X space: locally compact, homotopic to finite CW complex, closed embedding into a smooth mfd (e.g., complex or real variety).

$\boxed{\text{Definitions}}$ of BM homology ($\mathbb{C}$ coefficients):

① $\bar{X}$ compactification of X: $(\bar{X}, \bar{X}\setminus X)$ CW pair. Then $H_\bullet^{BM}(X) = H_\bullet(\bar{X}, \bar{X}\setminus X)$.
(e.g., $\bar{X} = X \cup \{\infty\}$).

② $H_\bullet^{BM}(X)$ is the homology of the cpx of locally finite singular chains (finite intersection with any compact).

(Recall $H_\bullet(X)$ is the homology of the cpx of finite singular chains.)

③ $X \subset M$ (smooth oriented m-mfd), proper retract of a closed nbhd. Then $H_\bullet^{BM}(X) \cong H^{m-\bullet}(M, M\setminus X)$.
In particular, $H_\bullet^{BM}(M) \cong H^{m-\bullet}(M)$.

For X cpct, $H_\bullet^{BM}(X) = H_\bullet(X)$.

$\boxed{\text{Properties w.r.t. maps}}$

$f: X \to Y$ proper $\rightsquigarrow f_*: H_\bullet(X) \to H_\bullet(Y)$ direct image (proper pushforward), e.g., $X \overset{\subset Y}{\text{closed}}$, $X \hookrightarrow Y$.

$U \hookrightarrow X$ (U open) $\rightsquigarrow H_\bullet(X) \to H_\bullet(U)$.

$F \subset X$ closed $\rightsquigarrow$ long exact sequence $\cdots \to H_p(F) \to H_p(X) \to H_p(X\setminus F) \to H_{p-1}(F) \to \cdots$
(comes from $\to H^k(M, M\setminus F) \to H^k(M, M\setminus X) \to H^k(M, M\setminus(X\setminus F)) \to \ldots$

$\boxed{\text{Fundamental class}}$ $[X] \in H_m(X)$ of a smooth oriented m-mfd X. (not necessarily cpct). For X irred. variety $/\mathbb{C}$, $H_m(X) \xrightarrow{\sim} H_m(X^{reg})$, so have $[X] \in H_m(X)$. In general, $[X]$ is the sum of $[X_i]$, X_i: top-dim components of X, and $[X_i]$ generate H_{top}.

$\boxed{\text{Intersection pairing}}$

$Z_1, Z_2 \subset M$ closed $\rightsquigarrow$ bilinear pairing $\cap: H_i(Z_1) \otimes H_j(Z_2) \to H_{i+j-m}(Z_1 \cap Z_2)$, dual to
$\cup: H^{m-i}(M, M\setminus Z_1) \otimes H^{m-j}(M, M\setminus Z_2) \to H^{2m-i-j}(M, (M\setminus Z_1)\cup(M\setminus Z_2))$. (e.g., $Z_1 = Z_2 = M$).

We can take ordinary homology: $H_i^{ord} \otimes H_j \to H_{i+j-m}^{ord}$. Recalling $H_c^{m-i}(M, M\setminus Z) \cong H_i^{ord}(Z)$, get Poincaré duality:

__Thm.__ M oriented connected sm. var.. $H_{m-j}^{ord}(M) \otimes H_j(M) \to H_0^{ord}(M)$ is nondegenerate.

Summarizing: $H_j(M) \cong H^{m-j}(M) \cong H_{m-j}^{ord}(M)^* \cong H_c^j(M)^*$.

Künneth formula: $H_\bullet(M_1 \times M_2) \xleftarrow{\boxtimes} H_\bullet(M_1) \otimes H_\bullet(M_2)$.

$\iota: N \overset{d}{\hookrightarrow} M$ closed embedding of mfds; for $Z \subset M$ closed, restriction with support $H_k(Z) \xrightarrow{i^*} H_{k-d}(Z \cap N)$,
$i^* = \cap[N]$. (Dual to $H^\bullet(M, M\setminus Z) \to H^\bullet(N, N\setminus Z)$. It depends on M!)

