Braid group actions on categories of $\mathfrak{g}$-modules

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1 Category $\mathcal{O}$.

1.1 Preliminaries

Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field $\Bbbk$ of characteristic 0, let $\mathfrak{b}$ be one of its Borel subalgebras, $\mathfrak{h} \subset \mathfrak{b}$ be its Cartan subalgebra, let $\mathfrak{n} \subset \mathfrak{b}$ be the corresponding maximal nilpotent subalgebra and $\mathfrak{n}^-$ be the nilpotent subalgebra given by the negative root subspaces of $(\mathfrak{b}, \mathfrak{h})$ so that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{b}$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$.

By $(x, y), x, y \in \mathfrak{g}$, we denote the Killing form on $\mathfrak{g}$. Let Φ be the root system of $\mathfrak{g}$, let Φ^+ be the choice of the subset of positive roots corresponding to $\mathfrak{b}$. For $\alpha \in \Phi$ let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ be a coroot of $\mathfrak{g}$. Define the lattice of *integral weights* as follows: $\Lambda = \{\lambda \in \mathfrak{h}^* | (\lambda, \alpha^\vee) \in \mathbb{Z}, \alpha \in \Phi\}$. Weight $\lambda \in \Lambda$ is called *integral dominant* if $(\lambda, \alpha^\vee) \geq 0$ for $\alpha \in \Phi^+$. These are highest weights of finite-dimensional irreducible representations. Analogously, $\lambda \in \Lambda$ is called *integral anti-dominant* if $(\lambda, \alpha^\vee) \leq 0$. The convex hull of the set of dominant weights in $\Lambda \otimes \mathbb{R}$ is called the fundamental Weyl chamber. Put $\rho = \dfrac{1}{2} \sum_{\Phi^+} \alpha$ so that $(\rho, \alpha^\vee) = 1$ for $\alpha \in \Phi$.

Note: if $\mathfrak{g}$ is of type A_n, $\mathfrak{h}^* = \{(\lambda_0, ..., \lambda_n) | \sum \lambda_i = 0\}$. $\Phi = \{1_i - 1_j\}$, where 1_i is a vector in $\mathfrak{h}^*$ with 1 on the i'th coordinate and 0s on others, $\Phi^+ = \{1_i - 1_j\}, i < j$, simple roots are of the form $\alpha_i = 1_i - 1_{i+1}$, the Killing form restricted to $\mathfrak{h}^*$ is given by $(x, y) = 2n \sum x_i y_j$, $\alpha^\vee = \alpha/2n$, $\Lambda = \{(\lambda_0, ..., \lambda_n) | \sum \lambda_i = 0, (\lambda_i - \lambda_j) \in \mathbb{Z}\}$, dominant weights are those with decreasing coordinates and $\rho = (n/2, (n-2)/2, ..., -(n-2)/2, -n/2)$.

1.2 Basic definitions and properties.

Let A be an algebra (sheaf of algebras). By $A-\mathrm{mod}$ we denote the category of left A-modules (sheaves of left A-modules). We will need the following subcategory of $U(\mathfrak{g})-\mathrm{mod}$:

Definition. Category $\mathcal{O}$ is a full subcategory of $U(\mathfrak{g})-\mathrm{mod}$ consisting of all modules M satisfying following axioms:

$\mathcal{O}1$. M is a finitely generated $U(\mathfrak{g})$-module.

$\mathcal{O}2$. M is $\mathfrak{h}$-semisimple, that is, $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, M_λ being the subspace of weight λ.

$\mathcal{O}3$. M is locally $\mathfrak{n}$-finite: for each $v \in M$, the subspace $U(\mathfrak{n})v$ is finite-dimensional.

We will need the following important examples of modules in category $\mathcal{O}$:

Verma modules. For each weight λ define $\Delta_\lambda = Ind_{U(\mathfrak{b})}^{U(\mathfrak{g})}\langle v_\lambda\rangle$, where $\langle v_\lambda\rangle$ is the one-dimensional $U(\mathfrak{b}) = U(\mathfrak{h} \oplus \mathfrak{n})$-module generated by a vector v_λ such that $hv_\lambda = \lambda(h)v_\lambda, h \in \mathfrak{h}$ and $nv_\lambda = 0, n \in \mathfrak{n}$. Modules Δ_λ are called Verma modules. Note that $\Delta_\lambda \cong U(\mathfrak{n}^-)$ as vector spaces.

