Differential operators in charachteristics p

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1 Differential operators in characteristic p

1.1 Frobenius twist

Fix an algebraically closed field $\Bbbk, char\Bbbk = p > 0$. Recall that for such a field the Frobenius map $x \mapsto x^p$ gives an automorphism. Let X be a reduced, smooth and irreducible scheme over $\Bbbk$. Consider the *Frobenius morphism* $Fr_p : X \to X, f \in \mathcal{O}_X \mapsto f^p$ (this is the morphism of schemes, not $\Bbbk$-schemes). Let $X^{(1)}$ be another scheme over $\Bbbk$, identical to X as a topological space but having a $\Bbbk$-action twisted by the Frobenius automorphism of $\Bbbk$: $k \cdot f = k^{1/p} f, k \in \Bbbk, f \in \mathcal{O}_X$. Then Fr_p gives an isomorphism of X and $X^{(1)}$ as abstract schemes. Fr_p is a bijection on $\Bbbk$-points. If X is defined over $\Bbbk$ Fr_p is the isomorphism of $\Bbbk$-schemes.

Let V, V' be vector spaces over $\Bbbk$. Define $V^{(1)}$ to be a vector space isomorphic to V as an abelian group but having a twisted $\Bbbk$-action as above. We call an additive map $A : V \to V'$ *p-linear*, if $A(kv) = k^p v$ for $k \in \Bbbk, v \in V$ – that is, this is a linear map from $V^{(1)}$ to V'.

1.2 Crystalline differential operators and their center

Representation theory and the theory of D-modules in positive characteristics are quite different from their characteristic 0 analogues. In positive characteristics, (finite-dimensional) representations of an algebraic group G no longer correspond to representations of $U(\mathfrak{g})$, $\mathfrak{g}$ being the Lie algebra of G (one should rather consider the universal enveloping algebra with divided powers). On the other hand, $U(\mathfrak{g})$ has a large center in positive characteristic ($U(\mathfrak{g})$ is finitely generated over the center) which makes it easier to study $U(\mathfrak{g})$-modules.

Recall that for an affine X and $\mathcal{O}_X$-modules M, N the ring of Grothendieck differential operators $D(M, N)$ is defined as follows. $D(M, N)$ is a filtered ring, having $D_{-1} = 0$ and $D_k = \{L \in Hom_\Bbbk(M, N) : [L, f] \in D_{k-1} \text{ for all } f \in \mathcal{O}_X\}$. Grothendieck differential operators in positive characteristic do not posses many properties of differential operators in characteristic 0. For example, there is no symbol map. Crystalline differential operators (see definition below) are in many ways similar to differential operators in characteristic 0. The main difference is, again, as follows: crystalline differential operators are coherent over their center. Moreover, over a suitable flat cover of the variety corresponding to the center, the sheaf of crystalline differential operators looks particularly simple – it is isomorphic to an endomorphism sheaf of a vector bundle. Sheaves of algebras that become endomorphism sheaves on a flat cover are called Azumaya algebras. We proceed with the formal definitions.

Definition. The sheaf of *crystalline differential operators* D_X is the sheaf of rings generated by $\mathcal{O}_X$ and $\mathcal{T}_X$, where the latter is a tangent sheaf of X subject to the relations: $\partial f - f\partial = \partial(f)$, $\partial'\partial'' - \partial''\partial' = [\partial', \partial'']$. D_X has a natural structure of an $\mathcal{O}_X$-module.

The action of crystalline differential operators on $\mathcal{O}$, unlike the characteristic 0 case, is not faithful: note that for the unital algebras in characteristic p, p's power of a derivation ∂ is again a derivation: $\partial^p(fg) = \sum C_p^i \partial^i(f)\partial^{p-i}(g) = \partial^p(f)g + f\partial^p(g)$. We will denote this derivation $\partial^{[p]}$. Consider a map $i : \mathcal{T}_X \to D_X, \partial \mapsto \partial^p - \partial^{[p]}$. This map is $\mathcal{O}_X^{(1)}$-linear, that is $i(f\partial) = f^p\partial$ and lands into the kernel of the action on $\mathcal{O}_X$. It gives a map $i : \mathcal{T}_X^{(1)} \to D_X$ (which we denote by the same letter - it will cause no confusion).

We have the standard geometric filtration $D_{\leq n,X}$ on D_X. It's properties are summarized in the following

Proposition 1. *(a)* $gr(D_X) \cong \mathcal{O}_{T^*X}$.