Special case $M \overset{\Delta}{\hookrightarrow} M \times M$, where, for $Z_1, Z_2 \subset M$ closed, $Z_1 \cap Z_2 = \Delta^*(Z_1 \boxtimes Z_2)$. [$H_\bullet(M\times M)$]

Pullbacks

$\tilde X \xrightarrow{p} X$ oriented locally trivial fibration with smooth fiber F, $\dim F = d \rightsquigarrow p^*: H_\bullet(X) \to H_{\bullet+d}(\tilde X)$.

Locally, $p^*: c \mapsto c \otimes [F]$ $(\tilde U = U \times F)$. General definition by derived sheaves.

$X \xrightarrow{i} \tilde X$ cts section $\rightsquigarrow$ Gysin pullback $i^*: H_\bullet(\tilde X) \to H_{\bullet-d}(X)$. Locally, $i^*: c \otimes [F] \to c$.

In general, have $i^* p^* = \mathrm{id}_{H_\bullet(X)}$. The i^*, p^* are "unique" among "natural" morphisms satisfying the local properties.

If $X \hookrightarrow (M$ sm. mfd$)$ and p is the restriction of $\bar p: \tilde M \to M$, then p^* comes from classical $\bar p^*: H^*(M, M \setminus X) \xrightarrow{\bar p^*} H^*(\tilde M, \tilde M \setminus \tilde X)$, similarly i^*, and Poincaré duality.

Then, let $X \subset M$ closed, $Y \subset \tilde M$. Have $p: \bar p^{-1}(Z) \to Z$.

If $\bar p^{-1}(X) \cap Y \to M$ is proper, then, for $c \in H_\bullet(X)$, $\tilde c \in H_\bullet(Y)$,

Thm. (Projection formula) $\bar p_*(p^* c \cap \tilde c) = c \cap (\bar p_* \tilde c) \in H_\bullet(\underbrace{\bar p(\bar p^{-1}(X) \cap Y)}_{X \cap \bar p(Y)})$,

where $p^*: H_\bullet(Z) \to H_\bullet(\bar p^{-1}(Z))$.

Specialization

$(S, 0)$ d-mfd smooth, $S^* = S \setminus \{0\}$, $\pi: Z \to S$, $Z_0 = \pi^{-1}(0)$. (Write $Z(S') = \pi^{-1}(S')$), π a locally trivial fibration over S^*. We will define $\lim: H_\bullet(Z(S^*)) \to H_{\bullet-d}(Z_0)$.

Locally, $(\mathbb{R}^d, 0)$. Let $\mathbb{H}^d = \overbrace{\mathbb{R}_{>0}}^{\mathbb{H}^1} \times \mathbb{R}^{d-1} \subset \mathbb{R}^d$. Then $H_d(\mathbb{H}^d) \cong H_1(\mathbb{H}^1)$. $Z(\mathbb{H}^d) \to \mathbb{H}^d$ can be assumed a trivial fibration with fiber F, so $H_\bullet(Z(\mathbb{H}^d)) \underset{\text{Kün}}{\cong} H_{\bullet-d}(F) \otimes H_d(\mathbb{H}^d) \cong H_{\bullet-d}(F) \otimes H_1(\mathbb{H}^1) \cong H_{\bullet-d+1}(Z(\mathbb{H}^1))$.

So, $\boxed{\text{first!}}$ enough to do it for $d=1$; have $H_{\bullet-d+1}(Z(\mathbb{H}^1)) \to H_{\bullet-d}(Z_0)$, from $H_j(Z_0) \to H_j(Z(\overline{\mathbb{H}}^1)) \to H_j(Z(\mathbb{H}^1)) \to H_{j-1}(Z_0) \rightsquigarrow \ldots$

Lemma. This is independent of the choice of chart.

For $S_1 \subset S$ k-submfd with $0 \in S_1$, have $\varepsilon^*: H_\bullet(Z(S^*)) \to H_{\bullet-k}(Z(S_1^*))$, since $\varepsilon: Z(S_1^*) \hookrightarrow Z(S^*)$ is locally a section of a sm. fibration.