Irreducible modules. One can show that each Verma module Δ_λ has a unique irreducible quotient which we denote L_λ. It is finite-dimensional if and only if λ is an integral dominant weight.

Other important properties of $\mathcal{O}$ we will use are given by the following

Theorem. *(a) $\mathcal{O}$ is abelian.*

(b) For $M \in \mathcal{O}$ and L being a finite-dimensional module, $M \otimes L \in \mathcal{O}$ and $? \otimes L$ is an exact functor.

(c) Any $M \in \mathcal{O}$ is a finitely generated $U(\mathfrak{n}^-)$-module.

(d) $\mathcal{O}$ is noetherian and artinian (this means, in particular, that any $M \in \mathcal{O}$ has finite length).

(e) All weight spaces of modules in $\mathcal{O}$ are finite-dimensional.

(f) The set of all weights of a module in $\mathcal{O}$ lies in the finite union of the sets of weights of Verma modules.

1.3 Blocks in category $\mathcal{O}$.

Let $Z = Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Let χ be some character of Z. For $M \in \mathcal{O}$ denote $M^\chi = \{v \in M | (z - \chi(z))^n v = 0, \text{for some } n = n(z) \in \mathbb{N}\}$. Let $\mathcal{O}_\chi$ be a full subcategory of $\mathcal{O}$ consisting of modules M such that $M^\chi = M$. It was shown by Jose during his talk that central elements act as constants on Verma modules. Set $\mathcal{O}_\lambda = \mathcal{O}_{\chi_\lambda}$, where χ_λ is the character of Z given by it's action on Δ_λ. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Note that $\chi_\lambda = \chi_\mu$ iff $\mu = w \cdot \lambda$, where $w\cdot$ denotes the so-called dot-action of W: $w \cdot \lambda = w(\lambda + \rho) - \rho$, so $\mathcal{O} = \bigoplus \mathcal{O}_\lambda$, where the sum is over W-dot-orbits in $\mathfrak{h}^*$.

2 Translation functors

We now describe how different blocks $\mathcal{O}$ relate to each other. To do this we introduce *translation functors* $T_{\lambda\to\mu}$.

Let $pr_\lambda : \mathcal{O} \to \mathcal{O}_\lambda$ be the projection functor and let L be some finite-dimensional $\mathfrak{g}$-module. We will be interested in functors of the form $M \mapsto pr_\lambda(M \otimes L)$ mapping $\mathcal{O}_\mu$ to $\mathcal{O}_\lambda$. These are obviously exact.

From now on we assume that all the weights under consideration are integral. Fix two weights λ, μ and set $\nu = \mu - \lambda$. There is a unique element $w \in W$ such that $\bar{\nu} = w(\nu)$ is dominant. We define functors $T_{\lambda\to\mu} : \mathcal{O}_\lambda \to \mathcal{O}_\mu, M \mapsto pr_\mu(L_{\bar{\nu}} \otimes M)$.

Proposition 1. *$T_{\lambda\to\mu}$ is biadjoint to $T_{\mu\to\lambda}$.*

We leave this proposition as an exercise to the reader.

To say something non-trivial about these functors we need to lay further restrictions on λ and μ.

2.1 Facets

Fix a decomposition of Φ into a disjoint union of sets Φ_0, Φ_+, Φ_-. *Facet* corresponding to this decomposition is the set of integral weights λ satisfying

(a) $(\lambda+\rho, \alpha^\vee) = 0, \alpha \in \Phi_0$;

(b) $(\lambda+\rho, \alpha^\vee) > 0, \alpha \in \Phi_+$;

(c) $(\lambda+\rho, \alpha^\vee) < 0, \alpha \in \Phi_-$.

Note that for some choice of subsets the corresponding facet may be empty.

