Dacha lectures.

Lecture 1.

Let V be a finite dimensional vector space / $\mathbb{C}$.
Let Γ be a finite subgroup of $Sp(V)$. We can form the semidirect product $H_0 = SV \# \Gamma$, which is the tensor product $SV \otimes \mathbb{C}\Gamma$, with multiplication rule $(f_1 \otimes \gamma_1)(f_2 \otimes \gamma_2) = f_1 \gamma_1(f_2) \otimes \gamma_1\gamma_2$. H_0 is a $\mathbb{Z}_+$-graded algebra, with Γ sitting in degree 0 and V sitting in degree 1. We would like to study algebras H which are filtered, and $\mathrm{gr} H \cong H_0$. To this end, let $\varkappa : \Lambda^2 V \to \mathbb{C}\Gamma$ be a linear map, and define $H_\varkappa$ to be the quotient of $TV \# \Gamma$ by the relation

$$[x, y] = \varkappa(x, y) \qquad (1)$$

for $x, y \in V$ (note that LHS has degree 2 and RHS has degree 0). The algebra $H_\varkappa$ has a natural incr. filtration with $\deg(V) = 1$, $\deg(\Gamma) = 0$, and we have a natural surjective homomorphism $\varphi : H_0 \to \mathrm{gr}(H_\varkappa)$. It is not always an isomorphism. Indeed, let $x, y, z \in V$, and let us write down the Jacobi identity:

$$0 = [[x,y],z] + [[y,z],x] + [[z,x],y] \qquad (2)$$

Substituting (1) into (2), we get

$$0 = [\varkappa(x,y), z] + [\varkappa(y,z), x] + [\varkappa(z,x), y]$$

Writing $\varkappa = \sum_{g\in\Gamma} \varkappa_g \cdot g$, we get

$$0 = \sum_{g\in\Gamma}\left(\varkappa_g(x,y)(z^g - z)g + \varkappa_g(y,z)(x^g - x)g + \varkappa_g(z,x)(y^g - y)g\right)$$

Thus for any $g\in\Gamma$ we must have

$$\varkappa_g(x,y)(z^g - z) + \varkappa_g(y,z)(x^g - x) + \varkappa_g(z,x)(y^g - y) \qquad (3)$$

in order for φ to be an isomorphism. (it's a necessary condition).

We claim that this implies $\varkappa_g(x,y) \equiv 0$ for any g with $\mathrm{rk}(g-1)|_V > 2$. Indeed, $x^g - x \in \mathrm{Im}(g-1)$, so if $\varkappa_g(x,y) \neq 0$ then

$$z^g - z = \frac{\varkappa_g(y,z)(x^g - x) + \varkappa_g(z,x)(y^g - y)}{\varkappa_g(x,y)},$$

hence $z^g - z$ runs over a space of dimension ≤ 2 as z varies. So $\dim \mathrm{Im}(g-1) \leq 2$, as claimed.

Note that if $g \neq 1$ then $\dim \mathrm{Im}(g-1) \geq 2$, since $\mathrm{Im}(g-1)$ is a symplectic vector space.

<u>Definition.</u> A semisimple element $g \in Sp(V)$ is a symplectic reflection if $\mathrm{rk}(g-1) = 2$.

So we get:

Prop 1.1. If φ is an isomorphism then $\varkappa_g(x,y)=0$ unless $g=1$ or g is a symplectic reflection.

So denoting the set of symplectic reflections in Γ by S, we get, when φ is an isom.

$$\varkappa(x,y)=\varkappa_1(x,y)+\sum_{g\in S}\varkappa_g(x,y)g. \qquad (4).$$

Moreover, we can get information on the properties of $\varkappa_g(x,y)$ if $g=1$ or $g\in S$. Indeed, first of all, conjugating (1) by $\gamma\in\Gamma$, we see that $\varkappa_1:\Lambda^2V\to\mathbb{C}$ is a Γ-invariant bilinear form. Also, if $g\in S$, $\varkappa_g(x,y)$ is g-invariant (for the same reason), so $\varkappa_g=\varkappa_g^{(1)}\oplus\varkappa_g^{(2)}$, where $\varkappa_g^{(1)}$ is a form on $\mathrm{Im}(g-1)$ and $\varkappa_g^{(2)}$ is a form on $\mathrm{Ker}(g-1)=\mathrm{Im}(g-1)^{\perp}$. But if $x,y\in\mathrm{Ker}(g-1)$ then by (3), taking $z\in\mathrm{Im}(g-1)$, we get that $\varkappa_g(x,y)=0$. So $\varkappa_g(x,y)=c_g\omega(P_gx,P_gy)$, where $c_g\in\mathbb{C}$, $\omega\in\Lambda^2V^*$ is the symplectic form, and $P_g:V\to\mathrm{Im}(g-1)$ is the projector along $\mathrm{Ker}(g-1)$. Note also that c_g are invariant with respect to conjugation.

These are necessary conditions for φ being an isomorphism, but they are also sufficient.

