2.7 Convolution in BM Homology

* Notation H_* is always for B-M homology, H_*^{ord} is for ordinary homology.

2.7.1 Toy Example (Motivation).

Convolution of functions on finite sets: $f_{12} \in \mathbb{C}(M_1 \times M_2)$, $f_{23} \in \mathbb{C}(M_2 \times M_3)$

(M_i are finite sets)

$\Rightarrow$ get $f_{12} * f_{23} \in \mathbb{C}(M_1 \times M_3)$: $(m_1, m_3) \to \sum_{m_2 \in M_2} f_{12}(m_1, m_2) f_{23}(m_2, m_3)$ (2.7.2)

(matrix multiplication)

From 2.7.2, generalize to diff forms on compact manifolds M_1, M_2, M_3

Let $p_{ij}: M_1 \times M_2 \times M_3 \to M_i \times M_j$, $d = \dim M_2$. $f_{12} \in \Omega^i(M_1 \times M_2)$, $f_{23} \in \Omega^j(M_2 \times M_3)$

Then:

$$f_{12} * f_{23} = \int_{M_2} p_{12}^* f_{12} \wedge p_{23}^* f_{23} \quad \in \Omega^{i+j-d}(M_1 \times M_3)$$

(similar to integration along fiber)

* Note: we have ~~sth~~ $-d$ for ~~if~~ the integral over M_2 only make sense when the wedge product has d variable of M_2 (i.e consist the top form of M_2), otherwise it is 0.

We get ~~di~~ convolution of diff forms, it induces convolution on De Rham cohomology.

(by compatibility: $d(f_{12} * f_{23}) = (df_{12}) * f_{23} + (-1)^j f_{12} * (df_{23})$)

(preserve exactness, closedness)

$\Rightarrow$ Get convolution on homology by Poincare duality.

2.7.5 Abstract definition for general case

M_1, M_2, M_3 oriented, connected C^∞ manifolds

$$Z_{12} \subset M_1 \times M_2, \quad Z_{23} \subset M_2 \times M_3$$

closed subsets. Convolution is defined on:

$$Z_{12} \circ Z_{23} = \{ (m_1, m_3) \in M_1 \times M_3 \mid \exists\, m_2 \in M_2 : (m_1, m_2) \in Z_{12},\ (m_2, m_3) \in Z_{23} \}$$

Example 2.7.7 : $f: M_1 \to M_2$, $g: M_2 \to M_3$
$\Rightarrow \mathrm{Graph}(f) \circ \mathrm{Graph}(g) = \mathrm{Graph}(g \circ f)$

Let $p_{ij}: M_1 \times M_2 \times M_3 \to M_i \times M_j$, assume

$$p_{13}: p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3 \text{ proper}$$

$\Rightarrow$ image of $p_{13} = Z_{12} \circ Z_{23}$, and is closed subset of $M_1 \times M_3$ (by properness)

To define convolution, first recall Kunneth formula ($\boxtimes$) for homology and intersection pairing ($\frown$)

Let $d = \dim_{\mathbb{R}} M_2$, convolution in BM homology is defined by:

$$H_i(Z_{12}) \times H_j(Z_{23}) \longrightarrow H_{i+j-d}(Z_{12} \circ Z_{23})$$

$$(c_{12}, c_{23}) \longrightarrow c_{12} * c_{23}$$

$$c_{12} * c_{23} = (p_{13})_* \big((c_{12} \boxtimes [M_3]) \frown ([M_1] \boxtimes c_{23}) \big)$$

~~with~~ by using Kunneth formula for c_{ij} and fundamental class of M_k and take intersection pairi

*Note: $Z_{12} \times M_3 \cap M_1 \times Z_{23}$ by p_{13} projects to $Z_{12} \circ Z_{23}$, so we get $H_{i+j-d}(Z_{12} \circ Z_{23})$ as targ

Remark: (i) In disconnected case, just change $[M_1]$, $[M_3]$ to sum of fundamental classes
(ii) If M_1, M_2, M_3 discrete set $\Rightarrow$ get 2.7.2 (proper $\Rightarrow$ finite fiber so sum well-defined)

2.7.10 Examples.

(i) $M_1 = M_2 = M_3 = M$ and $Z_{12}, Z_{23} \subset M_\Delta \hookrightarrow M \times M$,
which is diagonal embedding. Z_{12}, Z_{23} closed then p_{13} proper, and $Z_{12} \circ Z_{23} \subset M_\Delta$
$\Rightarrow$ $*$-convolution becomes intersection pairing in $H(M_\Delta)$.