The closure of a facet is, evidently, the set of the same form with $\leq$ and $\geq$ instead of $<$ and $>$. A facet is called a chamber if $\Phi_0 = \emptyset$. Fundamental Weyl chamber gives an example of the (closure of a) facet.

2.2 Equivalence of blocks

We will need the following important

Theorem 1. *Let L be a finite-dimensional module. Then $L \otimes \Delta_\lambda$ has a finite filtration with qoutients isomorphic to $\Delta_{\lambda+\mu} \otimes L[\mu]$, $L[\mu]$ being the corresponding weight component of L.*

Then one can prove the following

Proposition 2. *Let λ, μ be integral weights and assume that μ lies in the closure of the facet containing λ, $\nu = \mu - \lambda$, $\bar{\nu} = w\nu$ is dominant. Then $W \cdot \mu$ does not contain any weights $\lambda + \nu'$, where ν' are weights appearing in $L_{\bar{\nu}}$ except, maybe, $\lambda + \nu$.*

Corollary. *Let λ, μ be integral weights and assume that μ lies in the closure of the facet containing λ. Then $T_{\lambda\to\mu}\Delta_{w\cdot\lambda} = \Delta_{w\cdot\mu}$ for any $w \in W$.*

Indeed, this follows straightforwardly from Theorem and Proposition.

Corollary. *Let λ, μ be integral weights and assume that μ lies in the closure of the facet containing λ. Then $T_{\lambda\to\mu}L_{w\cdot\lambda} = L_{w\cdot\mu}$ or 0 for any $w \in W$.*

Corollary. *If λ and μ lie in the same facet then $T_{\lambda\to\mu} : \mathcal{O}_\lambda \to \mathcal{O}_\mu$ is an equivalence of categories.*

Indeed, $T_{\lambda\to\mu}$ induces the isomorphism on K-theory and takes irreducibles to corresponding irreducibles. One also has a biadjoint functor $T_{\mu\to\lambda}$ with the same properties. Now proceeding by induction on the length of the module one gets the equivalence.

2.3 Wall-crossing and reflection functors

In the previous section we dealt with the case of λ and μ lying in the same facet. Properties we mentioned involved the translation from the facet to its closure (or *wall*). Now we deal with the opposite case – translation from the wall.

Theorem. *Let λ, μ be anti-dominant integral weights and assume that μ lies in the closure of the facet containing λ. Let $W_\lambda \subset W_\mu$ be the stabilizers of λ and μ in the Weyl group (with respect to the dot-action). Then the following formula for characters holds:*
$chT_{\mu\to\lambda}\Delta_{w\cdot\mu} = \sum_{w'\in W_\mu/W_\lambda} ch\Delta_{ww'\cdot\lambda}$.

This theorem follows by combinatorial computation from the Theorem 1 in the previous section.

Now assume that λ is anti-dominant regular and μ lies in the closure of the chamber of λ having stabilizer $\{1, s\} \subset W$, where s is some simple reflection. The corresponding functor $\Theta_s = T_{\lambda\to\mu}T_{\mu\to\lambda} : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$ is called a *wall-crossing* functor. Theorem above gives us $ch\Theta_s\Delta_{w\cdot\lambda} = ch\Delta_{w\cdot\lambda} + ch\Delta_{ws\cdot\lambda}$. This motivates the following construction.

Now consider complexes of functors given by adjunction morphisms: $F_s : 0 \longrightarrow \Theta_s \longrightarrow id \longrightarrow 0$ and $F_{s^{-1}} : 0 \longrightarrow id \longrightarrow \Theta_s \longrightarrow 0$. These give us an endomorphism of $K_0(\mathcal{O}_\lambda)$ and an endofunctor of $D^b(\mathcal{O}_\lambda)$. In the latter case corresponding functors are called *reflection functors*. Note that from the previous paragraph it is obvious that these functors give an acion of W on $K_0(\mathcal{O}_\lambda)$. In the following sections we will give a construction of the functors $F_s : D^b(\mathcal{O}_\lambda) \to D^b(\mathcal{O}_\lambda)$ via D-modules and explain that these functors define the action of the braid group.