Lemma $\lim^{S_1} = \varepsilon^* \circ \lim^S$. [~~Intersection~~ Specialization compatible with restriction]

$M \xrightarrow{\pi} S$ $(M, S$ smooth$)$ l.t. fib.. Write $Z(S^*) = Z \cap \pi^{-1}(S^*)$ for $Z \subset M$. Suppose π is trivial over S^* and $Z_1(S^*) \to S^*$, $Z_2(S^*) \to S^*$ $(Z_1, Z_2 \subset M$ closed$)$ both trivialized.

Lemma [Specialization commutes with $\cap$]

$$\begin{array}{ccc} H_\bullet(Z_1(S^*)) \otimes H_\bullet(Z_2(S^*)) & \xrightarrow{\cap} & H_\bullet(Z_1(S^*) \cap Z_2(S^*)) \\ \downarrow{\scriptstyle \lim} & & \downarrow{\scriptstyle \lim} \\ H_{\bullet-d}((Z_1)_0) \otimes H_{\bullet-d}((Z_2)_0) & \xrightarrow{\cap} & H_{\bullet-d}((Z_1)_0 \cap (Z_2)_0) \end{array} \qquad \begin{array}{ccc} H_\bullet(M(S^*))^{\otimes 2} & \to & H_\bullet(M(S^*)) \\ \downarrow & & \downarrow \\ H_{\bullet-d}(M_0)^{\otimes 2} & \to & H_{\bullet-d}(M_0) \end{array}$$

$\boxed{\text{Cohomology action}}$ of $H^\cdot(Z)$ on $H_\cdot(Z)$, $\overset{a}{H^i(Z)} \otimes \overset{c}{H_k(Z)} \to \overset{a\circ c}{H_{k-i}(Z)}$ arises from

$Z \hookrightarrow (U \text{ mfd})$, with $H^\cdot(U) \cong H^\cdot(Z)$ and $\cup : H^i(U) \otimes H^{\dim U - k}(U, U \setminus Z) \to H^{\dim U - (k-i)}(Z)$.

and Z retract of U

This construction is independent of $Z \hookrightarrow U$.

For $Z_1, Z_2 \subset M$, $\omega \in H^\cdot(Z_1)$, $c_1 \in H_\cdot(Z_1)$, $c_2 \in H_\cdot(Z_2)$, $(\omega \circ c_1) \cap c_2 = \omega|_{Z_1 \cap Z_2} \circ (c_1 \cap c_2)$.

In particular, get the obvious $(\omega \circ c_1) \cap c_2 = \omega \circ (c_1 \cap c_2)$ for $c_1, c_2 \in H_\cdot(M)$, $\omega \in H^\cdot(M)$.

$\boxed{\text{Thom isomorphism}}$

$\pi : V \to X$ sm. r v.b. $\rightsquigarrow e(V) \in H^r(X)$ Euler class., i zero section.

Thm $H_\cdot(X) \underset{\pi^*}{\overset{i^*}{\leftrightarrows}} H_{\cdot + r}(V)$, and for $c \in H_\cdot(X)$, $i^* i_*(c) = e(V) \cup c$.

$\boxed{\text{Cly}}$ Let $N \overset{i}{\hookrightarrow} M$ sm. mfd.s, codim d. Then, for $c \in H_\cdot(N)$, $i^* i_*(c) = e(T_N M) \cup c$. (Proof sketch: $T_N M$ is diffeomorphic to a tubular nbhd of N in M.)

$W \overset{j}{\hookrightarrow} V$ subbundle, then $j_*[W] = p^* e(V/W) \cdot [V] \in H_\cdot(V)$. (Apply j^* on both sides.)

~~Access intersection formula~~

$Z_1, Z_2 \hookrightarrow M$ closed, $Z = Z_1 \cap Z_2$ smooth. Let $T_{1,2} = T_Z M / (T_Z Z_1 + T_Z Z_2)$ bundle on Z.

Theorem (Access intersection formula) Suppose $T_z Z_1 \cap T_z Z_2 = T_z Z$ $\forall z \in Z$. Then

$$[Z_1] \cap [Z_2] = e(T_{1,2}) \circ [Z].$$