(ii) $M_1 =$ point, $f: M_2 \to M_3$ proper map of connected varieties.
$Z_{12} = pt \times M_2$, $Z_{23} = \mathrm{graph}(f) \Rightarrow Z_{12} \circ Z_{23} = \mathrm{Im} f \subset pt \times M_3$
$c \in H_*(M_2) = H_*(Z_{12})$ then: $c * [\mathrm{Graph} f] = f_*(c)$

(iii) M_3 is point, $f: M_1 \to M_2$ smooth map of oriented, connected manifold. $Z_{12} = \mathrm{Graph} f$,
$Z_{23} = M_2 \times pt$

2.7.18, 2.7.19 Associativity of convolution.

Consider M_1, M_2, M_3, M_4, Z_{12}, Z_{23}, Z_{34} then:

$$(c_{12} * c_{23}) * c_{34} = c_{12} * (c_{23} * c_{34})$$

* Convolution and Specilization

Consider manifold S, and $M_i \to S$ $(i=1,2,3)$ as locally trivial fiberations over S.

Then we ~~form~~ consider $Z_{12} \subset M_1 \times_S M_2$, $Z_{23} \subset M_2 \times_S M_3$, $p_{ij}: M_1 \times_S M_2 \times_S M_3 \to M_i \times_S M_j$, and

$p_{13}: p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times_S M_3$. Take image as Z_{13}.

With same construction, we get $*$: $H_*(Z_{12}) \times H_*(Z_{23}) \to H_*(Z_{13})$

Let $o \in S$, $i: \{o\} \hookrightarrow S$. Put $S^* = S \setminus \{o\}$, for $f: Z \to S$ let $Z^o = f^{-1}(o)$, $Z^* = f^{-1}(S^*)$

Assume: (i) $Z_{ij}^* \to S^*$ locally trivial filbration

(ii)
$$\begin{tikzcd} p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \arrow[rr, "p_{13}"] \arrow[dr] & & Z_{13}^* \arrow[dl] \\ & S^* & \end{tikzcd}$$

with p_{13} is morphism of locally fibration

then Speciliztion and convolution commutes:

$$\begin{tikzcd} H_*(Z_{12}^*) \otimes H_*(Z_{23}^*) \arrow[r, "\lim_{t \to 0}"] \arrow[d, "\text{convolution}"] & H_*(Z_{12}^o) \otimes H_*(Z_{23}^o) \arrow[d, "\text{convolution}"] \\ H_*(Z_{13}^*) \arrow[r, "\lim_{t \to 0}"] & H_*(Z_{13}^o) \end{tikzcd}$$

7.10 (continued)

(iii) Then
$$Z_{12}\circ Z_{23} = M_1\times pt$$
$d = \dim M_1 - \dim M_2$ then map $c \longmapsto [\mathrm{Graph} f]\times c$ is smooth pull back.
$H_i(M_2) \to H_{i+d}(M_1)$ $\quad$ ($\mathrm{Graph} f \in H_{\dim M_1}(M_1\times M_2)$)

2.7.13 Alternating viewpoints; convolution with ordinary homology.

~~View~~ $Z_{12}\subset M_1\times M_2$ closed gives us map $H_*(Z_{12}) \longrightarrow H_*(M_1\times M_2) = H_*(M_1)\otimes H_*(M_2)$

