Lecture 12
Webster's functors

1) Grassmanian case
2) General case
3) Properties
4) Appl-n to Etingof's conj.

1) Goal $\mathfrak{sl}_2$-categ. action on $\bigoplus_{v=0}^{w} D^b(\mathcal{D}_{Gr(v,w)}\text{-mod})$ ($\mathcal{D}$-modules)

$$\begin{array}{ccc} & Fl(v,v+1;w) & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ Gr(v;w) & & Gr(v+1;w) \end{array}$$

– smooth, proper morphisms $\rightsquigarrow$

$$\pi_1^*: D^b(\mathcal{D}_{Gr(v,w)}\text{-mod}) \rightleftarrows D^b(\mathcal{D}_{Fl}\text{-mod}): \pi_{1*}$$

& π_2^*, π_{2*}

$$\rightsquigarrow F := \pi_{2*}\pi_1^*: D^b(\mathcal{D}_{Gr(v,w)}\text{-mod}) \rightleftarrows D^b(\mathcal{D}_{Gr(v+1,w)}\text{-mod}): E := \pi_{1*}\pi_2^*$$

• Passing to q-class. limit ($\operatorname{gr}\mathcal{D}_{Gr(v,w)} = T^*Gr(v,w)$) get CKL functors; non-triv. op-n(!);
• Functors upgrade to $GL(w)$-equiv. cat-s
• give rise to cat-l $\mathcal{U}(\mathfrak{sl}_2)$-action
• can consider twisted DO: $\mathcal{D}^\lambda_{Gr(v,w)}$; for $\lambda\in\mathbb{Z}$ those are DO in $\mathcal{O}(\lambda)$: $\mathcal{D}_{Gr(v,w)}\text{-mod} \xrightarrow[\mathcal{O}(\lambda)\otimes\bullet]{\sim} \mathcal{D}^\lambda_{Gr(v,w)}\text{-mod}$

So also get $\mathfrak{sl}_2 \curvearrowright \bigoplus_{v=0}^{w} D^b(\mathcal{D}^\lambda_{Gr(v,w)}\text{-mod})$ ($\lambda\in\mathbb{Z}$!!!)

2) $\lambda_i\in\mathbb{Z}$, $\theta_i\in\mathbb{Z}$ cat-l $\mathfrak{sl}_2$-action on $\bigoplus_v D^b(\mathcal{A}^\theta_\lambda(v)\text{-mod})$
(reminder: quiver Q, $v,w\in\mathbb{Z}^{Q_0}$, $\theta\in\mathbb{Z}^{Q_0} \rightsquigarrow$ quiv. var-ty $\mathcal{M}^\theta(v)$ $= \mu^{-1}(0)^{\theta\text{-ss}}/\!\!/GL(v)$, $\mu: T^*R\to\mathfrak{g}$ - moment map; $R = Rep(Q,v,w)$
$\lambda\in\mathbb{C}^{Q_0} \rightsquigarrow \mathcal{A}^\theta_\lambda(v) = [D_R/D_R\{x_R - \langle\lambda,x\rangle, x\in\mathfrak{g}\}|_{T^*R^{\theta\text{-ss}}}]^G$

Idea: $\mathcal{A}^\theta_\lambda(v)$-mod = quot. of cat-y $D_R\text{-mod}^{G,\lambda}$
Def: (G,λ)-equiv-t D_R-module: D_R-module M w. rat-l $G\curvearrowright M$

fin. gen modules | Rem $SS(M) \subset \mu^{-1}(0)$

s.t. $\chi_M = \chi_R$ ~~▪~~ $\langle \lambda, x \rangle$.

Ham. red functor: $\pi(v): \mathcal{D}_R\text{-mod}^{G,\lambda} \longrightarrow \mathcal{A}^\theta_\lambda(v)\text{-mod}$

$$M \longmapsto \left[M|_{T^*R^{\theta\text{-ss}}} \right]^G$$

-quot-t functor w. ker $= \{ M \mid SS(M) \subset T^*R \setminus (T^*R)^{\theta\text{-ss}} \}$

b/c $G \curvearrowright \mu^{-1}(0)^{\theta\text{-ss}}$ freely

- can use ~~[illegible]~~ const-n in 1) + reduction in stages: first $GL(v_i)$, then everything else

– reverse arrows so that i is sink in Q (& $Hom(V_i, W_i) \subset R$)