3 Reflection functors via D-modules

Let $\mathcal{B}$ be the flag variety G/B of a connected algebraic group G corresponding to $\mathfrak{g}$ where B stands for the Borel subgroup of G corresponding to $\mathfrak{b}$.

Note that the category $\mathcal{O}$ defined above does not fit to the context of the localization of D-modules described in Jose's talk. There, global sections of D_λ-modules (we recall the definition below) were U_λ-modules, that is, modules with regular central character. There were also no restrictions on the $\mathfrak{n}$-action. To work with category $\mathcal{O}$ we need to modify this context as follows.

Let $N \subset B$ be the unipotent subgroup corresponding to $\mathfrak{n}$, and let $T \subset B$ be the torus, $B = N \cdot T$. We have the T-torsor $\pi : \tilde{\mathcal{B}} = G/N \to \mathcal{B}$. Put $\tilde{D} = \pi_*(D_{\tilde{\mathcal{B}}})^T$. The action of $G \times T$ on $\tilde{\mathcal{B}}$ given by $(g, t)aN = gatN$ gives a map $U(\mathfrak{g}) \otimes U(\mathfrak{h}) \to D_{\tilde{\mathcal{B}}}$. Define $D_\lambda = \tilde{D} \otimes_{U(\mathfrak{h})} \Bbbk_\lambda$, where $\Bbbk_\lambda = U(\mathfrak{h})/m_\lambda$, m_λ being the maximal ideal corresponding to λ. This is the sheaf of twisted differential operators we had before. Now note that $U(\mathfrak{h}) = S(\mathfrak{h})$ and set $\hat{S(\mathfrak{h})} = \varprojlim S(\mathfrak{h})/m_\lambda^n$ (see [1]). Now global sections of $D_{\hat{\lambda}} = \tilde{D}\otimes_{S(\mathfrak{h})\hat{S(\mathfrak{h})}}$-modules are $U(\mathfrak{g})$-modules with the generalized central character χ_λ. We denote this category $U_{\hat{\lambda}} - \mathrm{mod}$.

Definition. B-equivariant D-module on $\mathcal{B}$ is a D-module with a structure of B-equivariant quasi-coherent sheaf such that the derivation of the B-action given by this structure corresponds to the $D \supset \mathfrak{b}$-action.

Let $D - \mathrm{mod}_N$ be a category of equivariant D-modules on $\mathcal{B}$. We now face the technical problem of defining the equivariant derived category of $D-\mathrm{mod}_N$. Note that it is **not** the derived category of the category $D - \mathrm{mod}_N$. See [3] for the correct definition.

Statements, parallel to the localizaion theorems for $(D_\lambda - \mathrm{mod}, U_\lambda - \mathrm{mod})$ hold for $(D_{\hat{\lambda}} - \mathrm{mod}, U_{\hat{\lambda}} - \mathrm{mod})$ and $(D_{\hat{\lambda}} - \mathrm{mod}_N, \mathcal{O}_\lambda)$.

Now we state the properties of the geometric version of translation functors. Again, we have the following two cases, translation to the wall and from the wall (compare with the previous section):

Theorem. *Assume that μ lies in the closure of the facet containing λ. The following diagram*

$$\begin{array}{ccc} D^b(D_{\hat{\lambda}} - \mathrm{mod}_N) & \xrightarrow{\otimes \mathcal{O}(\mu-\lambda)} & D^b(D_{\hat{\mu}} - \mathrm{mod}_N) \\ \Big\downarrow{\scriptstyle R\Gamma_\lambda} & & \Big\downarrow{\scriptstyle R\Gamma_\mu} \\ D^b(\mathcal{O}_\lambda) & \xrightarrow{T_{\lambda\to\mu}} & D^b(\mathcal{O}_\mu) \end{array} \tag{1}$$

commutes.

The following theorem is parallel to the character computation for the wall-crossing functor.