(b) $\mathcal{O}_{T^*X}$ *carries a Poisson algebra structure* $\{f_1, f_2\} = [\overline{f_1}, \overline{f_2}] mod D_{\leq deg f_1 + deg f_2 - 2, X}$ *where* $\overline{f_1}, \overline{f_2}$ *are some lifts of* f_1, f_2 *to* D_X

(c) The canonical map $D_{\leq p-1,X} \to End_{\Bbbk}(\mathcal{O}_X)$ *is an inclusion.*

First two properties are parallel to the case of characteristic 0 and the third one is a computation.

We shall see most of important properties of D_X in the case of the Weyl algebra.

1.3 Weyl algebra

Let $\mathbb{W} = \Bbbk\langle x_1, ..., x_n, \partial_1, ..., \partial_n\rangle$ be the Weyl algebra – an algebra generated by commuting $\{x_i\}$ and commuting $\{\partial_i\}$ with relations $[\partial_i, x_j] = \delta_{ij}$, $\mathbb{W} = D_{\mathbb{A}^n}$. Note that x_i^p for all i lies in the center of $\mathbb{W}$: $[x_i^p, \partial_j] = px_i^{p-1}\delta_{ij} = 0$. Note also that $[\partial_i^p, x_i] = p\partial^{p-1} = 0$. So the center $Z(\mathbb{W})$ of $\mathbb{W}$ contains $\Bbbk[x_1^p, ..., x_n^p, \partial_1^p, ..., \partial_n^p] = \mathcal{O}_{T^*\mathbb{A}^{n(1)}}$. Denote $\mathfrak{Z} = \Bbbk[x_1^p, ..., x_n^p, \partial_1^p, ..., \partial_n^p]$.

Proposition 2. *The center of* $\mathbb{W}$ *is isomorphic to* $\mathfrak{Z}$.

To show this we will show that fibers of $\mathbb{W}$ considered as a $\mathcal{O}_{T^*\mathbb{A}^{n(1)}}$-coherent sheaf of algebras have trivial center. We first note that there is an inclusion of $\mathfrak{Z} \to \Bbbk[x_1, ..., x_n, \partial_1^p, ..., \partial_n^p] = \mathcal{A}$. Note that $Spec\mathfrak{Z} = T * \mathbb{A}^{n(1)}, Spec\mathcal{A} = T^{*,1}\mathbb{A}^n$ is pullback of the twisted cotangent bundle to X from $X^{(1)}$.

Proposition 3. *Fibers of* $\mathbb{W}$ *considered as a* $\mathfrak{Z}$*-module are isomorphic to matrix algebras.*

Indeed, in the local coordinates at $\zeta \in Spec\mathfrak{Z}, \zeta = (a_1^p, ..., a_n^p, b_1^p, ..., b_n^p)$, $\mathbb{W}_\zeta$ has a basis of the form $x^\alpha\partial^\beta, \alpha, \beta$ being n-tuples of integers from $\{0, ..., p-1\}$, where $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}, \partial^\beta = \partial_1^{\beta_1}...\partial_n^{\beta_n}$ with $x_i^p = a_i^p, \partial_i^p = b_i^p$. So $\mathbb{W}_\zeta$ has dimension p^{2n}. Let ξ be a unique lift of a point ζ to $Spec\mathcal{A}$. Consider the module $\delta^\xi = \mathbb{W} \otimes_{\mathcal{A}} \mathcal{A}_\xi$. δ^ξ has a basis of the form $\partial^\beta, \beta \in \{0, ..., p-1\}^n$, $x_i\partial^\beta = [x_i, \partial^\beta] + a_i\partial^\beta, \partial_i^p = b_i^p$. Note that $\dim \delta^\xi = p^n$ so it is enough to show that δ^ξ is an irreducible $\mathbb{W}_\zeta$-module, which is standard.

We get that $\mathbb{W}$ is a sheaf of algebras over $T^*\mathbb{A}^{n,(1)}$ whose fibers are matrix algebras. Note that in this case we see that $\mathbb{W}$ itself is not isomorphic to an endomorphism ring of any bundle: endomorphism ring of every vector bundle (which is trivial over $\mathbb{A}^n$) have zero divisors, while $\mathbb{W}$ has none (because its associate graded has none).

1.4 General case

Set $\mathfrak{Z}_X = \pi_*(\mathcal{O}_{T^*X^{(1)}})$, where $\pi : T^*X^{(1)} \to X^{(1)}$ is the standard projection. Set $T^{*,1}X = X \times_{X^{(1)}} T^*X^{(1)}$, $\mathcal{A}_X = \pi'_*(\mathcal{O}_{T^{*,1}X})$, where $\pi' : T^{*,1}X \to X$ is a pullback of the standard projection.