$\tilde{W}_i = W_i \oplus \bigoplus_{a,\, t(a)=i} V_{h(a)}$, $\underline{R}$ – all arrows not adj to i:

$$R = Hom(V_i, \tilde{W}_i) \oplus \underline{R}.$$

$$\underline{G} := \prod_{j \neq i} GL(v_j)$$

$$T^*R /\!/\!/^{\theta_i} GL(v_i) = T^*Gr(v_i, \tilde{w}_i) \times T^*\underline{R}$$

$$\mathcal{D}_R /\!/\!/^{\theta_i}_{\lambda_i} GL(v_i) = \mathcal{D}^\lambda_{Gr(v_i, \tilde{w}_i)} \otimes \mathcal{D}_{\underline{R}}$$

$$\begin{tikzcd}
\mathcal{D}_R\text{-mod}^{G,\lambda} \arrow[r, "\pi_i(v)"] \arrow[dr, "\pi(v)"'] & \mathcal{D}^\lambda_{Gr(v_i,\tilde{w}_i)} \times \mathcal{D}_{\underline{R}}\text{-mod}^{\underline{G},\lambda} \arrow[d, "\underline{\pi}(v)"] \\
& \mathcal{A}^\theta_\lambda(v)\text{-mod}
\end{tikzcd}$$

Have functors E_i, F_i w.r.t 1st factor

Fact: ker $\underline{\pi}(v)$ is stable $\Rightarrow$ functors descend to $\mathcal{A}^\theta_\lambda(v)$-mod -functors we want.

12. 3) a) E_i, F_i give cat-l $\mathfrak{sl}_2$-action: enough to check in $GL(w)$-equiv. setting for Gr's.

b) E_i, F_i preserve $\bigoplus_v \mathcal{D}^b_{\rho^{-1}(0)}(\mathcal{A}^\theta_\lambda(v)\text{-mod})$ – cat-y we need for Etingof's conj.

c) $[E_i], [F_i]$ – op-rs on $\bigoplus K_0(\mathcal{A}^\theta_\lambda(v)\text{-mod}_{\rho^{-1}(0)}) \xrightarrow{CC} L_\omega$

$CC \circ [E_i] = e_i \circ CC$, $CC \circ [F_i] = f_i \circ CC$ – cat-n of Nakajima's constr-n

Reason: F_i -"convolution" w. suitable $\mathcal{A}_\lambda^\theta(v+\epsilon_i)$-$\mathcal{A}_\lambda^\theta(v)$-bimodule
convol-n commutes w. char. cycles

12.4) $\lambda_i \in \mathbb{Z} \rightsquigarrow$ functors $E_i, F_i \rightsquigarrow$ Im CC is stable under e_i, f_i
Need: real root α ~~⇒~~ ~~…~~ w. $\langle\lambda, \alpha\rangle \in \mathbb{Z} \Rightarrow$ Im CC is closed under $e_\alpha, f_\alpha \leftarrow ?$ functors E_α, F_α
Recall: quantum LMN isomorphisms : $\sigma \in W(Q)$
$\sigma : \mathcal{A}_\lambda^\theta(v)$ ~~-mod~~ $\xrightarrow{\sim} \mathcal{A}_{\sigma\cdot\lambda}^{\sigma\theta}(\sigma\cdot v)$
$\rightsquigarrow \sigma_* : \mathcal{A}_\lambda^\theta(v)\text{-mod} \xrightarrow{\sim} \mathcal{A}_{\sigma\cdot\lambda}^{\sigma\theta}(\sigma\cdot v)\text{-mod}$
Rem: $\langle\lambda,\alpha\rangle \in \mathbb{Z} \Leftrightarrow \langle\sigma\cdot\lambda, \sigma\alpha\rangle \in \mathbb{Z}$ (b/c $\sigma\cdot\lambda - \sigma\lambda$ is integral)
$\exists\, \sigma$ s.t. $\sigma\alpha = \pm\alpha_i$, $(\sigma\theta)_i > 0$
$\rightsquigarrow$ Webster's functors $E_i, F_i \curvearrowright \bigoplus_v D^b(\mathcal{A}_{\sigma\cdot\lambda}^{\sigma\theta}(v)\text{-mod})$
$\rightsquigarrow E_\alpha := \sigma_*^{-1} E_i \sigma_*$, $F_\alpha = \sigma_*^{-1} F_i \sigma_*$ (if $\sigma\alpha = \alpha_i$)
-preserve $\bigoplus_v D^b_{\rho^{-1}(0)}(\mathcal{A}_\lambda^\theta(v)\text{-mod})$
+ on $K_0(\mathcal{A}_\lambda^\theta(v)\text{-mod})$ have $[E_\alpha] = \pm e_\alpha$, $[F_\alpha] = \pm f_\alpha$ (e_α, f_α-op-rs from $\mathfrak{g}(Q)$)

Reason: LMN isom-s $\rightsquigarrow W(Q) \curvearrowright L_\omega = \bigoplus_v H_{top}(\mathcal{M}^\theta(v))$
canon. indep of θ

on the other hand $\mathfrak{g}(Q) \curvearrowright L_\omega \rightsquigarrow$ centr. ext of $W(Q) \curvearrowright L_\omega$.
two actions "coincide" (up to sign)