Theorem. *Assume that λ is regular and μ lies in the s-face of the closure of the chamber containing λ, $\mathcal{M} \in D^b(\mathcal{O}_\mu)$. If $\lambda > \lambda \cdot s$ we have the following exact triangle in $D^b(\mathcal{O}_\lambda)$:*

$$R\Gamma_{\lambda\cdot s}(\mathcal{M} \otimes \mathcal{O}(\lambda \cdot s - \mu)) \to T_{\mu\to\lambda} R\Gamma_\mu(\mathcal{M}) \to R\Gamma_\lambda(\mathcal{M} \otimes \mathcal{O}(\lambda - \mu)).$$

3.1 The action of generators of the braid group via comparing different localizations

Let λ and μ be some regular weights. One can define the intervening functor $\mathcal{F}_{\lambda\to\mu} : D^b(\mathcal{O}_\lambda) \to D^b(\mathcal{O}_\mu)$:

$$\mathcal{F}_{\lambda\to\mu}(M) = R\Gamma_\mu(L\mathcal{L}oc_\lambda M \otimes \mathcal{O}(\mu - \lambda)).$$

Theorem. *Assume that $\lambda \cdot s < \lambda$. Then $F_s \cong \mathcal{F}_{\lambda\to s\cdot\lambda}$, $F_{s^{-1}} \cong \mathcal{F}_{s\cdot\lambda\to\lambda}$.*

For the reduced expression $\underline{w} = s_1 ... s_k$ define $F_{\underline{w}} = F_{s_1} \circ ... \circ F_{s_k}$. Note that from this description of F_s given above it is evident that braid relations of the form $F_{w_1} \circ F_{w_2} = F_{w_2 \circ w_1}, l(w_2 w_1) = l(w_1) + l(w_2)$, are satisfied: $\mathcal{O}_{s\cdot\lambda-\lambda} \otimes \mathcal{O}_{ts\cdot\lambda - s\cdot\lambda} = \mathcal{O}_{ts\cdot\lambda-\lambda}$ so the functors on the right hand side and on the left hand side are intertwining for the same pair of equivalences of categories.

3.2 The action of generators of the braid group via Radon transform

We briefly mention another realization of the action of generators of the braid group on $D^b(\mathcal{O}_\lambda)$.

Assume that λ is as before. We will define a new intertwining functor $I_s : D^b(D_{\widehat{\lambda\cdot s}} - \mathrm{mod}) \to D^b(D_{\hat{\lambda}} - \mathrm{mod})$ such that $L\mathcal{L}oc_\lambda = I_s L\mathcal{L}oc_{\lambda\cdot s}$. As $w \cdot \lambda$ and λ give the same central character this intertwining functor will give us some autoequivalence of $D^b(U_{\hat{\lambda}} - \mathrm{mod})$.

Now assume $s \in W$ is a simple reflection. We have the Schubert cell $Y_s = \{(b, sb) \in X \times X\}$ and let pr_1 and $pr_2 : Y_s \to X$ denote projections to the first and the second factors respectively. Define

$$I_s = pr_{1*}(pr_2^*).$$

Here pr_{i*} and pr_i^* stand for derived direct and inverse image functors for D-modules. We sketch the definitions below, following [2].

Let $\pi : Y \to Z$ be an affine morphism. Let D be a t.d.o. on Z. Define $\pi^\circ(D)$ to be a sheaf of algebras on Y which is a sheaf-pullback of D. Define $D_{Y\to Z}$ to be a pull-back of D as a quasi-coherent sheaf. At last, define D^π to be a sheaf of differential endomorphisms of $\pi^*(D)$ commuting with the right action of $\pi^\circ(D)$. One can check that D^π is a t.d.o. By definition, $D_{Y\to Z}$ is a $D^\pi - \pi^\circ(D)$-bimodule and for a D-module M we define $\pi^*M = D_{Y\to Z} \otimes \pi^\circ M$ which is a D^π-module.

To define the direct image we define a $\pi^\circ(D) - D^\pi$-bimodule $D_{Z\to Y}$ and put $\pi_* M = \pi_\circ(D_{Z\to Y} \otimes M)$, where $\pi_\circ$ is a sheaf-direct image. This bimodule is defined as follows: $D_{Z\to Y} = \Omega_Y \otimes_{\mathcal{O}_Y} \pi^*(\Omega_Z^{-1} \otimes_{\mathcal{O}_Z} D^{op})$.

References

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