If $Z_{12}\to M_2$ is proper then we have map $H_*(Z_{12}) \longrightarrow H_*^{ord}(M_1)\otimes H_*(M_2)$

Similarly for $Z_{23}\subset M_2\times M_3$, then convolution:
$$\left(H_*^{ord}(M_1)\otimes \underline{H_*(M_2)}\right)\otimes\left(\underline{H_*^{ord}(M_2)}\otimes H_*(M_3)\right) \to H_*^{ord}(M_1)\otimes H_*(M_3)$$
Use Poincare duality here: $H_i^{ord}(Z) = H_c^{m-i}(M, M\setminus Z)$

* Use setting of Ex 2.7.10 (iii), Z_{12} arbitrary, $Z_{23} = M_2\times pt$, we get convolution.
$$H_i(Z_{12})\otimes H_j(M_2) \to H_{i+j-d}(M_1) \qquad (2.7.14)$$
Also we have $\quad H_i(Z_{12})\otimes H_j^{ord}(M_2) \to H^{ord}_{i+j-d}(Z_{12})$

(Also intersection pairing in 2.6.17)

Then with push forward by $Z_{12}\to M_1$, we get:
$$H_i(Z_{12})\otimes H_j^{ord}(M_2) \to H^{ord}_{i+j-d}(M_1) \qquad (2.7.15)$$
as 2.7.14, but in case of ordinary homology.

2.7.16, 2.7.17 Kunneth formula for convolution.

$$\begin{tikzcd}
 & H_*(Z_{12}\times \tilde Z_{12})\otimes H_*(Z_{23}\otimes \tilde Z_{23}) \arrow[dl, "\text{Kunneth}"'] \arrow[dr, "\text{Convolution}"] & \\
H_*(Z_{12})\otimes H_*(\tilde Z_{12})\otimes H_*(Z_{23})\otimes H_*(\tilde Z_{23}) \arrow[d, "\text{Convolution}"] & & H_*((Z_{12}\times\tilde Z_{12})\circ(Z_{23}\times\tilde Z_{23})) \arrow[d, equal] \\
H_*(Z_{12}\circ Z_{23})\otimes H_*(\tilde Z_{12}\circ\tilde Z_{23}) \arrow[rr, "\text{Kunneth}"] & & H_*((Z_{12}\circ Z_{23})\times(\tilde Z_{12}\circ\tilde Z_{23}))
\end{tikzcd}$$

Thus convolution map takes form: $H_*(Z) \times H_*(Z) \to H_*(Z)$

$\Rightarrow H_*(Z)$ ~~has~~ is associative algebra with unit, which is fundamental class of $M_\Delta \subset Z$

$x \in N$, let $M_x = \pi^{-1}(x)$, with $M_1 = M_2 = M$, ~~M_3~~ and $M_3 = pt$. $Z = Z_{12} = M \times_N M$ and

$Z_{23} = M_x \subset M \times \{pt\} \Rightarrow Z \circ M_x = M_x$

$\Rightarrow$ get $H_*(M_x)$ as left $H_*(Z)$-module

Example 2.7.43

Let N be point $\Rightarrow Z = M \times M$

$$H_\cdot(Z) \simeq H_\cdot(M) \otimes H_\cdot(M) \simeq H_\cdot(M) \otimes H_\cdot(M)^* \simeq \operatorname{End} H_\cdot(M)$$

Note: $H_\cdot(M) = H^\cdot(M) = H_\cdot(M)^*$ by Poincare duality.

From 2.7.14 (now $M_2 = M_1 = M$):

$$H_i(Z_{12}) \otimes H_j(M_2) \to H_{i+j-d}(M_1)$$

gives us algebra homomophism. $H_\cdot(Z) \to \operatorname{End} H_\cdot(M)$ which agree with chain of isomorphism above (Use formula in 2.7.13 to check).

$\Rightarrow H_\cdot(Z)$ is matrix algebra, $H_\cdot(M)$ is $H_\cdot(Z)$ simple-module.

