = 0$ for $i > 2\dim \mathcal{B}$, $\mathcal{M}$ a coherent $\mathcal{D}_\lambda$-module and $\mathcal{W}$ a quasi-coherent $\mathcal{D}_\lambda$-module. Since we're assuming λ is antidominant and regular, $\operatorname{Ext}^i_{\mathcal{U}_\lambda}(M, W) = 0$ for any finitely generated $\mathcal{U}_\lambda$-module M and any $\mathcal{U}_\lambda$-module W. Taking direct limits, it follows that $\operatorname{hd}\mathcal{U}_\lambda \leq 2\dim \mathcal{B}$. □

Remark 2.9 *If $\lambda \in \mathfrak{h}^*$ is an integral regular weight, then actually* $\operatorname{hd}\mathcal{U}_\lambda = 2\dim \mathcal{B}$.

2.2 Derived Beilinson-Bernstein Theorem.

Assume $\lambda \in \mathfrak{h}^*$ is regular. Then, $\mathcal{U}_\lambda$ has finite homological dimension, so the localization functor has a left derived functor $\mathcal{D}_\lambda \overset{L}{\otimes}_{\mathcal{U}_\lambda} \bullet : D^b(\mathrm{Mod}\text{-}\mathcal{U}_\lambda) \to D^b(\mathrm{Mod}_{qc}(\mathcal{D}_\lambda))$. Note that the global sections functor admits a right derived functor $R\Gamma : D^b(\mathrm{Mod}_{qc}(\mathcal{D}_\lambda)) \to D^b(\mathrm{Mod}\text{-}\mathcal{U}_\lambda)$.

Theorem 2.10 *Let $\lambda \in \mathfrak{h}^*$ be a regular integral weight. Then, $\mathcal{D}_\lambda \overset{L}{\otimes}_{\mathcal{U}_\lambda} \bullet$ and $R\Gamma$ are quasi-inverse equivalences of triangulated categories.*

Remark 2.11 *We remark that Theorem 2.10 is valid in a greater generality for $\lambda \in \mathfrak{h}^*$ regular but not necessarily integral.*

Let P be a projective $\mathcal{U}_\lambda$-module. Recall that this means that P is a direct summand of a free module $\mathcal{U}_\lambda^{\oplus |I|}$, for some set I so, in particular, P is flat. Note that, by Theorem 1.1, the adjunction morphism $P \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda \otimes_{\mathcal{U}_\lambda} P)$ is an isomorphism. By the same Theorem, $\mathcal{D}_\lambda \otimes_{\mathcal{U}_\lambda} P$ is a direct summand of $\mathcal{D}_\lambda^{\oplus |I|}$. Note that $\mathcal{O}_{T^*\mathcal{B}}$ has no higher cohomology, this is a consequence of the Grauert-Riemenschneider Theorem applied to $T^*\mathcal{B} \to \mathcal{N}$. Since $\operatorname{gr}\mathcal{D}_\lambda = \mathcal{O}_{T^*\mathcal{B}}$, it follows that $\mathcal{D}_\lambda$ is Γ-acyclic. Hence, $\mathcal{D}_\lambda \otimes_{\mathcal{U}_\lambda} P$ is Γ-acyclic.

Now, let $V^\cdot$ be a complex in $D^b(\mathrm{Mod}\text{-}\mathcal{U}_\lambda)$, with $\lambda \in \mathfrak{h}^*$ a regular weight. By Proposition 2.8, $V^\cdot$ is quasi-isomorphic to a complex $P^\cdot$ of projective $\mathcal{U}_\lambda$-modules, and $\mathcal{D}_\lambda \overset{L}{\otimes}_{\mathcal{U}_\lambda} V^\cdot = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\lambda} P^\cdot$. It follows that $R\Gamma(\mathcal{D}_\lambda \overset{L}{\otimes}_{\mathcal{U}_\lambda} V^\cdot) = \Gamma(\mathcal{D}_\lambda \otimes_{\mathcal{U}_\lambda} P^\cdot) \cong P^\cdot \cong V^\cdot$. We have proved the following.

Lemma 2.12 *Let $\lambda \in \mathfrak{h}^*$ be a regular integral weight. Then, $R\Gamma(\mathcal{D}_\lambda \overset{L}{\otimes}_{\mathcal{U}_\lambda} \bullet) : D^b(\mathrm{Mod}\text{-}\mathcal{U}_\lambda) \to D^b(\mathrm{Mod}\text{-}\mathcal{U}_\lambda)$ is isomorphic to the identity functor.*

Note that it follows that $R\Gamma$ is a quotient functor of triangulated categories. Then, to prove Theorem 2.10, it suffices to show that $R\Gamma(\mathcal{M}^\cdot) = 0$ only when $\mathcal{M}^\cdot = 0$. We will follow a strategy that appears in [2, Section 3]. We will need the following result, due to Kontsevich (see e.g. [2, Theorem 3.5.1]):

Lemma 2.13 *Let $X \subseteq \mathbb{P}^n_\mathbb{C}$ be a smooth closed subscheme. Then, $\mathcal{O}_X(i)$, $-n \leq i \leq 0$ generate $D^b(\operatorname{coh} X)$ under shifts, cones, and direct summands.*