<u>Theorem</u> 1.2. If $\varkappa(x,y) = \varkappa_1(x,y) +$

$$+ \sum_{g\in S} c_g \omega(P_g x, P_g y),$$

where $\varkappa_1$ is invariant and c_g are invariant under conjugation, then φ is an isomorphism.

<u>Definition</u>. In the situation of Th. 1.2 the algebra $H_\varkappa$ is called the symplectic reflection algebra.

Theorem 1.2. is the PBW theorem for symplectic reflection algebras.

There are two proofs of Thm 1.2. One proof is based on the Koszul deformation principle of Drinfeld (also Braverman–Gaitsgory, Beilinson–Ginzburg–Soergel). Namely, the algebra H_0 is Koszul, so to check flatness of a deformation of this algebra, it suffices to check flatness in degrees up to 3 inclusively, which gives exactly the conditions above.

I will explain another proof, based on classical deformation theory and calculation of the Hochschild cohomology of H_0.

Theorem 1.3.

$$HH^i(H_0,H_0)=\Big(\bigoplus_j \bigoplus_{g\in\Gamma:\ \mathrm{rk}(g-1)=i-j} SV^g\otimes\Lambda^j(V^g)^*\Big)^\Gamma$$

Proof. $HH^i(H_0,H_0)=\mathrm{Ext}^i_{H_0\text{-bimod}}(H_0,H_0)=$

$$=\mathrm{Ext}^i_{(SV\#\Gamma)\otimes(SV\#\Gamma)}(SV\#\Gamma,SV\#\Gamma)\overset{\text{Shapiro L.}}{=}$$

$$\mathrm{Ext}^i_{(SV\otimes SV)\#\Gamma}(SV,SV\#\Gamma)=\mathrm{Ext}^i_{SV\otimes SV}(SV,SV\#\Gamma)^\Gamma$$

$$=\Big(\bigoplus_{g\in\Gamma}\mathrm{Ext}^i_{SV\otimes SV}(SV,SV\cdot g)\Big)^\Gamma$$

Now, $\mathrm{Ext}^*_{SV\otimes SV}(SV,SV\cdot g)=\bigotimes_{i=1}^{\dim V}\mathrm{Ext}^*_{\mathbb{C}[x]\text{-bimod}}\ \mathbb{C}[x]^{\lambda_i}$

where $\mathbb{C}[x]^\lambda$ is the bimodule with action $(f_1\circ h\circ f_2)(x)=f_1(x)h(x)f_2(\lambda x)$, and λ_i are the eigenvalue of g on V.

We have a resolution of $\mathbb{C}[x]$ as a $\mathbb{C}[x]$-bimodule:

$$0\to\mathbb{C}[x_1,x_2]\xrightarrow{(x_1-x_2)\cdot}\mathbb{C}[x_1,x_2]\xrightarrow{|_{x_1=x_2}}\mathbb{C}[x]\to 0.$$

Taking Hom from this to $\mathbb{C}[x]^\lambda$, we get

$$0 \to \underset{\text{starts in deg } 0}{\mathbb{C}[x]} \xrightarrow{(1-\lambda)x} \underset{\text{starts in deg } -1}{\mathbb{C}[x]} \to 0.$$

Its cohomology is:

1) $\mathbb{C}$ in degree 1, 0 in degree 0 if $\lambda \neq 1$

2) $\mathbb{C}[x]$ in degrees 0 and 1 if $\lambda = 1$.

So altogether we get

$$HH^i(H_0, H_0) = \Big(\bigoplus_j \bigoplus_{\substack{g \in \Gamma: \\ rk(g-1) = i-j}} SV^g \otimes \Lambda^j (V^g)^* \Big)^\Gamma$$

as desired. ■

This answer can be written as

$$HH^i(H_0, H_0) = \bigoplus_j \bigoplus_{\substack{g \in \Gamma/\mathrm{Conj} \\ rk(g-1) = i-j}} \left(SV^g \otimes \Lambda^j (V^g)^* \right)^{Z_g},$$

where Z_g is the centralizer of g.

The grading on HH^i is as follows: $\deg(V^g) = 1$, $\deg(V^{g*}) = -1$, and also terms corresponding to $g \in \Gamma$ have overall degree $-rk(g-1)$.

We are interested in HH^2 and HH^3 (since we want to study deformations).

We get

$$HH^2 = (SV \otimes \Lambda^2 V^*)^\Gamma \oplus \Big(\bigoplus_{g \in S} SV^g\Big)^\Gamma$$

In particular, we'll be interested in

$$HH^2[-2] = (\Lambda^2 V^*)^\Gamma \oplus \mathbb{C}[S]^\Gamma = E$$

We have a first order deformation of H_0 parametrized by this space. Moreover, obstructions to this deformation lie in $HH^3[\leq -4] = 0$, according to our computation. Thus, the deformations defined by E are unobstructed, and we have a universal graded deformation H of H_0 over $\mathcal{O}(E)$, such that $\deg(E^*) = 2$. Hence, given $x \in E$, we have specialization $\overline{H}_x$ of H at x, a filtered algebra with $gr(\overline{H}_x) = H_0$.