Example 2.7.44

Let Y be smooth, compact, N smooth connected, and $M = Y \times N \Rightarrow Z = Y \times Y \times N$,

similar to 2.7.43, we have:

$$H_\cdot(Z) \simeq H_\cdot(Y) \otimes H_\cdot(Y)^* \otimes H_\cdot(N)^* \simeq (\operatorname{End} H_\cdot(Y)) \otimes H^\cdot(N)$$

(isomorphism as algebra isomorphism)

For any $x \in N \Rightarrow M_x = \pi^{-1}(x) \simeq Y$, so $H_\cdot(Y) = H_\cdot(M_x)$ is $H_\cdot(Z)$-module and we futher get:

$$H_\cdot(Z) \xrightarrow{\sim} (\operatorname{End} H_\cdot(Y)) \otimes H^\cdot(N) \xrightarrow{Id \otimes \varepsilon} (\operatorname{End} H_\cdot(Y)) \otimes \mathbb{C} = \operatorname{End} H_\cdot(Y)$$

ε is just $H^\cdot(N) \to H^0(N)$.

.25 Explicit formula

Consider X_1, X_2, X_3 complex manifolds, $Y_{12} \subset X_1 \times X_2$, $Y_{23} \subset X_2 \times X_3$ submanifolds.

Let $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$, $pr_{ij} : T^*(X_1 \times X_2 \times X_3) \to T^*(X_i \times X_j)$
for cotangent bundle.

Let $Y_{13} = Y_{12} \circ Y_{23}$ and $Z_{ij} = T^*_{Y_{ij}}(X_i \times X_j)$
(conormal bundle of Y_{ij})

Theorem 2.7.26:
Assume conditions:
(a) $p_{12}^{-1}(Y_{12})$, $p_{23}^{-1}(Y_{23})$ intersects transversely
(b) $p_{13} : p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \to Y_{13}$ is smooth locally trivially oriented fibration (base Y_{13}) with compact & smooth fiber F

Then: (i). $Z_{12} \circ Z_{23} = Z_{13}$ (set-theoretic)
(ii) $pr_{13} : pr_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to Z_{13}$ is smooth locally trivially oriented fibration with fiber F.
(iii) In $H_*(Z_{13})$: $[Z_{12}] * [Z_{23}] = \chi(F) \cdot [Z_{13}]$
where χ is Euler characteristic of F.

2.7.27 Remarks
i) When $X_1 = X_3 = pt$, $Y_{23} = X_2 \times pt$, to have (a) let $F \subset X_2$ be compact submanifold then $Y_{12} = pt \times F$. Then (iii) $\Leftrightarrow$ $\chi(F) = \int_{T^*F} s_0(F) \wedge s_0(F)$ (self intersection index of zero section $F \to T^*F$)
ii) If $Y_{12} \to X_2 \leftarrow Y_{23}$ has surjective differential $\Rightarrow$ (a) holds
iii) If map in (b) is bijective $\Rightarrow$ $pr_{12}^{-1}(Z_{12})$ and $pr_{23}^{-1}(Z_{23})$ transverse
$\Rightarrow$ pr_{13} in (ii) is isomorphism.

.7.40 Convolution Algebra

M be smooth complex manifold, and N a vareity: $\pi\ M \to N$
$M_1 = M_2 = M_3 = M$, $Z = Z_{12} = Z_{23} = M \times_N M$
$\Rightarrow Z \circ Z = Z = \{(m_1, m_2) \mid \pi(m_1) = \pi(m_2)\}$

2.7.45 Base locality

In 2.7.40, Let $U \subset N$ open, $M_U = \pi^{-1}(U)$

$$Z_U = M_U \times_U M_U = Z \cap (M_U \times M_U)$$

Lemma 2.7.46

(a) restriction to $H_*(Z) \to H_*(Z_U)$ is an algebra homomorphism

(b) $x \in U$ then $H_*(Z_U)$-module structure of $H_*(M_x)$ is from the restriction $H_*(Z) \to H_*$