Corollary 2.14 *There exists a finite set of dominant weights $\mathcal{S}$ such that $\mathcal{L}(\mu), \mu \in \mathcal{S}$, generate $D^b(\operatorname{coh}\mathcal{B})$ under shifts, cones, and direct summands.*

We will also need a derived version of the splitting method used in the proof of Theorems 2.2, 2.6. Recall that, if M is a (sheaf of) module(s) on which the center $\mathfrak{z}$ of $\mathcal{U}(\mathfrak{g})$ acts locally finitely, then by $[M]_\lambda$ we denote the generalized eigenspace (resp. generalized eigensheaf) with generalized eigencharacter χ_λ. For integral weights λ, μ with $\mu - \lambda$ dominant, define the translation functor $T^\mu_\lambda : \mathrm{Mod}\text{-}\mathcal{U}_\lambda \to \mathrm{Mod}\text{-}\mathcal{U}_\mu$ by $T^\mu_\lambda(M) = [L^+(\mu - \lambda) \otimes M]_\mu$, where $L^+(\mu - \lambda)$ is the simple finite dimensional module with highest weight $\mu - \lambda$.

Now, let $\mathcal{M}^\cdot$ be in $D^b(\mathrm{Mod}_{qc}(\mathcal{D}_\lambda))$. Then, using the notation on the previous paragraph, $L^+(\mu-\lambda)\otimes\mathcal{M}^\cdot$ is a complex of $\mathcal{U}^\circ$-modules. Moreover, by the construction before Lemma 2.1, $\mathfrak{z}$ acts locally finitely on $L^+(\mu-\lambda)\otimes\mathcal{M}^\cdot$. So one can talk about $T_\lambda^\mu[\mathcal{M}^\cdot]$. We have that translation functors commute with $R\Gamma$, that is,

$$T_\lambda^\mu[R\Gamma_\lambda\,\mathcal{M}^\cdot]\cong R\Gamma_\mu([T_\lambda^\mu\,\mathcal{M}^\cdot]). \tag{1}$$

We'll use the following Lemma, that is parallel to Lemmas 2.1, 2.5. The proof is also similar. To get the desired combinatorial relations between the weights, it uses [4, Lemma 7.7].

Lemma 2.15 *Assume λ, μ are in the same chamber. Then, for $\mathcal{M}^\cdot\in D^b(\mathrm{Mod}_{qc}(\mathcal{D}_\lambda))$, $T_\lambda^\mu(\mathcal{M}^\cdot)=\mathcal{L}(\mu-\lambda)\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{M}^\cdot$.*

Finally, Theorem 2.10 follows from the next result.

Lemma 2.16 *Let λ be an integral and regular weight, and let $\mathcal{M}^\cdot\in D^b(\mathrm{Mod}_{qc}(\mathcal{D}_\lambda))$ be such that $R\Gamma(\mathcal{M}^\cdot)=0$. Then, $\mathcal{M}^\cdot=0$.*

Proof. Let μ be a dominant weight such that λ, $\lambda+\mu$ are in the same chamber. It then follows from Equation (1) that $0=T_\lambda^{\lambda+\mu}[R\Gamma_\lambda\,\mathcal{M}^\cdot]=R\Gamma_\mu(\mathcal{L}(\mu)\otimes_{\mathcal{O}_\mathcal{B}}\mathcal{M}^\cdot)$. By Corollary 2.14, it follows that for λ deep in its chamber, $R\Gamma(\mathcal{F}^\cdot\otimes\mathcal{M}^\cdot)=0$ for all $\mathcal{F}^\cdot\in D^b(\mathrm{coh}\,\mathcal{B})$. Then, $\mathcal{M}^\cdot=0$.

The case for any integral regular weight λ follows again from (1) but, to pass from an integral regular weight λ to another (integral and regular) weight deep into the chamber of λ, we need to extend the definition of translation functors to allow the case when the difference $\mu-\lambda$ is not dominant. Here, define $T_\lambda^\mu(M):=[L(\mu-\lambda)\otimes M]_\mu$, where $L(\mu-\lambda)$ is a finite dimensional $\mathfrak{g}$-module with extremal weight $\mu-\lambda$. Again, we can extend this functor to $D^b(\mathrm{Mod}_{qc}(\mathcal{D}_\lambda))$, and Equation (1) is valid. Finally, Lemma 2.15 is also valid in this more general setting, with same proof. It follows that, for λ,μ in the same chamber and $\mathcal{M}^\cdot\in D^b(\mathrm{Mod}_{qc}(\mathcal{D}_\lambda))$, $T_\lambda^\mu(\mathcal{M}^\cdot)=0$ only when $\mathcal{M}^\cdot=0$. $\square$

References

[1] A. Beilinson, J. Bernstein, *Localisation de $\mathfrak{g}$-modules.* C. R. Acad. Sci. Paris Ser. I Math. 292 (1981) no.1, 15-18

[2] R. Bezrukavnikov, I. Mirković, D. Rumynin, *Localization of modules for a semisimple Lie algebra in prime characteristic.* arXiv:math/0205144v5

[3] R. Hotta, K. Takeuchi, T. Tanisaki, *D-modules, Perverse Sheaves and Representation Theory.* Birkhäuser Progress in Mathematics 236.

[4] J. Jantzen, *Representations of algebraic groups.* Pure and Applied Math. 131, Academic Press.

[5] D. Miličić, *Localization and Representation Theory of Reductive Lie Groups.*