Proposition 4. *Map i extends to a map $\mathfrak{Z}_X \to D_X$ by sending $Fr_X(f)$ to f^p.*

Image of i lies in the center of $Z(D_X)$: locally $[f^p, \partial] = pf^{p-1}\partial(f) = 0$, and $[i(\partial), f], [i(\partial), \partial']$ are both in $D_{\leq p-1,X}$ and in the kernel of the action on $\mathcal{O}_X$, so are equal to 0. Indeed, the first commutator is in $D_{\leq p-1,X}$ by definition and the second one is there because, for any Poisson algebra in characteristic p, p^{th} power of any element lies in the Poisson center: $\{f^p, g\} = -\{g, f^p\} = -pf^{p-1}\{g, f\}$.

Proposition 5. $\mathcal{A}_X = \mathcal{O}_X \cdot \mathfrak{Z}_X$ *(considered in D_X).*

We can now view D_X as a $\mathfrak{Z}_X$- and $\mathcal{A}_X$-module and, hence, as a sheaf of algebras over $T^*X^{(1)}$ and a coherent sheaf over $T^{*,1}X$.

For a point $\zeta = (b, \omega) \in T^*X^{(1)}$ by δ_ζ we denote a point module $D_X \otimes_{\mathfrak{Z}} \mathcal{O}_\zeta$. $\Bbbk$ is algebraically closed so there exists a unique pullback $a \in X$ of the point $b \in X^{(1)}$. Set $\xi = (a, \omega) \in T^*X^{(1)}, \delta^\xi = D_X \otimes_{\mathcal{A}_X} \mathcal{O}_\xi = \delta_a \otimes_{\mathfrak{Z}_X} \delta_\zeta$.

Proposition 6. $D_{X,\zeta} = End_\Bbbk(\Gamma(X, \delta^\xi))$

This is a local computation which we already did for $\mathbb{W}$.

Proposition 7. *i extends to an isomorphism $\mathfrak{Z}_X = Z(D_X)$*

This is because, as a sheaf of algebras on $T^*X^{(1)}$, D_X has stalks isomorphic to matrix algebras, hence image of the center in these stalks is canonically isomorphic to $\Bbbk$.

2 Azumaya property

2.1 Central simple and Azumaya algebras

Let $\Bbbk$ be an arbitrary field.

Definition. Unital algebra A over $\Bbbk$ is called a simple central algebra if A is simple as a ring and $Z(A) = \Bbbk$.

Proposition 8. *Let $\overline{\Bbbk}$ be an algebraic closure of $\Bbbk$ and let A be a unital central algebra. Then $A \otimes_\Bbbk \overline{\Bbbk} \cong Mat_n(\overline{\Bbbk})$, where $Mat_n(\overline{\Bbbk})$ is an algebra of $n \times n$ matrices over $\overline{\Bbbk}$.*

Example. Consider the ring of quaternions $\mathbb{H}$ as an algebra over $\mathbb{R}$. Then $\mathbb{H}$ becomes a simple central algebra. It is well-known that $\mathbb{C} \otimes_\mathbb{R} \mathbb{H} \cong Mat_2\mathbb{C}$.

Let X be a scheme over $\Bbbk$. An $\mathcal{O}_X$-algebra is called an Azumaya algebra over X if it is locally-free and coherent as an $\mathcal{O}_X$-module and A_x is a central simple algebra for all geometric points $x \in X$.

Remember that a morphism $X \to Y$ is called a flat cover if it is flat, finitely presented (ring of functions of X is locally finitely-presented over the ring of functions of Y), and quasi-finite (fibers are finite).

Proposition 9. *The following properties are equivalent.*

(a) *A is an Azumaya algebra over X.*

(b) *There is a flat cover $(U_i \to X)$ such that for any i there exists r_i such that $A \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong M_{r_i}(\mathcal{O}_{U_i})$, where $M_{r_i}(\mathcal{O}_{U_i})$ denotes the sheaf of matrix algebras over U_i.*

2.2 Splitting of D_X over the flat cover

In this section we prove that D_X splits over $\mathcal{A}_X$.

Proposition 10. *$D_X \otimes_{\mathfrak{Z}_X} \mathcal{A}_X \cong End_{\mathcal{A}_X}((D_X)_{\mathcal{A}_X})$, where by $(D_X)_{\mathcal{A}_X}$ we mean D_X viewed as an $\mathcal{A}_X$-module – D_X is a vector bundle over $\mathcal{A}_X$.*

This Proposition is local and parallel to the case of $\mathbb{W}$.