2.7.47 Dimension property

M_1, M_2, M_3 smooth varieties of real dim m_1, m_2, m_3, $Z_{12} \subset M_1 \times M_2$, $Z_{23} \subset M_2 \times M_3$

and
$$p = \frac{m_1 + m_2}{2}, \quad q = \frac{m_2 + m_3}{2}, \quad r = \frac{m_1 + m_3}{2}$$

then convolution gives:

$$H_p(Z_{12}) \times H_q(Z_{23}) \to H_r(Z_{12} \circ Z_{23})$$

Now we take $M_1 = M_2 = M_3 = M$, $\dim_{\mathbb{R}} M = m$, $Z = M \times_N M$, $H(Z) = H_m(Z)$

then $H(Z)$ is subalgebra of $H_*(Z)$

Lemma 2.7.49: $\{Z_w\}_{w \in W}$ be irreducible components of Z indexed by finite set W.

If all components same dimension $\Rightarrow$ $[Z_w]_{w \in W}$ form basis of convolution algebra $H(Z)$

Next with $x \in N$, form M_x, and $H(Z) \subset H_*(Z)$ acts on $H_*(M_x)$ preserve the degree

$$H.(Z) * H_j(M_x) \subset H_j(M_x)$$

Corollary 2.7.50: Action of subalgebra $H(Z) \subset H_*(Z)$ on $H_*(M_x)$ preserves degree, i.e.

$$H(Z) * H_j(M_x) \subset H_j(M_x)$$

And we have analogue of middle dimension preserving in symplectic geometry:

Let (M_i, ω_i) be symplectic algebraic manifold then $M_i \times M_j$ with form $\omega_{ij} = p_i^* \omega_i - p_j^* \omega_j$ is symplectic.

Proposition 2.7.51, If $Z_{12} \subset M_1 \times M_2$ and $Z_{23} \subset M_2 \times M_3$ are both isotropic algebraic subvarieties

then ~~M_{12}~~ $Z_{12} \circ Z_{23}$ is isotropic in $M_1 \times M_3$

*) 2.7.27, remark (i) with more detailed explanation:

When $Y_{12} = pt \times F$, $Y_{23} = X_2 \times pt$ we get $Z_{12} = T^*_F(X_2)$ and $Z_{23} = X_2$
(both as submanifold of T^*X_2)

Then $[Z_{12}] * [Z_{23}] = [Z_{12}] \frown [Z_{23}] = T^*_F(X_2) \frown [X_2]$ which can be considered as
(in T^*X_2)

intersection of two copies of $S_0(F) \subset T^*_F$ $\Rightarrow$ we get $\chi(F) \cdot [Z_{13}]$ (Z_{13} is just a point and $\chi(F)$ counts self-intersection idex)

*) Proof of Associativity in 2.7.18

We have: $(c_{12} * c_{23}) * c_{34} = (p_{14})_* \left(\left((p_{13})_* (c_{12} \boxtimes [M_3] \frown [M_1] \boxtimes c_{23}) \boxtimes [M_4] \right) \frown ([M_1] \boxtimes c_{34}) \right)$

$= (p_{14})_* \left((p_{134})_* \left(c_{12} \boxtimes [M_3] \boxtimes [M_4] \frown M_1 \boxtimes c_{23} \boxtimes M_4 \right) \frown ([M_1] \boxtimes c_{34}) \right)$

$= (p_{14})_* \left((p_{134})_* \left(c_{12} \boxtimes [M_3] \boxtimes [M_4] \frown M_1 \boxtimes c_{23} \boxtimes M_4 \frown p_{134}^* ([M_1] \boxtimes c_{34}) \right) \right)$

(we use projection formula here)

$= (p_{14})_* \left(c_{12} \boxtimes [M_3] \boxtimes [M_4] \frown M_1 \boxtimes c_{23} \boxtimes M_4 \frown \boxtimes [M_1] \boxtimes [M_2] \boxtimes c_{34} \right)$

$= c_{12} * (c_{23} * c_{34})$ (by similar calculation)