We claim that $\overline{H}_x$ coincides with H_x defined above. Indeed, let $x, y \in V$, and consider the commutator $[x, y]$ in $\overline{H}_x$. It has degree 0, so it must have the form $[x, y] = x(x, y)$, as in the beginning of the lecture (by looking at the first order terms w.r.t. parameters). This gives Th. 1.2.

Remark. We can assume, essentially without loss of generality, that Γ is generated by symplectic reflections. Indeed, if $\overline{\Gamma} = \langle S \rangle \subset \Gamma$, then $H = \overline{H} \otimes_{\mathbb{C}\overline{\Gamma}} \mathbb{C}\Gamma$, where $\overline{H}$ is the SRA associated to $\overline{\Gamma}$. Also, it suffices to assume that Γ is symplectically irreducible, i.e. $(\Lambda^2 V^*)^\Gamma = \mathbb{C}$.

Note that groups generated by symplectic reflections can be classified. This was done by A. Cohen in 1980. Here are some examples.

1. Complex reflection groups, i.e. $\Gamma \subset GL(\mathfrak{h})$ generated by $\gamma \sim \begin{pmatrix} \lambda & & \\ & 1 & \\ & & 1 \end{pmatrix}$. Such $\Gamma \subset Sp(V)$, $V = \mathfrak{h} \oplus \mathfrak{h}^*$, and any complex reflection in Γ is a symplectic reflection in this repr, so Γ is generated by symplectic reflections inside $Sp(V)$. This includes

1a) Coxeter groups, in particular Weyl groups. Notably the symmetric group S_n, $\mathfrak{h} = \mathbb{C}^{n-1}$ (or $\mathbb{C}^n$)

1b) Cyclotomic groups:

$\Gamma = S_n \ltimes \mathbb{Z}_\ell^n$ acting naturally on $V = \mathbb{C}^n$.

2. Finite subgroups of $SL_2(\mathbb{C})$:
ADE classification: $\mathbb{Z}_\ell$ (type $A_{\ell-1}$), $\ell \geq 1$
Dihedral gp $D_{4\ell}$ (type $D_{\ell+2}$) $\ell \geq 2$
Tetrahedral gp T_{24} (E_6), Cube group C_{48} (E_7)
Icosahedral gp I_{120} (E_8).

Higher rank version:

$\Gamma = S_n \ltimes G^n$, $G \subset SL_2(\mathbb{C})$, acting naturally on $(\mathbb{C}^2)^n$. If $G = \mathbb{Z}_\ell$, get cyclotomic groups from previous example.

Examples of SRA:

$\Gamma \subset SL_2(\mathbb{C})$, $H = \mathbb{C}\langle x, y \rangle \# \Gamma \Big/ [x,y] = \sum_{\gamma \in \Gamma} c_\gamma \gamma$

E.g. $\Gamma = 1 \Rightarrow$ Weyl algebra $[x,y] = t$
$\Gamma = \mathbb{Z}/2 \Rightarrow$ Cherednik algebra for $\mathbb{Z}_2$.
$[x,y] = t + Cs$, $s^2 = 1$, $sx = -xs$, $sy = -ys$.

Spherical SRA. Let $e = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma$ be the symmetrizer.

Definition. The spherical SRA is the algebra eHe (universal) or $eH_x e$ (specialized).

We have $gr(eH_x e) = e(SV\#\Gamma)e = (SV)^\Gamma$, a commutative algebra.

So $eH_x e$ is a filtered deformation (quantization) of the comm. algebra $(SV)^\Gamma$.

Theorem 1.4. The Poisson bracket on $(SV)^\Gamma$ of degree -2 defined by $eH_x e$ is given by $x_1 \in (\Lambda^2 V^*)^\Gamma$.

Proof. Direct calulation.

Corollary 1.5. If $x_1 = 0$ then $eH_x e$ is commutative.

Proof. Lemma 1.6. Any Poisson bracket on $(SV)^\Gamma$ of degree ≤ -3 is zero.

Pf. We have $(SV)^{\Gamma} \overset{-\Pi}{=} \mathcal{O}(V^*/\Gamma)$. Let $V^{*\circ}$ be the set of pts of V^* with trivial stabilizer. A Poisson bracket on V^*/Γ defines a bivector field π on $V^{*\circ}/\Gamma$, hence a Γ-invariant bivector field π on $V^{*\circ}$. Since $V^* \setminus V^{*\circ}$ has codim 2, π extends to a bivector field on V^*. If $\deg(\pi) \leq -3$ then $\pi = 0$. ∎

Now, assume $\varkappa_1 = 0$, and let d be the smallest integer such that $\exists\, f_1 \in F_i(H_{\varkappa})$, $f_2 \in F_j(eH_{\varkappa}e)$ such that $[f_1, f_2]$ has degree exactly $i+j-d$. Consider the degree $-d$ Poisson bracket induced by $eH_{\varkappa}e$ on $(SV)^{\Gamma}$. Since $d \geq 3$, this bracket is zero. So d does not exist, i.e. $eH_{\varkappa}e$ is commutative. ∎