3 Universal enveloping algebra in positive characteristics

Let G be an algebraic group and let $\mathfrak{g}$ be its Lie algebra. By definition, $\mathfrak{g}$ is isomorphic to the algebra of G-(right)invariant derivations of $\mathcal{O}_G$, that is derivations commuting with the right G-action ([5]). As we have seen before, if ∂ is a derivation, ∂^p is a derivation too, and if ∂ is G-invariant ∂^p is, obviously, also G-invariant. This gives $\mathfrak{g}$ a structure of a *restricted* Lie algebra, that is a Lie algebra equipped with a map $x \mapsto x^{[p]}$ (we won't need a precise definition, see [5]). Consider a p-linear map $i : \mathfrak{g} \to U(\mathfrak{g}), x \mapsto x^p - x^{[p]}$. As we have seen before, this maps $\mathfrak{g}$ to the center of D_G and hence to the center of $U(\mathfrak{g})$. The image of this map is called the *Frobenius center* and is denoted $\mathfrak{Z}_{Fr}$. Note that $U(\mathfrak{g})^G$ is also contained in the center and is called the Harish-Chandra center, denoted $\mathfrak{Z}_{HC}$.

A prime p is called good if it does not coincide with a coefficient of a simple root in the maximal root, and p is called very good if it is good and G does not contain a factor isomorphic to $SL(mp)$. Let p be very good.

Proposition 11 ([3])**.** *$Z(U(\mathfrak{g}))$ is generated by $\mathfrak{Z}_{Fr}$ and $\mathfrak{Z}_{HC}$. Moreover, it is isomorphic to $\mathfrak{Z}_{Fr} \otimes_{\mathfrak{Z}_{Fr}^G} \mathfrak{Z}_{HC}$.*

Now note that $\mathfrak{Z}_{Fr} \cong \mathcal{O}_{\mathfrak{g}^{(1)}} = S(\mathfrak{g}^{*(1)})$, $\mathfrak{Z}_{HC} \cong S(\mathfrak{h}^*)^W$.

Proposition 12. *$S(\mathfrak{g}^{*(1)})^G \cong S(\mathfrak{h}^{*(1)})^W$.*

Finally we get

Theorem 1. *$Spec(Z(U(\mathfrak{g}))) \cong \mathfrak{g}^{*(1)} \times_{\mathfrak{h}^{*(1)}/W} \mathfrak{h}^*/W$, where the map $\mathfrak{h}^*/W \to \mathfrak{h}^{*(1)}/W$ comes from a map $S(\mathfrak{h}^{(1)}) \to S(\mathfrak{h})$ given by $h \mapsto h^p - h^{[p]}$ (this map is called an Artin-Schrieir map).*

3.1 Comoment maps

Let $\mathcal{B}$ be the flag variety $\mathcal{B} = G/B$. Now let N be a unipotent subgroup corresponding to $\mathfrak{n}$. For $\tilde{\mathcal{B}} = G/N$ we have a T-torsor $\pi : \tilde{\mathcal{B}} \to \mathcal{B}$. Put $\tilde{D} = \pi_*(D_{\tilde{\mathcal{B}}})^T$. One has $T^*\tilde{\mathcal{B}} = \tilde{\mathfrak{g}}^* = \{(b, x) : b \in \mathcal{B}, x \in \mathfrak{g}^*, x(\operatorname{rad} b) = 0\}$, so that

$\tilde{\mathfrak{g}}^* = \mathfrak{g}^* \times_{\mathfrak{h}^*/W} \mathfrak{h}^*$. Calculation analogous to the one we did for D shows that the center of $\tilde{D}$ is isomorphic to $\tilde{\mathfrak{g}}^{*(1)} \times_{\mathfrak{h}^{(1)}/W} \mathfrak{h}$, where the map $\mathfrak{h} \to \mathfrak{h}^{(1)}/W$ comes from the Artin-Schrieir map.

Proposition 13. *The map $U(\mathfrak{g}) \otimes U(\mathfrak{h}) \to \tilde{D}$ factors through $U(\mathfrak{g}) \otimes_{\mathfrak{Z}_{HC}} U(\mathfrak{h})$.*

Proposition 14. *The comoment map $\mathfrak{g} \to \mathcal{T}_{\mathcal{B}}$ commutes with $\cdot^{[p]}$, that is preserves restricted structure.*

So the map $\mathfrak{g} \to \mathcal{T}_{\mathcal{B}}$ gives the map $\mathfrak{Z}_{HC} = \mathcal{O}_{\mathfrak{g}^{*(1)}} \to Z(\tilde{D})$.

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